EXPLICIT FORMULA FOR VIRASORO SINGULAR VECTOR

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We denote by \( Vir \) the Virasoro algebra with generators \( \{ C, L_i, \ i \in \mathbb{Z} \} \) and \( V(h, c) \) stands for a Verma module over \( Vir \). A nontrivial vector \( w \in V(h, c) \) is called singular, if \( L_iw = 0 \) for all positive \( i \). There is a singular vector \( w \) with grading \( n \) in a Verma module \( V(h, c) \) if and only if there exist two positive integers \( p \) and \( q \) and a complex number \( t \neq 0 \) such that \( pq = n \) and

\[
c = c(t) = 13 + 6t + 6t^{-1}, \quad h = h_{p,q}(t) = \frac{1 - p^2}{4} t + \frac{1 - pq}{2} + \frac{1 - q^2}{4} t^{-1}.
\]

For fixed positive integers \( p \) and \( q \) and a fixed complex number \( t \) the Verma module \( V(h_{p,q}(t), c(t)) \) contains a homogeneous singular vector \( w_{p,q}(t) \) of degree \( pq \) defined uniquely up to multiplication by some scalar:

\[
w_{p,q}(t) = S_{p,q}(t)v = \sum_{i_1 + \ldots + i_s = pq} P_{p,q}^{i_1, \ldots, i_s}(t)L_{-i_1} \ldots L_{-i_s}v,
\]

We assume that the coefficient \( P_{p,q}^{i_1, \ldots, i_s}(t) \) is equal to one.

In 1988 Benoist and St-Aubin found an explicit expression for the series \( S_{1,p}(t) \):

\[
S_{1,p}(t) = \sum_{i_1 + \ldots + i_s = p} \frac{(p-1)!^2 t^{s-p}}{\prod_{l=1}^{s-1} ((\sum_{q=1}^{l} i_q)(p-\sum_{q=1}^{l} i_q))} L_{-i_1} \ldots L_{-i_s},
\]

Later Bauer, Di Francesco, Itzykson and Zuber, using the formula (1) as an initial step, proposed an algorithm for finding all singular vectors. But in practice, their algorithm meets with serious computational difficulties and it is still unclear how to get with its help an explicit formula for all \( S_{p,q}(t) \). However, the expressions for \( S_{2,2}(t) \) and \( S_{2,3}(t) \) found by means of it allowed us to guess the general formula for singular vectors of the entire series \( S_{2,p}(t) \).

We prove that for a Verma module \( V(h, c) \) over the Virasoro algebra with

\[
c = c(t) = 13 + 6t + 6t^{-1}, \quad h = h_{2,p}(t) = -\frac{1}{4} (p-1+t)(t^{-1}(p+1)+3), \quad t \in \mathbb{C}, t \neq 0.
\]

we have the following formula for \( S_{2,p}(t) \)

\[
S_{2,p}(t) = \sum_{i_1 + \ldots + i_s = 2p} f_{2,p}(t; i_1, \ldots, i_s)L_{-i_1} \ldots L_{-i_s},
\]

where the sums are over all partitions of \( 2p \) by positive numbers without any ordering restrictions and the coefficients \( f_{2,p}(t; i_1, \ldots, i_s) \) are defined by the formulas

\[
f_{2,p}(t; i_1, \ldots, i_s) = \frac{(2p-1)!^2(2t)^{s-2p}}{\prod_{r=1}^{2p-1} (p-t-r)} \prod_{m=1}^{s} \left( (2t-1)(i_m-1)+2p-1-2 \sum_{n=1}^{m-1} i_n \right) \prod_{k=0}^{p-1} (2p-1-2k) \prod_{l=1}^{s} \left( (\sum_{n=1}^{l} i_n)(2p-1-\sum_{n=1}^{l} i_n)(p-t-\sum_{n=1}^{l} i_n) \right).
\]

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