ACCL Lecture 4: Infinitary classical propositional logic: Axiomatization and Completeness

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Advanced Course in Classical Logic
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Other connectives ($\wedge, \vee, \to$) are introduced as abbreviations; in particular:

$$\bigvee \Phi := \neg \bigwedge \{\neg A \mid A \in \Phi\}.$$

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A set $\Gamma \subseteq \mathrm{Fm}$ is satisfiable if $v \models \Gamma$ for some valuation v.

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A derivation from a set of hypothesis $\Gamma \vdash A$.
(here Γ is at most countable)

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Lemma (Deductive closure of a set of formulas)

For any set Γ, the set Γ_{ω_1} is closed under the rules (MP) *and* $(\mathsf{R}\bigwedge)$.

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Take any $\vec{a} = (a_0, a_1, \dots)$ from X_{ω_1}.
Then, for each $i \in \mathbb{N}$, $a_i \in X_{\lambda_i}$ for some countable ordinal $\lambda_i < \omega_1$.

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Lemma (The closure of X is reached at step ω_1)

For any set X, the set X_{ω_1} is closed under all function from $\mathcal{S}$.

Proof.

Take any $\vec{a} = (a_0, a_1, \ldots)$ from X_{ω_1}.
Then, for each $i \in \mathbb{N}$, $a_i \in X_{\lambda_i}$ for some countable ordinal $\lambda_i < \omega_1$.
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Then $f(\vec{a}) \in X_{\lambda+1}$.
Therefore, $f(\vec{a}) \in X_{\omega_1}$. So X_{ω_1} is closed under f. $\square$

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(a) $\vdash A \iff A$ *is valid.*

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$(a) \Rightarrow (b)$ by Deduction theorem (2). So we prove only (a).

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Then H is satisfiable. (Lemma 2)
Therefore, A is not valid.
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Why $\mathrm{Sub}(\Gamma)$ is again at most countable?

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Corollary. *If* $|\Gamma| \leqslant \omega$, *then* $|\mathrm{Sub}(\Gamma)| \leqslant \omega$.

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  (!) Such a formula B exists by the rule $(R \bigwedge)$.

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The remainder of the proof is in the lecture notes.
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This derivation uses only countable subset $\Gamma' \subseteq \Gamma$.

Theorem (Completeness)

(a) $\vdash A \iff A$ *is valid.*
(b) $\Gamma \vdash A \iff \Gamma \models A$. *(here Γ is at most countable)*

This theorem (b) cannot be generalized to uncountable sets Γ!
Moreover, no rules with the countable number of premises can make the calculus complete! We need rules with uncountably many premises!

Recall the set $\Gamma \subseteq \mathrm{Fm}$ such that
– each countable subset of Γ is satisfiable,
– the whole set Γ is not satisfiable.

Then $\Gamma \models \bot$.

However, $\Gamma \nvdash \bot$. Indeed, otherwise $\bot$ has a countable derivation from Γ. This derivation uses only countable subset $\Gamma' \subseteq \Gamma$. But we know that $\Gamma' \not\models \bot$.