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Behavior of solutions of an acoustic wave diffraction problem on a set of small obstacles

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Abstract. The article discusses the problem of diffraction of an acoustic wave by a multitude of obstacles (or cavities) enclosed in a finite subdomain of a homogeneous material domain. It is assumed that the distance between obstacles (cavities) is much greater than the size of each obstacle (cavity). Conditions are indicated under which the total influence of obstacles on the transmitted wave is expressed as an additional term of the potential type in the wave equation. Conditions at the boundary of the cavities are accepted as boundary conditions of the third kind. The results of the work can be used in the development of non-destructive testing procedures.

1. Introduction

In this paper, we consider the problem of diffraction of an acoustic wave at an object consisting of a large number of small obstacles or cavities. It is assumed that the diameter of each of these obstacles is much smaller than the distance to the "nearest neighbor". A neighbor is an obstacle located at a minimum distance from a given one. Obstacles can also be interpreted as cavities of small diameter in the domain of three-dimensional space occupied by homogeneous material. Due to the large number of specific problems in mechanics, physics, and biology, such problems are considered by many authors. One example of an application is the problem of developing methods for non-destructive testing of materials samples: it is necessary to evaluate how the appearance of cavities or impurities inside a sample will modify the transmitted acoustic or electromagnetic field [1,2,3]. The mathematical foundations of analyzing the magnitude of changes in solving boundary value problems when small obstacles or cavities appeared were first developed in [4] and subsequently actively developed by a number of Russian and foreign researchers. In this paper, boundary conditions of the third kind on the obstacles' surfaces (Fourier condition) are considered. As was established in [4] for the Dirichlet and Neumann boundary conditions on cavities, in a limit or averaged model, under certain conditions, a potential or "additional term" arises. The condition for its appearance is a certain rate of tending to zero of the ratio of the obstacle diameter and the distance to the "nearest neighbor", as well as the type of boundary condition. These conditions were obtained earlier for the Dirichlet conditions in terms of the cavities' electrostatic capacity [4] and for the third boundary condition [5]. The corresponding boundary value problems were considered in bounded domains. In this paper, we consider the diffraction problem in an unbounded domain with radiation conditions at infinity. Consideration of radiation conditions is the problem that has an important place in this work. We study the existence and asymptotic behavior of a solution by reducing the diffraction problem in an

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unbounded domain to a problem in a bounded one, but for a nonlocal operator. This method is described in [6], and presented for the first time in [7].

2. Problem specification

We formally define what "obstacles" are and give the conditions under which the appearance of an additional term in the limit equation will be established below. Consider *M* different sets of the next form $Y^j = Q \setminus \overline{G}^j$, j = 1, 2, ..., M; *Q* is an unit cube, $\overline{G}^j \in Q$, each connected component G^j is diffeomorphic to a ball in \mathbb{R}^n . We denote $Y_{\varepsilon}^j = \varepsilon Q \setminus a_{\varepsilon}^j \overline{G}^j$, $a_{\varepsilon}^j \leq \varepsilon$, where a_{ε}^j is some constant depending on j = 1, 2, ..., M and ε . We define functions Θ_{ε}^j as solutions to problems:

$$\Delta \Theta_{\varepsilon}^{j} = \mu_{\varepsilon}^{j} \text{ at } x \in Y_{\varepsilon}^{j}; \qquad \frac{\partial \Theta_{\varepsilon}^{j}}{\partial v} = 1 \text{ on } S_{\varepsilon}^{j} = a_{\varepsilon}^{j} \partial G^{j};$$

$$\frac{\partial \Theta_{\varepsilon}^{j}}{\partial v} = 0 \text{ on } \gamma_{\varepsilon}^{j} \equiv \partial Y_{\varepsilon}^{j} \setminus S_{\varepsilon}^{j}; \quad \left\langle \Theta_{\varepsilon}^{j} \right\rangle = 0, \text{ where } j = 1, 2, \dots, M.$$
(1)

The constant μ_{ε}^{j} is determined from the condition of solvability of problems (1) (and these are Neumann problems and they are solvable only with a certain relation of the boundary condition and the right side of the equation):

$$\mu_{\varepsilon}^{j} = \frac{|\partial G^{j}|}{\left(a_{\varepsilon}^{j}\right)^{1-n} \varepsilon^{n} \left(1 - \left(a_{\varepsilon}^{j} \varepsilon^{-1}\right)^{n} |G^{j}|\right)}.$$
(2)

Let be $\mu_{\varepsilon}^{j} \to \mu$ if $\varepsilon \to 0$, and μ is independent of *j*. Then we get the following relation: $(a_{\varepsilon}^{j})^{1-n}\varepsilon^{n} \to const$. When n = 3, a_{ε}^{j} is independent of *j* and the next equality holds $a_{\varepsilon}^{j} = \varepsilon^{\alpha}$, it follows from our condition that $\alpha = \frac{3}{2}$.

Let also ω_{ε} is an unlimited domain in \mathbb{R}^n , which can be divided into cells, and each of the cells geometrically coincides with one of the sets Y_{ε}^j , j = 1, 2, ..., M. Let a region Ω contains a set Ω_* , that consists of a finite number of connected domains Ω^i , i = 1, ..., l. Let be $\Omega_{\varepsilon}^i = \Omega_i \cap \omega_{\varepsilon}$, $\Omega_*^{\varepsilon} = \bigcup_{i=1}^l \Omega_{\varepsilon}^i$. We denote $\Omega_{\varepsilon} = (\Omega \setminus \Omega_*) \cup \Omega_*^{\varepsilon}$. We consider the following diffraction problem in an unbounded domain:

$$(\Delta + \omega^2) u_{\varepsilon,\omega} = f \text{ in } (\mathbf{R}^3 \setminus \Omega_*) \cup \Omega^{\varepsilon}_*, \tag{3}$$

$$\frac{\partial u_{\varepsilon,\omega}}{\partial v} + \alpha u_{\varepsilon,\omega} = 0 \text{ on } S_{\varepsilon} = \partial \Omega^{\varepsilon}_{*}, \alpha = const, \alpha > 0, \qquad (4)$$

$$\frac{\partial u_{\varepsilon,\omega}}{\partial R} - i\omega u_{\varepsilon,\omega} = \overline{o}\left(\frac{1}{R}\right), \text{ at } R \to \infty,$$
(5)

where $f(x) \in L_2(\mathbf{R}^3)$, supp f(x) = K, K is a compact in \mathbf{R}^3 . Supp means the support of function, that is, the set where its values are nonzero. We understand equation (3) and boundary condition (4) in the generalized sense, i.e., for an arbitrary function $\varphi \in C_0^{\infty}(\mathbf{R}^3)$ the integral identity must be satisfied

$$\int_{\Omega_{\varepsilon}} (\Delta u_{\varepsilon,\omega}, \Delta \varphi) dx + \alpha \int_{\partial \Omega_{\varepsilon}} u_{\varepsilon,\omega} \varphi ds - \omega^2 \int_{\Omega_{\varepsilon}} u_{\varepsilon,\omega} \varphi dx = -\int_{\Omega_{\varepsilon}} f \varphi dx$$

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Since f(x) is equal to zero for sufficiently large values |x|, by virtue of theorems on the internal smoothness of solutions of elliptic equations $u_{\varepsilon,\omega} \in C^{\infty}(\mathbb{R}^3 \setminus K)$ and the radiation condition (5) can be understood in the classical sense. We define the function $u_{0,\omega}$ as the solution to the following problem:

$$(\Delta + \omega^2) u_{0,\omega} = f \text{ in } \mathbf{R}^3 \setminus \Omega_*, \qquad (6)$$

$$(\Delta + \omega^2 + \mu)u_{0,\omega} = f \text{ in } \Omega_*, \qquad (7)$$

$$[u_{0,\omega}] = \left[\frac{\partial u_{0,\omega}}{\partial n}\right] = 0 \text{ on } \partial \Omega_*, \tag{8}$$

$$\frac{\partial u_{0,\omega}}{\partial R} - i\omega u_{0,\omega} = \overline{o} \left(\frac{1}{R}\right) \text{at } R \to \infty.$$
(9)

This problem corresponds to the "average" solution, which should well describe the wave field at low ε values. Equations (6) - (7) and the boundary condition (8) are also understood in a generalized sense, that is, in the sense of fulfilling the integral identity

$$\int_{\mathbf{R}^3} (\Delta u_{0,\omega}, \Delta \varphi) dx + \int_{\mathbf{R}^3} K(x) u_{0,\omega} \varphi dx = -\int_{\Omega_{\varepsilon}} f \varphi dx$$

for arbitrary function $\varphi \in C_0^{\infty}(\mathbf{R}^3)$, and

$$K(x) = \begin{cases} \omega^2 & \text{in } \mathbf{R}^3 \setminus \Omega_* \\ \omega^2 + \mu & \text{in } \Omega_* \end{cases}$$

Here, the radiation condition (9) must also be understood in the classical sense, since f(x) = 0 at large |x| values, and $u_{0,\omega}$ is a smooth function for large |x|.

Our goal is to compare problem (3)-(5) and (6)-(9) solutions and prove their proximity in the corresponding functional space. Our task is to investigate the limit transition at $\varepsilon \to 0$ in the problem (3)-(5) and to show that in a certain sense $u_{\varepsilon,\omega}$ is close to $u_{0,\omega}$. First, we investigate the solvability of the problems under consideration.

3. Existence of solutions to problems (3) - (5) and (6) - (9)

For the solvability of the corresponding diffraction problems, we use the method described in [3, 7]. We will seek a solution to the problem in the form of a convolution of the function of a point source that describes a diverging wave and some "density" of the distribution of such sources in the volume.

Let be
$$w_{\varepsilon} = g_{\varepsilon} * \psi^+$$
, where $g_{\varepsilon} \in L_2(\mathbf{R}^3)$, $\sup g_{\varepsilon} \in \Omega_{\varepsilon}^{\rho+1} = \{|x| < \rho+1\} \setminus (\Omega_* \setminus \Omega_*^{\varepsilon}), \psi^+ = -\frac{\exp(i\omega r)}{4\pi r}$

(here * means a convolution operation). Then w_{ε} satisfies condition of diverging radiation and equation [3]:

$$(-\Delta - \omega^2) w_{\varepsilon} = g_{\varepsilon} \text{ in } \Omega_{\varepsilon}^{\rho+1}.$$
(10)

We define an auxiliary function $v_{\varepsilon} \in H^1(\Omega_{\varepsilon}^{\rho+1})$ as a solution to the problem:

$$(-\Delta - \omega^2) v_{\varepsilon} = (-\Delta - \omega^2) w_{\varepsilon} \text{ in } \Omega_{\varepsilon}^{\rho+1}.$$
(11)

$$\left(\frac{\partial}{\partial v} + \alpha\right) v_{\varepsilon} = 0 \text{ on } S_{\varepsilon}.$$
(12)

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$$v_{\varepsilon} = w_{\varepsilon} \text{ on } \partial \Omega_{\varepsilon}^{\rho+1}.$$
(13)

We will further consider the function $u_{\varepsilon,\omega} = w_{\varepsilon} - \gamma(w_{\varepsilon} - v_{\varepsilon})$, where $\gamma = \gamma(r) \in C^{\infty}(\mathbf{R}_{+})$, $0 \le \gamma(r) \le 1$ is "shear" function defined as follows:

$$\gamma(r) = \begin{cases} 1, \text{ if } 0 < r < \rho \\ 0, \text{ if } r > \rho + 1 \end{cases}$$
(14)

The function $u_{\varepsilon,\omega}$ satisfies the condition of divergent radiation, since it coincides with the function w_{ε} for large r and satisfies the condition $\left(\frac{\partial}{\partial v} + \alpha\right)u_{\varepsilon,\omega} = 0$ on S_{ε} (the last equality holds because $u_{\varepsilon,\omega} \equiv v_{\varepsilon}$ on S_{ε}). In order for the equation to hold $(\Delta + \omega^2)u_{\varepsilon,\omega} = 0$ we will choose such a function $g_{\varepsilon} \in L_2(\mathbb{R}^3)$, supp $g_{\varepsilon} \in \Omega_{\varepsilon}^{\rho+1}$, so that the operator equation is satisfied:

$$g_{\varepsilon} + T_{\varepsilon}(\omega)g_{\varepsilon} = -f(15)$$

in domain $\Omega_{\varepsilon}^{\rho+1}$, where operator $T_{\varepsilon}(\omega): L_2(\Omega_{\varepsilon}^{\rho+1}) \to L_2(\Omega_{\varepsilon}^{\rho+1})$ defined by the formula:

$$T_{0}(\omega)g_{0} = (\Delta\gamma)(w_{0} - v_{0}) + 2(\nabla\gamma, \nabla(w_{0} - v_{0})).$$
(16)

According to our method, the compactness of the operator $T_{\varepsilon}(\omega)$, established in [3] allows us to prove the solvability of the corresponding problem. Operator $T_{\varepsilon}(\omega)$ can be considered as an operator acting on a function g_{ε} . Indeed, the functions w_{ε} and v_{ε} , defined earlier, were constructed by a given function g_{ε} and therefore can be considered as the values of some linear operators of the function g_{ε} . Function $u_{0,\omega} = w_0 - \gamma(w_0 - v_0)$ (where $w_0 \equiv g_0 * \psi^+$) satisfies the first two relations of the problem

(6)-(9), if $g_0 \in L_2(\mathbf{R}^3)$ satisfies the operator equation

$$g_0 + T_0(\omega)g_0 = -f$$
(17)

in $\Omega_{\varepsilon}^{\rho+1}$, where the operator $T_0(\omega): L_2(\Omega^{\rho+1}) \to L_2(\Omega^{\rho+1})$ defined by the formula

$$T_{0}(\omega)g_{0} = (\Delta\gamma)(w_{0} - v_{0}) + 2(\nabla\gamma, \nabla(w_{0} - v_{0})), \qquad (18)$$

and v_0 is a solution to the following problem:

$$(-\Delta - \omega^2)v_0 = (-\Delta - \omega^2)w_0 \text{ in } \Omega^{\rho+1} \setminus \Omega_*, \qquad (19)$$

$$(-\Delta - \omega^2 + \mu)v_0 = (-\Delta - \omega^2)w_0 \text{ in } \Omega_*, \qquad (20)$$

$$[v_0] = \left[\frac{\partial v_0}{\partial v}\right] = 0 \text{ on } \partial \Omega_*, \qquad (21)$$

$$v_0 = w_0 \text{ on } \partial \Omega^{\rho+1} \,. \tag{22}$$

Functions u_0 and v_0 can also be interpreted as the result of the action of some linear operators on the function g_0 .

The following lemma was proved in [3].

Lemma. Operators $T_{\varepsilon}(\omega), T_0(\omega)$ are compact in $L_2(\Omega_{\varepsilon}^{\rho+1}), L_2(\Omega^{\rho+1})$ respectively, and equations (17) and (15) are solvable.

Further, if g_{ε}, g_0 are the solutions to corresponding equations (17) and (15), then as shown in [3], functions $u_{\varepsilon,\omega}, u_{0,\omega}$ are solutions to problems (6) - (9) and (3) - (5).

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The following statement about the proximity of solutions of problems (3)-(5) and (7)-(10) is valid:

Solutions of equations (18) and (20) satisfy the next estimate: $\|g_{\varepsilon} - g_0\|_{L_2(\Omega_{\varepsilon}^{\rho+1})} \leq C \cdot \sqrt{\varepsilon}$, if $a_{\varepsilon}^j \cong C \cdot \varepsilon^{\frac{3}{2}}$;

(C is constant).

In this paper, we omit the mathematical proofs of this statement. We only note that it is based on the result of work [5], where the proximity of boundary value problems solutions in a bounded domain (without radiation conditions) is estimated in the presence of a set of small obstacles or cavities in the domain, and for a homogeneous region, but for equations with additional term.

Such a reduction of the averaging problem of the diffraction problem in an unbounded region with radiation conditions at infinity to the averaging problem in a bounded region is an essential result of this paper.

4. Conclusion

It is clear from the present paper that the physical interpretation of the "critical" parameter value α is as follows. If alpha is less than 1.5, then the wave does not practically pass into the area occupied by "obstacles", and with alpha greater than 1.5, almost nothing is reflected, the wave passes through the zone occupied by obstacles, almost not noticing them. If alpha is 1.5, then the influence of many obstacles appears "on average" in the form of the appearance of "potential" and in the limit (averaged) model.

It should be noted that previously averaging problems were considered for various boundary value problems and for various assumptions regarding the geometric characteristics of material inhomogeneities. A variety of results are presented in monographs [4,6-11]. An important characteristic of the geometric structure of inhomogeneities or material defects is the relation between the dimensions of the inhomogeneities and the distance from an individual defect to the "nearest neighbor". If the indicated values have the same order of smallness, then the averaging effect usually leads to in the averaged differential equation or system to a change in the highest term, if the distance to the nearest neighbor significantly exceeds the size of the defect itself (and this is a completely natural condition), then this will be a condition for the appearance of a zero-order term having the form of a potential in the averaged model [4,5,12-20]. The conditions for the appearance of the potential and its expression also depend on the type of boundary conditions at surface defects. In the present paper, boundary conditions of the third kind are considered, which describe the processes of interaction of acoustic and electromagnetic waves from the boundary of the region rather well. A feature of this work is the presence of radiation conditions. Here we use the method of reducing a boundary-value problem in an unbounded domain with radiation conditions to a boundary-value problem for a nonlocal operator, but in a bounded domain, proposed in [7].

The statement we have proved about the influence of small obstacles can be used to describe the propagation of waves in media with small impurities, and it can also be useful in identifying defects in a homogeneous medium, that is, for developing procedures for non-destructive testing of materials.

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