Some general properties of resonant rotational motions of celestial bodies

Yu. V. Barkin

N. É. Bauman State Technical University, Moscow
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The mean resonant rotational motions of some celestial bodies are studied using the method of construction of conditionally periodic solutions of Hamiltonian systems with total resonance between fundamental frequencies. It is shown that the mean resonant rotational motion of a celestial body corresponds to a certain steady-state motion that, while close to the corresponding Cassini’s motion, does differ from it. The following mechanical phenomena have been detected and described, in particular: noncoincidence of the mean angular velocity of rotation of a body with its exact resonant value; splitting of Cassini’s node; a constant displacement of the pole of the rotation axis of a celestial body relative to the pole of its polar inertial axis; constant angular displacements of the inertial axes of a body (relative to Cassini’s positions) at the time of passage through the pericenter or node of the orbit, etc. Quantitative estimates of these effects in the motion of the moon and Phobos are given.

1. INTRODUCTION

Many authors have devoted works to the investigation of resonant rotational motions of celestial bodies (see, e.g., Refs. 1-8). In those works, the rotation of celestial bodies has been investigated, as a rule, on the basis of simplified model problems. For the orbital motion of these bodies, for example, their motion in a circular or elliptical Keplerian orbit or motion in a uniformly precessing elliptical orbit has been taken. The authors have usually been confined to the analysis of solutions of averaged equations of motion, devoting particular attention to the study of steady-state solutions. The motion corresponding to these steady-state solutions has been called motions under Cassini’s generalized laws.

If the problem of the rotation of a celestial body is formulated more completely and precisely (with allowance for the actual properties of its perturbed orbital motion, for example) and the corresponding solutions are analyzed more deeply, however, one discovers certain general tendencies and phenomena due to the resonant nature of the motion and the influence of various perturbing factors.

The essence of these phenomena consists in the fact that the mean resonant rotational motion of a celestial body corresponds to a certain steady-state motion that, while close to the corresponding Cassini’s motion, does differ from it.9-10 Let us enumerate the main mechanical effects that the mean resonant motion enables one to detect: 1) noncoincidence between the mean angular velocity of rotation of a body and its exact resonant value; 2) splitting of Cassini’s node; 3) a displacement of the pole of the rotation axis of a celestial body relative to the pole of its polar inertial axis; 4) constant angular displacements of the inertial axes of a body (relative to Cassini’s positions) at times of passage through the pericenter or node of the orbit, etc.

These effects are due to the asphericity of the corresponding celestial body, the resonant nature of its motion, the influence of the second and higher harmonics of its force function, and other perturbing factors. Statements (1)-(4) actually give refinements and supplements to the corresponding statements of Cassini’s generalized laws, so we can say that they describe subtle relationships in the resonant rotational motions of celestial bodies.

It is also appropriate to note that resonant effects essentially similar to (1)-(4) should occur in the resonant motion of various celestial mechanical systems (e.g., in the orbital motion of certain groups of resonant asteroids and planetary satellites, in the motion of the synchronous components of binary asteroids, the Pluto—Charon system, etc.).

The purpose of the present paper is to provide a method of describing these subtle relationships in the motion of celestial bodies and to estimate them quantitatively in the motion of specific synchronous planetary satellites: the moon and Phobos.

2. MEAN RESONANT MOTION OF CELESTIAL MECHANICAL SYSTEMS AND ITS PROPERTIES

The motion of many celestial mechanical systems can be described by the canonical differential equations12,13

$$\frac{d\mathbf{A}}{dt} = \frac{\partial F}{\partial \mathbf{q}^*}, \quad \frac{d\mathbf{q}}{dt} = -\frac{\partial F}{\partial \mathbf{q}^*},$$

(1)

$$F = F_p(I) + \mu F_s(p, q, \tau) + \mu^2 F_{ss}(p, q, \tau) + \ldots,$$

(2)

$$F_s(p, q, \tau) = \sum_{1 \leq k + l = 0} F_{k, l} (p, q) \exp(ikq + lv\tau).$$

(3)

The following matrix definitions of variables are used in (1)-(3):

$$\mathbf{I} = (p_1, p_2, \ldots, p_n)^{\tau}, \quad \mathbf{J} = (p_{1, n}, p_{2, n}, \ldots, p_{n, n})^{\tau}, \quad \mathbf{p} = (I, J)^{\tau},$$

$$q = (q_1, q_2, \ldots, q_n)^{\tau}, \quad \mathbf{q} = (q_{1, n}, q_{2, n}, \ldots, q_{n, n})^{\tau}, \quad \tau = (\tau_1, \tau_2, \ldots, \tau_n)^{\tau}.$$

Here $p$ and $q$ are conjugate canonical variables, $N$ is the number of degrees of freedom of the respective mechanical system, and $N_0$ is the number of arguments $\tau_s$, which are linear functions of time, $\tau_s = \Omega_s t + \tau_{s0}$ ($\Omega_s$ are constant...
given frequencies and \( \tau_0^{(0)} \) are the initial values of the angular variables \( \tau_j \); \( s = 1, 2, \ldots, N_0 \). For the integer summation indices in (3) we use the notation

\[
\kappa^* = (k_1, k_2, \ldots, k_s), \quad \nu^* = (\nu_1, \nu_2, \ldots, \nu_s), \quad ||\kappa|| = |k_1| + |k_2| + \ldots + |k_s|, \quad ||\nu|| = |\nu_1| + |\nu_2| + \ldots + |\nu_s|.
\]

The Hamiltonian (2), (3) of Eqs. (1) can be represented by a convergent series in powers of the small parameter \( \mu \), the coefficients of which are represented by Fourier series (3) in the multiple angular variables \( \kappa \) and \( \tau \). The unperturbed Hamiltonian \( F_0 \) depends on only some of the positional variables \( I \). The Hamiltonian of the problem and the right-hand sides of Eqs. (1)-(3) are thus conditionally periodic functions of time with a frequency basis \( \Omega = (\Omega_1, \Omega_2, \ldots, \Omega_{N_0}) \). If \( N_0 = 1 \), then the right-hand sides of these equations will naturally be periodic functions of time.

Equations like (1)-(3) are encountered in many problems of celestial mechanics, including problems of the rotational motion of a rigid celestial body having a small dynamical flattening, the center of mass of which moves along a conditionally periodic trajectory in the Newtonian force field of the central body and other bodies.

We assume that for \( \mu = 0 \), Eqs. (1)-(3) have a family of periodic solutions:

\[
\begin{align*}
I &= a_1, \quad J = a_2, \quad \varphi = n(a_1) + \omega_1, \quad \psi = \omega_2, \\
n(a_1)T &= 2\pi K, \quad \Omega_k T = 2\pi K_0.
\end{align*}
\]

Here \( a_1, a_2, \omega_1, \) and \( \omega_2 \) are integration constants,

\[
\begin{align*}
a_1 &= (p_1^{(0)}, p_2^{(0)}, \ldots, p_s^{(0)}), \\
a_2 &= (p_{s+1}^{(0)}, p_{s+2}^{(0)}, \ldots, p_{N_0}^{(0)}), \\
\omega_1 &= (q_1^{(0)}, q_2^{(0)}, \ldots, q_s^{(0)}), \\
\omega_2 &= (q_{s+1}^{(0)}, q_{s+2}^{(0)}, \ldots, q_{N_0}^{(0)}),
\end{align*}
\]

and \( p_j^{(0)} \) and \( q_j^{(0)} \) (\( j = 1, 2, \ldots, N_0 \)) are the initial values of the canonical variables. The constants \( a_1 \) satisfy the condition (5) of commensurability of fundamental frequencies [the \( n(a_1) \) are their unperturbed values], \( K = (K_1, K_2, \ldots, K_{N_0}) \) are integer commensurability indices, and \( T \) is the period of the solution (4), (5) (in H. Poincaré’s sense\(^{13} \)).

We determine the time-averaged value of a certain function \( \Phi(p, q, \tau) \), calculated for the generating values of the variables (4) and (5) from the equation

\[
\langle \Phi \rangle = \frac{1}{(2\pi)^N} \int_0^{2\pi} \int_0^{2\pi} \ldots \int_0^{2\pi} \Phi \left( a_1, a_2, \frac{k}{K}, \tau_0^{(0)} - \tau_0^{(0)} \right) + \omega_1, \omega_2, \\
\tau_1, \tau_2, \ldots, \tau_s d\tau_1 d\tau_2 \ldots d\tau_s.
\]

In calculating the integral (6) we allow for the commensurability of frequencies \( n(a_1) \) and \( \Omega_{N_0} \) of (5) (the frequencies \( \Omega_1, \Omega_2, \ldots, \Omega_{N_0} \) are not rationally commensurable). We then have the following theorem.\(^{12} \)

**Theorem.** If the constants \( a_1, a_2, \omega_1, \) and \( \omega_2 \) of the generating periodic solution (4) satisfy the group of conditions

\[
\begin{align*}
n(a_1)T &= 2\pi K, \quad \Omega_k T = 2\pi K_0, \\
\partial \langle F \rangle &= \partial \langle F \rangle = 0, \\
\frac{\partial \langle F \rangle}{\partial a_1} &= 0, \\
\frac{\partial \langle F \rangle}{\partial a_1^*} &= 0, \\
\frac{\partial^2 \langle F \rangle}{\partial a_1 \partial a_1^*} &= 0, \\
\frac{\partial^2 \langle F \rangle}{\partial a_1 \partial a_1^*} &= 0, \\
\frac{\partial^2 \langle F \rangle}{\partial a_1 \partial a_1^*} &= 0, \\
\frac{\partial^2 \langle F \rangle}{\partial a_1 \partial a_1^*} &= 0, \\
\frac{\partial^2 \langle F \rangle}{\partial a_1 \partial a_1^*} &= 0, \\
\frac{\partial^2 \langle F \rangle}{\partial a_1 \partial a_1^*} &= 0, \\
\frac{\partial^2 \langle F \rangle}{\partial a_1 \partial a_1^*} &= 0,
\end{align*}
\]

then the method of successive approximations in the vicinity of this solution can be used to construct formal asymptotic series in integer powers of the small parameter \( \mu \),

\[
\begin{align*}
I &= a_1 + \sum_{t=1}^{\infty} \mu^t \left( I_1^{(0)} + I_2^{(0)} \right), \\
J &= a_2 + \sum_{t=1}^{\infty} \mu^t \left( J_1^{(0)} + J_2^{(0)} \right), \\
\varphi &= n(a_1) + \sum_{t=1}^{\infty} \mu^t \left( \varphi_1^{(0)} + \varphi_2^{(0)} \right), \\
\psi &= \omega_2 + \sum_{t=1}^{\infty} \mu^t \left( \psi_1^{(0)} + \psi_2^{(0)} \right),
\end{align*}
\]

which satisfy Eqs. (1)-(3). The coefficients of these series are determined uniquely as conditionally periodic functions of time with the frequency basis \( \Omega \). The constants \( I_1^{(0)}, J_1^{(0)}, \varphi_1^{(0)}, \) and \( \psi_1^{(0)} \) (\( a = 1, 2, \ldots \)) are determined successively by solving uniform systems of algebraic equations, and \( I_2^{(0)}, J_2^{(0)}, \varphi_2^{(0)}, \) and \( \psi_2^{(0)} \) (\( a = 1, 2, \ldots \)) are determined by calculating the integrals of known functions of time. For \( \mu = 0 \), the conditionally periodic solution (11) goes over into the corresponding periodic solution of the unperturbed system.

The motion described by the solution (4) in which the constants \( a_1, a_2, \omega_1, \) and \( \omega_2 \) are determined by Eqs. (5) and (8) will be called an unperturbed resonant solution (in the problems of the rotation of celestial bodies under consideration, it corresponds to motion under Cassini’s generalized laws).

The motion of the corresponding system of celestial bodies described by the mean values of the variables (11),

\[
\begin{align*}
\langle I \rangle &= a_1 + \sum_{t=1}^{\infty} \mu^t I_1^{(0)}, \\
\langle J \rangle &= a_2 + \sum_{t=1}^{\infty} \mu^t J_1^{(0)}, \\
\langle \varphi \rangle &= n(a_1) + \sum_{t=1}^{\infty} \mu^t \varphi_1^{(0)}, \\
\langle \psi \rangle &= \omega_2 + \sum_{t=1}^{\infty} \mu^t \psi_1^{(0)},
\end{align*}
\]

will be called the mean resonant motion.

The motion described by Eqs. (12) is also periodic motion that is close to the corresponding unperturbed resonant...
motion (Cassini’s motion). We note that it is advantageous to use Eqs. (11) and (12) to describe perturbed motions of specific celestial bodies when their motion is fairly close to the unperturbed periodic motion described by Eqs. (4) and (5)-(10).

Along with the variables $I$, $J$, $\psi$, and $\varphi - R(0)_{I}$, we can introduce into the analysis other characteristics of the motion of celestial mechanical systems of the type

$$G = G(I, J, \psi, \varphi - R(0)_{I}).$$  \hspace{1cm} (13)

In the problem of lunar rotation, for example, such characteristics include the rotation angular velocity vector $\omega$, the projections of $\omega$ onto the moon’s principal central axes of inertia and onto the axis of Cassini’s coordinate system, and the angle between the lunar equator and the ecliptic plane.

The mean values of the characteristics (13),

$$G_{m} = \langle G \rangle + \theta_{1},$$

$$_{m} \theta = \mu \left\{ \frac{\partial \langle G \rangle}{\partial a_{1}} I_{1}^{(1)} + \frac{\partial \langle G \rangle}{\partial a_{2}} J_{2}^{(1)} + \frac{\partial \langle G \rangle}{\partial a_{3}} \psi^{(1)} + \frac{\partial \langle G \rangle}{\partial \omega_{2}} \omega_{2}^{(1)} + \frac{\partial \langle G \rangle}{\partial \omega_{3}} \omega_{3}^{(1)} \right\} + \ldots$$

(14)

are of particular interest for the description of resonant motions of celestial bodies.

Here $G$ is the conditionally periodic component of the function $G$, calculated for the generating values of the variables (4)-(9).

The constant components of perturbations of different orders in Eqs. (11) and (12), as mentioned earlier, are determined successively from systems of algebraic equations. The constructive aspect of the method of constructing a conditionally periodic solution (12) has been elucidated in detail in Refs. 11 and 12. Explicit equations for the constant and purely conditionally periodic components of first-, second-, and third-order perturbations, in particular, have been obtained in Ref. 12.

The constant components of first-order perturbations in the solution (11), for example, are determined by the equations

$$I_{1}^{(1)} = - \left( \frac{\partial^{2} F_{1}}{\partial a_{1} \partial a_{1}} \right)^{-1} \frac{\partial F_{1}}{\partial a_{1}},$$  \hspace{1cm} (15)

$$J_{1}^{(1)} = \left| \begin{array}{cc}
\frac{\partial^{2} F_{1}}{\partial a_{1} \partial a_{1}} & - \frac{\partial^{2} F_{1}}{\partial a_{1} \partial a_{2}} I_{1}^{(1)} \\
- \frac{\partial^{2} F_{1}}{\partial a_{1} \partial a_{2}} & \frac{\partial^{2} F_{1}}{\partial a_{2} \partial a_{2}}
\end{array} \right|,$$

$$\psi_{1}^{(1)} = \left| \begin{array}{cc}
\frac{\partial^{2} F_{1}}{\partial a_{1} \partial a_{1}} & - \frac{\partial^{2} F_{1}}{\partial a_{1} \partial a_{3}} I_{1}^{(1)} \\
- \frac{\partial^{2} F_{1}}{\partial a_{1} \partial a_{3}} & \frac{\partial^{2} F_{1}}{\partial a_{3} \partial a_{3}}
\end{array} \right|,$$

(16)

in which the matrix $B$ is defined by Eq. (10).

From Eq. (16) it follows that second-order perturbing factors in the small parameter $\mu$ lead to first-order effects in the solutions under consideration.

From an analysis of Eqs. (12) and (14)-(16), it is easy to establish the following properties of the mean resonant motion:

1. In mean resonant motion, the values of the positional variables $I$ and $J$ and the angular variables $\varphi - R(0)_{I}$ and $\psi$ differ from the unperturbed resonant values $a_{1}$, $a_{2}$, $a_{3}$, $\omega_{1}$, and $\omega_{2}$ by small constant amounts $\mu_{I}^{(1)} + \ldots$, $\mu_{J}^{(1)} + \ldots$, $\mu_{\varphi}^{(1)} + \ldots$, and $\mu_{\psi}^{(1)} + \ldots$, determined by Eqs. (15), (16), etc., and they describe constant dynamical displacements (in problems of rotational motion of celestial bodies, these are constant angular displacements of the inertial axes of a body in Cassini’s coordinate system or the orbital coordinate system, as well as constant angular displacements of the angular momentum vector of rotational motion and the angular velocity vector of this body, both in the principal central axes of inertia and in Cassini’s coordinate system).

2. In contrast to the unperturbed frequencies $n(\omega_{1})$, the mean frequencies $\langle n \rangle$ do not satisfy the condition of their rational commensurability with the frequency $\Omega_{N0}$, since

$$\langle n \rangle T = 2n_{k} - \mu \left\{ \frac{\partial n}{\partial a_{1}} I_{1}^{(1)} + \frac{\partial n}{\partial a_{2}} J_{2}^{(1)} + \frac{\partial n}{\partial a_{3}} \psi_{1}^{(1)} \right\} + \ldots$$

3. The mean values of the kinematic and dynamical characteristics of a celestial mechanical system in its perturbed, conditionally periodic motion (13) differ from their unperturbed values by small constant quantities $\theta$ determined by Eq. (14).

The enumerated properties of resonant motions of celestial mechanical systems are displayed with particular clarity in the rotational motion of the moon, and should also be displayed in the motion of Phobos and other synchronous planetary satellites.

Below we give a brief description of some of the basic effects in the rotational motion of the moon and Phobos that can be established by an analysis of their mean resonant rotational motions.

3. SOME RESONANT EFFECTS IN THE ROTATIONAL MOTION OF THE MOON

Modern, high-precision, laser ranging observations of the moon, which have been conducted systematically since August 1969, confirm that its rotational motion is close to that of a rigid body described by Cassini’s three laws. In Refs. 9 and 12 the author has developed a high-precision analytic theory of the rotational motion of the moon that allows for the influence of various perturbing factors and the actual features of the orbital motions of the moon, the sun, and other solar system bodies. This theory is based on a model problem in which the moon and the earth are treated as perfectly rigid bodies and the sun and planets are treated as particles. The orbital motions of all those bodies are specified and described by current theories. The earth undergoes specified rotational motion. The equations of rotational motion of the moon in canonical Andoyer variables, relative to the moving ecliptic plane for the date, are reduced to the standard form of system (1)-(3).

In the first stage of construction of the theory of physical libration of the moon (PLM), the unperturbed resonant motion was found [by solving Eqs. (5) and (8)] and studied.
This made it possible to describe in detail all of the statements of Cassini’s laws, and to supplement them with information about the stability of motion, about the alignment of the rotational angular momentum vector, etc. In contrast to Refs. 4–8, in which a simplified model of the lunar orbital motion was chosen, in Refs. 9 and 12 a mechanical explanation of Cassini’s laws was given in the case of the actual orbit of the moon’s center of mass, described by Brown’s improved theory.

In the second stage of the investigation of lunar rotation, we constructed the conditionally periodic motion of the moon about its center of mass. Here, in particular, the mean resonant rotational motion (12) of the moon was constructed in analytic form to the third order in the small parameter $\mu$ with complete allowance for all perturbing factors of the problem (the parameter $\mu$ is of the same order as the dynamical flattening of the moon, i.e., $\mu = 10^{-4}$). Omitting the lengthy analytic equations and their construction here, we give a brief description of the main mechanical phenomena in the mean resonant motion of the moon.

Splitting of Cassini’s node. One of Cassini’s laws reads: "The nodes of the orbit of the moon and the equator of its figure in the ecliptic plane coincide, with the longitude of the descending node of the equator equal to the longitude of the ascending node of the orbit." This means that the lines of intersection of the equatoral plane and the mean plane of the lunar orbit with the ecliptic plane coincide, and the aforementioned nodes of these planes on the celestial sphere merge into a point (we arbitrarily call it Cassini’s node). The fact that Cassini’s node undergoes uniform motion along the ecliptic with a period $T_\Omega = 18.5995$ yr has always evoked the particular delight of investigators of the rotational motion of the moon.

Such an ideal coincidence of the lines of nodes does not occur in the actual motion of the moon, of course, and the moon itself undergoes complicated librational motion. But Cassini’s statement does reduce to the fact that the aforementioned relationship does occur in the mean position of these lines of nodes (with averaging of librational perturbations).

Although the mean resonant motion of the moon is close to Cassini’s motion, they do differ, and its properties depend on all the perturbing factors of the problem (see Sec. 2 and Refs. 9 and 10).

For convenience in the analysis, we introduce the following coordinate planes: $Q_G$ is the plane orthogonal to the rotational angular momentum vector $G$ of the moon; $Q_\omega$ is the plane orthogonal to the vector of the absolute angular velocity of rotation $\omega$; $Q_\xi$ is the equatorial plane of the moon ($\xi$, $\eta$, and $\zeta$ are the principal central axes of inertia, $\eta$ being the polar axis). In the unperturbed resonant motion of the moon (in Cassini’s motion), these coordinate planes $Q_G$, $Q_\omega$, and $Q_\xi$ coincide.

The results of the present work show that in the mean resonant motion of the moon, the lines of intersection of the $Q_G$, $Q_\xi$, and $Q_\omega$ planes and of the mean plane of the lunar orbit do not coincide in the ecliptic plane but form small angles with each other. In other words, we observe a splitting of Cassini’s node. For example, the mean longitude of the descending node of the intermediate plane $Q_G$ in the ecliptic plane is somewhat larger than the mean longitude of the ascending node of the lunar orbit in the same plane. The difference between these longitudes is 0.33, and it depends primarily on the coefficients $C_{nm}$ and $S_{nm}$ of the third and fourth harmonics of the lunar force function, but it also depends on other perturbing factors.

**Mean pole of the rotation axis of the moon.** Cassini’s first law is one of the most interesting and difficult to understand: "The moon rotates from west to east at constant angular velocity about the polar axis, which remains fixed in the body of the moon, and the rotational period of the moon coincides with the mean period of orbital revolution of the center of mass of the moon." As Khabibullin recently pointed out, such a treatment of the law is ambiguous. It is not clear, for example, which period of orbital motion of the moon is meant (the sidereal, draconic, or another) or in which coordinate system the lunar motion is being considered. Nor is it clear which axis, remaining fixed in the body of the moon, is meant.

In Ref. 12 it was shown that such an axis is the polar axis of inertia of the moon, and the period of its rotation about that axis in fixed axes associated with the ecliptic and the equinox of epoch equals the sidereal month. One of the conclusions of Refs. 9 and 12 is that the moon rotates in such a way that, on the average, the position (in the axes associated with the moon) of the vector of its absolute angular velocity coincides with its polar axis of inertia. In other words, the pole of the polar axis of inertia of the moon and the pole of its instantaneous rotational axis (we consider the north pole for definiteness) coincide on the average. To this it must be added that in Cassini’s motion, the vector $G$ of the moon’s rotational angular momentum also coincides with its polar axis of inertia. We emphasize once again that the vectors $\omega$ and $G$ correspond to lunar rotation in the selenocentric coordinate system associated with the equinox and ecliptic plane of epoch, and the mean positions of those vectors are determined in the principal central axes of inertia of the moon. In the actual complicated librational motion of the moon, of course, this coincidence does not occur, but in accordance with Cassini’s statement, it occurs on the average.

An analysis of the analytic expressions for the projections of the angular velocity of rotation of the moon onto its principal central axes of inertia $\xi_\eta_\zeta$, obtained using the procedure in Sec. 2 (see Ref. 13), enables us to make this statement more precise. It turns out that these equations determine the constant angular displacements of the vector $\omega$ in the $\xi_\eta_\zeta$ axes relative to Cassini’s position, i.e., all three projections $p$, $q$, and $r$ of the vector $\omega$ onto the respective $\xi$, $\eta$, and $\zeta$ axes have constant components. The analytic expressions for these displacements enable us to explain their nature. It consists in the resonant nature of the lunar motion and is due primarily to the influence of the third and higher harmonics of the lunar force function, as well as other perturbing factors.

Calculations show that the mean pole $P_m$ of the vector $\omega$ is displaced relative to Cassini’s pole (or the pole of the polar axis of inertia $\eta$) along the meridian located in the $\xi_\eta$ plane of inertial axes of the moon (the $\xi$ axis is directed toward the earth) in the direction away from the earth at an angle $\xi_{P_m} = 74^\circ.72$, as well as at the small angle $\zeta_{P_m} = 0^\circ.025$ in the $\eta_\zeta$ plane of the meridian.
The aforementioned effects were obtained by using the values of the parameters of the lunar gravitational field \( (C_{nm} \text{ and } S_{nm}, n \geq 3) \) adopted in current work on the theory of physical libration of the moon (Refs. 16, 17, and others).

It must be noted, however, that the values of \( \xi_{pm} \) and \( \xi_{pm} \) depend to a considerable extent on the choice of model of the lunar gravitational field. For example, numerical calculations carried out for the well-known models of the lunar gravitational field of Williams et al., Calame, and Segran et al.\(^{12,18} \) yield \( \xi_{pm} \in (76^\circ-92^\circ.0) \). These angular displacements \( P_{pm} \) of the pole correspond to linear displacements \( \xi_{pm} \in (643.0-775.2 \text{ m}) \) on the lunar surface.

Mean angular velocity of rotation of the moon. Difficulties with the concept of the mean angular velocity of rotation of the moon arise even in the consideration of the basic laws of its motion. In Cassini’s first law it is noted that the time of a complete rotation of the moon about its axis equals the mean period of revolution of the center of mass of the moon in orbit. Above we introduced some clarity into the treatment of this law, but its formulation is still difficult to link up with the kinematic significance of Cassini’s third law, which actually describes the regular precession of the moon.

In Ref. 9 the present author showed that in unperturbed resonant motion, the moon rotates uniformly about its polar axis of inertia \( \eta \) with an angular velocity \( \omega_0 = n = \eta \) equal to the angular rate of change of the mean longitude \( \lambda \) of the moon. As we noted above, the position in the body of the moon of the vector of its absolute angular velocity coincides, on the average, with its inertial axis \( \eta \).

In the mean resonant motion of the moon, the absolute angular velocity is determined by the equation\(^{12,12} \)

\[
\langle \omega \rangle = n \left\{ 1 + \mu \frac{G_\eta}{G_0} + \mu^2 \frac{G_\eta}{G_0} + \ldots \right\},
\]

where \( \mu \) is a small parameter of the order of the dynamic flattening of the moon, \( G_0 \) is the unperturbed value of the magnitude \( G \) of the angular momentum \( G \), and \( \mu G_1 \) and \( \mu^2 G_2 \) are the constant components of the first- and second-order perturbations of the variable \( G \) in the conditionally periodic solution describing the perturbed rotational motion of the moon.

Numerical calculations yield the values

\[
\mu \frac{G_\eta}{G_0} = -25^\circ.25611 \text{ per Julian year},
\]

\[
\mu^2 \frac{G_\eta}{G_0} = 1^\circ.185071^\circ.62688 \text{ per Julian year}.
\]

We thus have

\[
\langle \omega \rangle = 17,325,620^\circ.000-17,325,620^\circ.431
\]

per Julian year.

In calculating \( \mu^2 G_2 / G_0 \) and \( \langle \omega \rangle \) we used analytic equations from Ref. 12 and the parameters \( C_{nm} \) and \( S_{nm} \) of the lunar gravitational field obtained by Williams et al., Calama, and Segran et al.\(^{12,18} \).

The result obtained reflects the complicated mechanical effect in the lunar motion due to the three-dimensional and resonant nature of its rotation and the influence of various perturbing factors.

Constant angular displacements of the inertial axes of the moon. The mean resonant rotational motion of the moon enables us to describe one other important phenomenon. It turns out that on the average, the position of the semimajor axis \( \xi \) of the lunar inertia ellipsoid is displaced by a constant angle \( \alpha_\xi \) in longitude toward the east and by a constant angle \( \beta_\xi \) in latitude toward the south relative to the direction of this axis in its unperturbed resonant motion.

The calculated values of the angles \( \alpha_\xi \) and \( \beta_\xi \) obtained from the equations of the theory of Ref. 12 for different models of the lunar gravitational field lie in the ranges

\[
\alpha_\xi = 177^\circ.8-264^\circ.4; \beta_\xi = 76^\circ.3-92^\circ.0.
\]

In recent years, these angular displacements of the lunar inertial axis \( \xi \) have been confirmed by ground-based photometric and astrometric observations. The angles \( \alpha_\xi = 195^\circ \) and \( \beta_\xi = 60^\circ \) were calculated in Ref. 19, for example, by a photometric method using large-scale photographs of the moon against the star background. The angles \( \alpha_\xi = 289^\circ \pm 25^\circ \) and \( \beta_\xi = 80^\circ \pm 12^\circ \) were obtained in Ref. 20 by analyzing multiyear astrometric observations of the moon, and the values \( \alpha_\xi = 226^\circ \pm 10^\circ \) and \( \beta_\xi = 64^\circ \pm 10^\circ \) were found in Ref. 21 using a photometric method.

4. RELATED RESONANT PHENOMENA IN THE ROTATIONAL MOTION OF PHOBOS

Phobos undergoes synchronous motion in the gravitational field of Mars. In one revolution in orbit, Phobos makes one rotation about its axis. To first order, the motion of Phobos is similar to that of the moon (although the orbit is nearly circular with a radius \( r = 9300 \text{ km} \), it undergoes rapid precessional motion). The main distinctive feature of Phobos is related to its shape and dynamical structure. An approximate model of Phobos\(^{22} \) consists of a homogeneous ellipsoid with semiaxes \( a = 13.5 \text{ km} \), \( c = 11.5 \text{ km} \), and \( b = 9.5 \text{ km} \). A more accurate homogeneous model of Phobos with the shape of a polyhedron, the faces of which are 288 triangles, has been constructed in Ref. 23. This geometrical model was constructed by analyzing television photographs of the satellite from on board the Mariner 9 station from different aspects. The error in determining the radius vectors of the apices of the polyhedron was 60 m.

On the basis of that model of Phobos, a model of its gravitational field was constructed in Ref. 24. That is, the values of the normalization constants \( C_{nm} \) and \( S_{nm} \) for the gravitational field of Phobos (for the second, third, and fourth harmonics) and their rms errors were found. From the normalization factors, the values of the standard parameters \( C_{nm} \) and \( S_{nm} \) of the gravitational field were calculated in Ref. 25. The latter quantities enable us to estimate certain effects in the rotational motion of Phobos due to the resonant properties of its motion.

Displacement of the mean pole of the rotation axis of Phobos. On the average, the position of the pole \( P_m \) of the vector of angular velocity of rotation of Phobos is displaced relative to the pole of the solar axis of inertia \( \eta \) by an angle \( \xi_{pm} = 14^\circ.13 \) in the \( \xi \eta \) plane of the prime meridian toward Mars.
On the mean angular velocity of Phobos. Because of the asphericity of Phobos and the resonant nature of its motion, the mean absolute value of the angular velocity vector of this celestial body differs from its exact resonant value by the small quantity

$$\mu \langle G_i \rangle / G_i \approx n = 2.346 \cdot 10^{-4} n,$$

where \( n \) is the angular rate of change of the mean longitude of Phobos.

Constant angular displacements of the inertial axes of Phobos. In the mean resonant motion, the semimajor axis \( \xi \) of the ellipsoid of inertia of Phobos is displaced by an angle \( \alpha_\xi = 19^\circ.99 \) in longitude toward the east and by a constant angle \( \beta_\xi = 14^\circ.13 \) in latitude toward the north relative to the direction of this axis in Cassini’s unperturbed motion.

As in the case of the moon, these effects in the rotation of Phobos are due to the influence of higher harmonics of its force function, other perturbing factors, and the resonant nature of its motion.

In conclusion, we note that essentially similar effects should occur in the rotational motion of Mercury and Venus, as well as in the motion of other celestial mechanical systems.

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