ACCL Lecture 2: Classical Propositional Logic: Set semantics. Boolean algebras. Algebraic semantics

Evgeny Zolin

Department of Mathematical Logic and Theory of Algorithms
Faculty of Mechanics and Mathematics
Moscow State University

Advanced Course in Classical Logic
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Solving set equations and inclusions

Does the following always hold? Is it "the law of sets"?

$$X \cup \overline{Y \cap (X \cup Z)} \quad \subseteq \quad Z \cup (\overline{X} \cap \overline{(Y \cap Z)})$$

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In our example:

$$X \vee \neg(Y \wedge (X \vee Z)) \quad \to \quad Z \vee (\neg X \wedge \neg(Y \wedge Z))$$

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- the symbols $\bot$ and $\top$ are formulas,
- every variable p_i is a formula,
- if A is a formula, then $\neg A$ is a formula,
- if A, B are formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ are formulas.

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This definition can be written concisely:

$$A, B \ ::= \ \bot \ | \ \top \ | \ p_i \ | \ \neg A \ | \ (A \wedge B) \ | \ (A \vee B) \ | \ (A \to B).$$

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The set of all formulas is denoted by Fm.

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Valid and satisfiable; consequence relation

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Hence $\exists e \in W\colon\ e \in \mathcal{I}(\neg A)$. Of course, $e \in \mathcal{I}(B)$ for all $B \in \Gamma$. $\square$

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Equivalence of 2-valued and set semantics

Theorem (Completeness of CPL for set semantics)

For every formula $A \in \mathsf{Fm}$ and every set of formulas $\Gamma \subseteq \mathsf{Fm}$

(1) *A is derivable in CPC* $\iff$ *A is valid* $\iff$ *A is set-valid:*

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So, it remains to prove (3).

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So Γ is set-satisfiable. $\square$

Remark: If $W = \{e\}$, then $\mathcal{I}(A)$ has 2 values: $\varnothing$ and W.
So, it behaves like a valuation with 2 values: 0 and 1.

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<u>**Conclusion:**</u> The Classical Propositional Logic axiomatizes not only the laws of $\neg, \wedge, \vee, \to$, but also the laws of set operations $\overline{}, \cap, \cup, \subseteq$.

Definition

Boolean algebra: $\mathcal{B} = (D, \wedge, \vee, -, 0, 1)$, where $D \neq \varnothing$ is a set, $0, 1 \in D$, the operations $\wedge, \vee \colon D \times D \to D$ and $- \colon D \to D$ satisfy the laws:

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We can define implication $\to \colon D \to D$ by putting $a \to b := (\overline{a} \vee b)$.

Example (a)

$\mathcal{B} = (2^W, \cap, \cup, -, \varnothing, W)$ is a Boolean algebra. Here $\overline{a} := X \setminus a$.

Boolean algebras: examples

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Later in this course we will show that:

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Example

$S = \{X \subseteq \mathbb{N} \mid X \text{ is finite or co-finite}\}$ is a Boolean algebra.

Boolean algebras: examples

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$\mathcal{B} = (2^W, \cap, \cup, -, \varnothing, W)$ is a Boolean algebra. Here $\overline{a} := X \setminus a$.

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Assume that $S \subseteq 2^W$ is closed under $\cup, \cap, -$. Clearly, $\varnothing, W \in S$.
Then $(S, \cap, \cup, -, \varnothing, W)$ is a Boolean algebra, it is called a set algebra.

Later in this course we will show that:

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Hence it is not isomorphic to any algebra of the form (a).

Propositional Logic: Algebraic semantics

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If $D = 2^W$, then this semantics = set semantics $\mathcal{I}(p) \subseteq W$.

Algebraic semantics: Validity, satisfiability

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A set Γ BA-implies (or BA-entails) a formula A, in symbols: $\Gamma \models_{\mathrm{a}} A$, if

for every BA-interpretation i such that $i \models \Gamma$, we have $i \models A$.

Equivalence of 2-valued and algebraic semantics

Theorem (Completeness of CPL for algebraic semantics)

For every formula $A \in \mathsf{Fm}$ *and every set of formulas* $\Gamma \subseteq \mathsf{Fm}$

(1) *A is derivable $\Leftrightarrow$ A is valid $\Leftrightarrow$ A is set-valid $\Leftrightarrow$ A is BA-valid:*

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(2) *Γ proves A* $\Leftrightarrow$ *Γ implies A* $\Leftrightarrow$ *Γ set-implies A* $\Leftrightarrow$ *Γ BA-implies A:*

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For every formula A, the following are equivalent:

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So it will be one counter-model to all non-theorems. □

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What about $\Gamma \vdash A \Longleftrightarrow \Gamma \models_{\mathsf{a}} A$?

Lindenbaum algebra for Γ

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The end of lecture 2. Thank you!