ACCL Lecture 3: Infinitary classical propositional logic: Syntax, semantics, non-compactness. The cardinality of the set of formulas

Evgeny Zolin

Department of Mathematical Logic and Theory of Algorithms
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Advanced Course in Classical Logic
March 10th, 2021

Warming up

$f(x_1)$ —

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$f(x_1, x_2, \ldots)$ — infinitary function (инфинитарная, бесконечноместная)

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[файнайт]	[инфинит]
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Formulas of the infinitary propositional language:
the set Fm is the smallest set, such that

- every variable $p_i \in \mathrm{Var}$ is a formula: $p_i \in \mathrm{Fm}$,

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Other connectives are introduced as abbreviations:

$$(A \wedge B) := \bigwedge\{A, B\}, \quad (A \vee B) := \neg(\neg A \wedge \neg B),$$
$$(A \to B) := \neg(A \wedge \neg B), \quad \bigvee \Phi := \neg \bigwedge \{\neg A \mid A \in \Phi\}.$$

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Also we abbreviate $\top := \bigwedge \varnothing$, $\bot := \neg \top$.

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Exactly continuum or more?

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A set $\Gamma \subseteq \mathrm{Fm}$ is satisfiable if $v \models \Gamma$ for some valuation v.

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For every valuation $v = \sigma$ we have $v \not\models \neg A_\sigma$.

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But every proper subset of Γ is satisfiable!

Representing Boolean functions

In the usual finitary logic:
How many functions $f(x_1, \ldots, x_n)\colon \{0,1\}^n \to \{0,1\}$ does there exist?

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Task. Try to build a concrete example of an non-expressible function.

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Answer: at $\lambda = \omega_1$ (the first uncountable ordinal).

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A derivation from a set of hypothesis $\Gamma \vdash A$ — is defined similarly.
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Theorem (Completeness)

(a) $\vdash A \iff A$ *is valid.*

(b) $\Gamma \vdash A \iff \Gamma \models A$. *(here* Γ *is at most countable)*

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Thank you! Questions?