These problems are about determinants and linear algebra.

1. Compute $\left|\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}\right|$

(First row/column consist of 1's, any number which is neither in the first row nor in the first column, is sum of its neighbors from above and from the left).

2. Consider two quadratic polynomials, ax^2+bx+c, dx^2+ex+f, where $a,d \neq 0$. Prove that they have the common root if and only if the matrix

$$\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{pmatrix}$$

is degenerate.

3. A quadric in plane is a locus of zeroes of an equation of order 2: $ax^2+bxy+cy^2+dx+ey+f=0$. (At least one coefficient should be nonzero)

a. Show that for each 5 points there exists a quadric containing all of them.

b. Show that if two quadrics have only finite number of common points, than it isn't bigger than 4.

4. Find the roots of the polynomials:

a. $\left|\begin{pmatrix} x & 1 & 0 & 0 & 1 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 1 & 0 & 0 & 1 & x \end{pmatrix}\right|$ b. $\left|\begin{pmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{pmatrix}\right|$

5. Write an equation which holds if and only if the four points (x_1,y_1), (x_2,y_2), (x_3,y_3) (x_4,y_4) lie on one circle or one straight line.

These problems are about determinants and linear algebra.

1. Compute $\left|\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}\right|$

(First row/column consist of 1's, any number which is neither in the first row nor in the first column, is sum of its neighbors from above and from the left).

Solution. This determinant is one, it is shown by some cunning version of Gauss method. Det is not changed when You subtract one line from another. So, subtract line 4 from line 5, then line 5 becomes line shifted left by 1, precisely: 0 1 5 15 35.
Now subtract line 3 from line 4, line 4 will become 0 1 4 10 20
Now subtract line 2 from line 3, line 3 it also shifts by 1.
Then subtract line 1 from line 2. What You get is:

$$\left|\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 & 10 \\ 0 & 1 & 4 & 10 & 20 \\ 0 & 1 & 5 & 15 & 35 \end{pmatrix}\right|$$

Subtracting one column from another also keeps determinant.
So, subtract column 4 from column 5. Column 5 moves down.
Subtract column 3 from column 4. Column 4 moves down.
Subtract column 2 from column 3. Column 3 moves down.
Subtract column 1 from column 2. Column 2 moves down. Now we get:

$$\left|\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 & 10 \\ 0 & 1 & 4 & 10 & 20 \end{pmatrix}\right|$$

Now again, perform similar actions on rows and then on columns, three times.
In the end, you will get the unit matrix. And det is still the same.

2. Consider two quadratic polynomials, ax^2+bx+c, dx^2+ex+f, where $a,d \neq 0$. Prove that they have the common root if and only if the matrix

$$\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{pmatrix}$$

is degenerate.

Proof. Suppose they do have a common root α. Then multiplying this matrix by $\begin{pmatrix} \alpha^3 \\ \alpha^2 \\ \alpha \\ 1 \end{pmatrix}$ will give You 0 vector. So, if we have a common root, than a matrix is degenerate.
We need to prove the other direction also.
Denote the 2 (complex) roots of the first polynomial are x_1, x_2 and of the second polynomial y_1, y_2. Then by Vieta theorem we get:

$$\left|\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{pmatrix}\right| = \left|\begin{pmatrix} a & a(x_1+x_2) & ax_1x_2 & 0 \\ 0 & a & a(x_1+x_2) & ax_1x_2 \\ d & d(y_1+y_2) & dy_1y_2 & 0 \\ 0 & d & d(y_1+y_2) & dy_1y_2 \end{pmatrix}\right| =$$

$$= a^2d^2 \left|\begin{pmatrix} 1 & x_1+x_2 & x_1x_2 & 0 \\ 0 & 1 & x_1+x_2 & x_1x_2 \\ 1 & y_1+y_2 & y_1y_2 & 0 \\ 0 & 1 & y_1+y_2 & y_1y_2 \end{pmatrix}\right|$$

If You sum up all ways to place 4 rooks on this matrix, and You can express this determinant as a polynomial in x_1, x_2, y_1, y_2.
The degree of the polynomial is 4 (why?) and it is divisible by $(y_1-x_1)\cdot(y_2-x_1)\cdot(y_1-x_2)\cdot(y_2-x_2)$ because if some polynomial of several variables is a constant zero on the zeroes of some linear polynomial, such as y_1-x_1, then it is divisible by that linear polynomial (and why am I so sure of this?).
So, the polynomial I get is $C(y_1-x_1)\cdot(y_2-x_1)\cdot(y_1-x_2)\cdot(y_2-x_2)$ where *C* is a polynomial of degree 0, i. e. a number.

Notice that both the determinant and $(y_1-x_1)\cdot(y_2-x_1)\cdot(y_1-x_2)\cdot(y_2-x_2)$ contain a monomial $y_1^2y_2^2$ so $C = 1$.
That's it.

Remarks. (1) In precisely the same way You can see that the determinant of

$$\begin{vmatrix}\begin{pmatrix} a_n & & \dots & a_1 & a_0 & & \\ & \ddots & & & \ddots & \ddots & \\ & & a_n & & \dots & a_1 & a_0 \\ b_m & \dots & b_1 & b_0 & & & \\ & b_m & \dots & b_1 & b_0 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & b_m & \dots & b_1 & b_0 \end{pmatrix}\end{vmatrix}$$

(first parallelogram consists of *m* lines, second of *n* lines), is 0 iff the polynomials $a_nx^n+\ldots+a_2x^2+a_1x+a_0, b_mx^m+\ldots+b_2x^2+b_1x+b_0$ have a common root.
I didn't write it in the generic form from all the time for two obvious reasons.
Firstly, I was lazy to draw complicated determinants; secondly, You might prefer to do the general case by Yourself.

(2) This determinant is called resultant of two polynomials.

(3) The resultant of a polynomial with itself (sometimes divided by the first coefficient) is called a discriminant of the polynomial. You probably learned it in kita het for quadratic polynomials.

3. A quadric in plane is a locus of zeroes of an equation of order 2:
$ax^2+bxy+cy^2+dx+ey+f=0$. (At least one coefficient should be nonzero)
a. Show that for each 5 points there exists a quadric containing all of them.
b. Show that if two quadrics have only finite number of common points, then it isn't bigger than 4.

a. A condition that given quadric has a certain point imposes a linear condition on its six coefficients (given the point) 5 points give us 5 linear conditions. 6 variables, 5 linear condition, homogenous system ➔ it must have a nontrivial solution.

b. First solution. Assume we have two quadrics which have more than 4, but a finite number of intersections. So, there are finite number of lines passing through 2 intersections, and we can rotate this picture so that none of those lines will be horizontal.

After rotation quadrics will remain quadrics (since the formulas for the rotation are linear). Let us say that their equations will be:

$$a_1x^2 + (b_1y + d_1)x + (c_1y^2 + e_1y + f_1) = 0$$

$$a_2x^2 + (b_2y + d_2)x + (c_2y^2 + e_2y + f_2) = 0$$

To have an intersection on the level of certain y means to have a common root x for these two polynomials. Which means there will be a zero resultant for this y :

$$0 = \left| \begin{pmatrix} a_1 & b_1y + d_1 & c_1y^2 + e_1y + f & 0 \\ 0 & a_1 & b_1y + d_1 & c_1y^2 + e_1y + f \\ a_2 & b_2y + d_2 & c_2y^2 + e_2y + f & 0 \\ 0 & a_2 & b_2y + d_2 & c_2y^2 + e_2y + f \end{pmatrix} \right|$$

Number of root of this resultant, which is polynomial in y, is a number of intersections. But we see it is polynomial of degree no more than 4.

Remark. This proof is easily generalizable, and we get a theorem about any two algebraic curves: if their degrees are M and N and they have only finite number of common points, then it is not bigger than M·N. This fact is called Bezout theorem. Here comes a second solution, which is more elementary, but it is not generalizable (as far as I know).

Second solution. In analytic geometry there is a classification of quadrics: nondegenerate quadric is either non-degenerate (an ellipse, a parabola, a hyperbola) or degenerate (union of two lines, a line, an isolated point, an empty set).
Degenerate cases are easily verified. For example, a line can have no more than 2 intersections with quadric, since it is a solution of quadratic equation.
So, two lines can have no more than 4 intersections with a quadric.

Hence, it is sufficient to consider the case in which both quadrics are non-degenerate.
Choose a tangent to the first quadric at some point which is not their intersection.
Perform a projection of this plane which will send this tangent to infinity.
(By the way, why does a projection send quadrics into quadrics???)
The first quadric will become a parabola.
After some stretching and rotating, the first quadric will become $y = x^2$, so it will have only one y for each x. The second quadric will be

$ax^2 + bxy + cy^2 + dx + ey + f = 0$. Substitute $y = x^2$, You get a polynomial of degree 4. so only 4 values of x allow intersection, and for each x there is just one y.

4. Find the roots of the polynomials:

a. $\left\|\begin{pmatrix} x & 1 & 0 & 0 & 1 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 1 & 0 & 0 & 1 & x \end{pmatrix}\right\|$ b. $\left\|\begin{pmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{pmatrix}\right\|$

a. Solution. Consider a matrix $R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

It is rotating the standard basis. We can guess its eigenvectors.

Let $\xi = e^{2\pi i/5}$ ("a root of degree 5 of 1"). Then the vector $\begin{pmatrix} 1 \\ \xi^k \\ \xi^{2k} \\ \xi^{3k} \\ \xi^{4k} \end{pmatrix}$, when You multiply R by it, is multiplied by ξ^k. For k=0, 1, 2, 3, 4 we get in this way 5 different eigenvectors and 5 different eigenvalues. Eigenvectors corresponding to different eigenvalues are linearly independent. Let switch to this eigenbasis.

Our original matrix is actually $R+R^{-1}+x\mathrm{E}$, where E is a unit matrix. So, in eigenbasis,

R= $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 & 0 \\ 0 & 0 & 0 & \xi^3 & 0 \\ 0 & 0 & 0 & 0 & \xi^4 \end{pmatrix}$ so the original matrix is

$$\begin{pmatrix} x+2 & 0 & 0 & 0 & 0 \\ 0 & x+\xi^4+\xi & 0 & 0 & 0 \\ 0 & 0 & x+\xi^3+\xi^2 & 0 & 0 \\ 0 & 0 & 0 & x+\xi^2+\xi^3 & 0 \\ 0 & 0 & 0 & 0 & x+\xi^4+\xi \end{pmatrix},$$

and its determinant is

$$\left(x+2\right)\left(x+\xi^4+\xi\right)\left(x+\xi^3+\xi^2\right)\left(x+\xi^2+\xi^3\right)\left(x+\xi^4+\xi\right)=$$

$$\left(x+2\right)\left(x+2\cos\left(\frac{2\pi}{5}\right)\right)^2\left(x+2\cos\left(\frac{4\pi}{5}\right)\right)^2$$

And roots are $-2,-2\cos\left(\frac{2\pi}{5}\right),-2\cos\left(\frac{4\pi}{5}\right)$ (the last two are double roots).

Remark. This matrix R is a well know mathematical object, and its eigenbasis is even more famous. Passing to this basis is called discreet Fourier Transform, and it has a lot of magical properties (did You ever hear, for example, about multiplying numbers of length N in O(N·logN) operations?).

b. Here again, we shall guess the eigenvector. The key to guessing is a nice trigonometric formula: sin x + siny.
A shall allow myself to show You its proof (not only because in some schools they don't prove formulas, but also because I have a special prove which hints the solution).
Take two unit vector (cos *x*, sin *x*) and (cos *y*, sin *y*) and sum them. We get a rombus. The angle of the sum vector is *(x+y)/2* since it is a bisector. The length of the diagonal of the rhombus (מעוין) is 2cos(*(x-y)/2*). So,
cos *x* + cos y = 2 cos(*(x-y)/2*) cos(*(x+y)/2*)
sin *x* + sin y = 2 cos(*(x-y)/2*) sin(*(x+y)/2*)

So, we have

$$\begin{pmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{pmatrix}\begin{pmatrix} \sin(k\pi/6) \\ \sin(2k\pi/6) \\ \sin(3k\pi/6) \\ \sin(4k\pi/6) \\ \sin(5k\pi/6) \end{pmatrix}=\left(x+2\cos(k\pi/6)\right)\begin{pmatrix} \sin(k\pi/6) \\ \sin(2k\pi/6) \\ \sin(3k\pi/6) \\ \sin(4k\pi/6) \\ \sin(5k\pi/6) \end{pmatrix}$$

For k = 1, 2, 3, 4, 5. And since 5×5 matrix can have only 5 distinct eigenvalues, the only answers are $-2\cos(k\pi/6)$ for 1, 2, 3, 4, 5.

Remark. Of course, all this can be said for each N and not just for 5.
The polynomial related to trigonometry were have a lot of nice properties, some of them were studied by Chebyshev and bear his name (the particular polynomials we showed are not Chebyshev's, but are closely related to them).

5. Write an equation which holds if and only if the four points (x_1,y_1), (x_2,y_2), (x_3,y_3) (x_4,y_4) lie on one circle or one straight line.

First solution

An equation of a line or a circle is of a form $a(x^2 + y^2) + bx + cy + d = 0$

And it should hold for all 4 points, so the condition is that the matrix

$$\begin{pmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{pmatrix}$$

is degenerate. So a condition is a determinant of this matrix = 0. That's it.

Remarks. Denote those points A, B, C, D and O the zero. Then this formula has some geometric meaning. Develop it with respect to the first column, you get

$$0 = \left|\begin{pmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{pmatrix}\right| = \left(x_1^2 + y_1^2\right)\left|\begin{pmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{pmatrix}\right| - \left(x_2^2 + y_2^2\right)\left|\begin{pmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{pmatrix}\right| +$$

$$+\left(x_3^2 + y_3^2\right)\left|\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{pmatrix}\right| - \left(x_4^2 + y_4^2\right)\left|\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|$$

But those 3×3 determinants have an obvious meaning as twice the area of the triangle, (the area of the triangle which is oriented, i. e. has minus sign iff its coordinates are mentioned clockwise) so our condition takes form

$$OA^2 \cdot S_{BCD} - OB^2 \cdot S_{ACD} + OC^2 \cdot S_{ABD} - OD^2 \cdot S_{ABC} = 0$$

So, we get a geometric theorem – ABCD is inscribed, iff for any point O the above condition holds. If O is the center of the circle, OA=OB=OC=OD=R, we get a trivial condition: $S_{BCD} + S_{ABD} = S_{ACD} + S_{ABC}$.

On the contrary, if O=A we get $AB^2 \cdot S_{ACD} + AD^2 \cdot S_{ABC} = AC^2 \cdot S_{ABD}$

If ABCD is really inscribed, all triangles are inscribed in the same circle, so each area is are product of their sides divided by 4R. So, if we multiply by 4R we get:

$$AB^2 \cdot AC \cdot CD \cdot DA + AD^2 \cdot AB \cdot BC \cdot CA = AC^2 \cdot AB \cdot BD \cdot DA$$

Divide it by $AB \cdot AC \cdot AD$ and You get the famous Ptolemy's theorem, which holds for every inscribed quadrilateral: $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

Second solution.

Consider points A, B, C, D as complex numbers.

Then the argument of a complex number (A-B)/(C-B) is precisely the angle which is needed to rotate a vector a vector BA to the direction of vector BC.
The argument of complex number (A-D)/(C-D) is precisely the angle which is needed to rotate a vector a vector DA to the direction of vector DC.

Those two angles should be either equal (if B, D are on the same side of line AC) or should be opposite (if they are on different sides), so anyway, the condition is that the ratio of the two above ratios $\frac{A-B}{C-B}:\frac{A-D}{C-D}=\frac{(A-B)(C-D)}{(C-B)(A-D)}$ is real.

Remark. This ratio is famous in projective geometry and it is called cross ratio.
It is famous for being an invariant of projective transformations of projective line, and if You consider complex projective line, $\mathbf{CP}^1$ (if You know such words) our conclusions become quite obvious.

So, let us multiply nominator and denominator by the conjugate of denominator,

$$\frac{(A-B)(C-D)\overline{(C-B)(A-D)}}{real}=real$$

And the condition is: $\operatorname{Im}\left((A-B)(C-D)\overline{(C-B)(A-D)}\right)=0$.

Now we can tight it in coordinates, if need be.

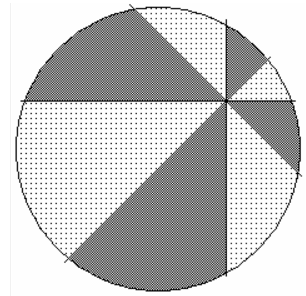
$$0=\operatorname{Im}\Big(\big((x_1-x_2)+i(y_1-y_2)\big)\big((x_3-x_4)+i(y_3-y_4)\big)+$$
$$+\big((x_3-x_2)-i(y_3-y_2)\big)\big((x_1-x_4)-i(y_1-y_4)\big)\Big)=$$
$$=\operatorname{Im}\Big(\Big(\big((x_1-x_2)(x_3-x_4)-(y_1-y_2)(y_3-y_4)\big)+$$
$$+i\big((x_1-x_2)(y_3-y_4)+(y_1-y_2)(x_3-x_4)\big)\Big)\times$$
$$\times\Big(\big((x_3-x_2)(x_1-x_4)-(y_3-y_2)(y_1-y_4)\big)-$$
$$-i\big((x_3-x_2)(y_1-y_4)+(y_3-y_2)(x_1-x_4)\big)\Big)\Big)=$$
$$=-\big((x_1-x_2)(x_3-x_4)-(y_1-y_2)(y_3-y_4)\big)$$
$$\big((x_3-x_2)(y_1-y_4)+(y_3-y_2)(x_1-x_4)\big)+$$
$$+\big((x_1-x_2)(y_3-y_4)+(y_1-y_2)(x_3-x_4)\big)$$
$$\big((x_3-x_2)(x_1-x_4)-(y_3-y_2)(y_1-y_4)\big)$$

Well, if I didn't make a mistake in the computation.
Anyway, like in first solution, it is a polynomial of order 4 and it is 0 when 2 points coincide.

This time the problems are about differentiating/integrating in geometrical context.

1. Consider a sphere (the surface of a ball) of radius *R*, and cut it by a horizontal plane at distance *h* from its north pole. Compute the area of the part of the sphere which is above the plane (spherical kipa).

2. A circle was cut by 4 lines, passing through the same interior point of the circle and having forming 8 equal angles at that point (of 45 degrees each).
So, the inner part of the circle is divided into 8 parts, which are painted in 2 colors, so that neighboring parts have different colors (like the chessboard).
Show, that each color takes the same part of the area.



3. Develop a formula for an area of a polygon, whose vertices are (x_1,y_1) (x_2,y_2), …, (x_n,y_n) (in this order).

4. Two convex polygons P, Q are given, such that Q is strictly inside P.
A point X on the boundary of Q is called good, if there are two points A, B on the boundary of polygon P such that line AB doesn’t cut the polygon Q in two parts, and X is a middle point of the interval AB.
Show that there are at least 2 good points.

5. Consider a billiard ball on a table bounded by a convex smooth wall without pockets. The ball is so small that we consider it to be a point. The trajectory of the ball is a straight line, until it hits the wall, at the wall it is reflected so that the angle between the wall and the trajectory is preserved. So the trajectory is a broken line, consisting on intervals.

A convex line inside the table is called *caustics*, if the following property holds:
If the trajectory of the billiard ball touches this line once, then it touches it on with interval.
We are given a smooth convex line inside the billiard table which is caustics.
For each point A of the edge of the table, let p(A) be the perimeter of the convex hull of A and the given caustics.
Show that p(A) doesn’t depend on A.

This time the problems are about differentiating/integrating in geometrical context.

1. Consider a sphere (the surface of a ball) of radius R, and cut it by a horizontal plane at distance h from its north pole. Compute the area of the part of the sphere which is above the plane (spherical kipa).

Solution: the answer is surprising. It is proportional to h.

For unit sphere, it is $2\pi h$, for sphere of radius R it is $2\pi hR$.

Consider a huge number of densely distributed horizontal planes. They cut the spherical surface into a lot of circular tilted bands.

Consider plane at distance h from the north pole. If the center of the sphere is (0,0,0), this plane can be described analytically as $z = R - h$.

The thin spherical band, between z and $z+dz$, is circular and its length is $2\pi r$ where r I a radius of the circle, $r^2 + z^2 = R^2$.

The height of this spherical band is dz. But its width is bigger, because it is tilted. The quotient between the width and the height is R/r, since triangles are similar.

So the area of a thin spherical band of height dz is length times width = $2\pi r \times dz \times R/r =$

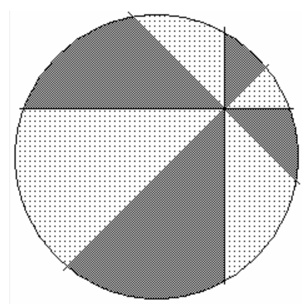
$2\pi R dz$. It is a constant function!

Integrate it over an interval of length h, you get $2\pi Rh$.

2. A circle was cut by 4 lines, passing through the same interior point of the circle and having forming 8 equal angles at that point (of 45 degrees each).

So, the inner part of the circle is divided into 8 parts, which are painted in 2 colors, so that neighboring parts have different colors (like the chessboard).

Show, that each color takes the same part of the area.



First solution. Rotate this set of lines around their common intersection with a constant angular speed v. When one of the lines will be passing through the center of the circle, both colors will occupy the same area (due to symmetry w. r. t. diameter).

So, all we have to prove is that the difference between dark area and light area will be preserved during the rotation.

The area swept by an interval of length l rotated around its end by an infinitesimal angle $d\alpha$ is $l^2 d\alpha/2$.

In our picture, when all 8 the intervals are rotated, 4 of them turn dark area into light, and another for turn light area into dark. Denote the lengths of 8 intervals, a, e, b, f, c, g, d, h clockwise. We compute the derivative of "dark area – light area". During time dt it will sweep angle $d\alpha = v\,dt$ and area $((a^2+b^2+c^2+d^2) - (e^2+f^2+g^2+h^2))d\alpha/2$, so derivative (when you divide it by dt) will be $((a^2+b^2+c^2+d^2) - (e^2+f^2+g^2+h^2))/2$.

We want to prove it is 0, so it is enough to prove a lemma:

Lemma. Inside a circle, two orthogonal chords cut each other. The length of 4 subintervals which are formed are a, b, c, d. Then $a^2+b^2+c^2+d^2$ doesn't depend on the tilt of this cross, but only on the radius of the circle and the position of the intersection point. Actually, it doesn't even depend on the position of the intersection, and it is always equal to the square of circle's diameter.

Proof of lemma. Let O be the intersection of the chords, and their ends A, B, C, D (named clockwise). By Pythagoras theorem,

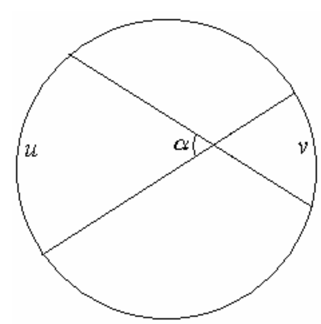
$a^2 + b^2 + c^2 + d^2 = OA^2 + OB^2 + OC^2 + OD^2 = AB^2 + CD^2$

Draw a parallel line to AC through D. It will cut the circle in 2 points: D and E.
BDE is a right angle, so BE is a diameter of the circle, so BAE is also right angle.
AE = CD since because of the symmetry, hence we can continue our computation so:

$AB^2 + CD^2 = AB^2 + AE^2 = BE^2 = \text{diameter}^2$

That's it.

Second solution. Start reducing the circle concentrically, until the circle gets to the intersection. Both dark and light area are being reduced at the same rate. Here we are using:

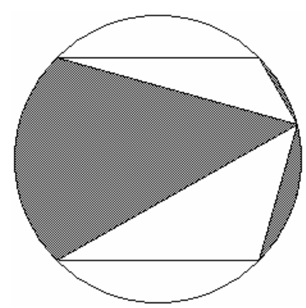


Lemma. The angle between the chords of a circle is average of two arcs: in the picture on the right $\alpha = (u + v) / 2$.

This is known from elementary geometry, but if you don't know it, take it as an exercise.

So, in our problem, sum of dark and light arcs is the same. When we reduce the radius, at each moment dark are is reduced by *dr* times sum of dark arcs, while light are is reduced by *dr* times sum of light arcs, which is the same.

When the circle contains the intersection of 4 lines, we stop and look. Of the 8 rays, only 4 intersect the boundary, and 4 others look outside the circle. The 4 intersection points with the boundary cut the circle in the vertices of the square. (Since inscribed angles are 45° so intercepted arcs are 90°). So, the circle is split into 5 parts: 3 of one color, and 2 of another. The 2 parts can be divided by chords into 2 segments and 2 triangles. This triangles have bases equal to the side of the square and sum of their heights is the same, so if we will move their common vertex to the center of the circle, their area will be the same. But then, it will definitely become half a circle.



Third solution. (No derivatives, by V. Proizvolov, works for any form with 4 axes of symmetry, taken from the famous Russian "Quant", 1992/10 solutions to quant problems, http://kvant.mirror1.mccme.ru/1992/10/resheniya_zadachnika_kvanta_ma.htm). Would it be a square instead of circle, it would be an easy exercise. So it is enough to prove it for the difference between the bounding square and its circle. Now, just relax and watch:

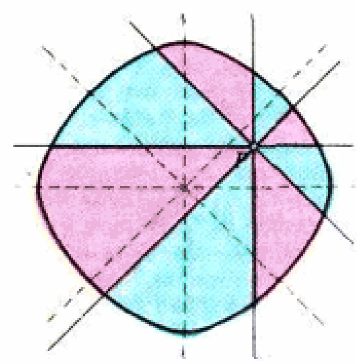


Рис. 1.

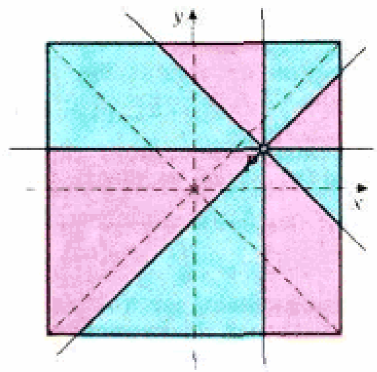


Рис. 2.

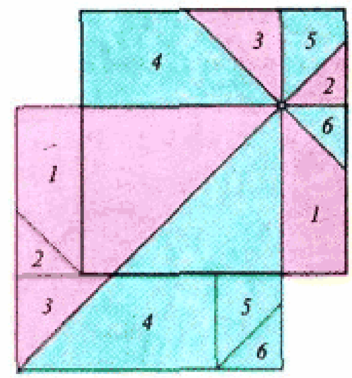
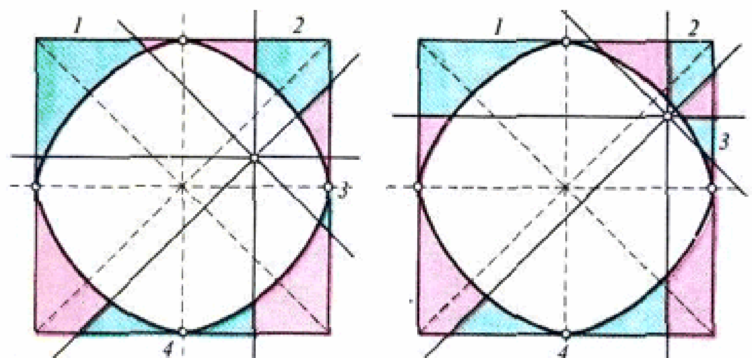


Рис. 3.



а) б)

Рис. 4.

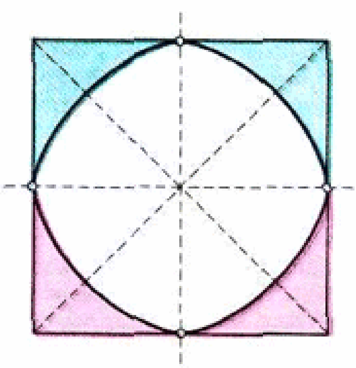


Рис. 5. В. Произволов

3. Develop a formula for an area of a polygon, whose vertices are (x_1,y_1) (x_2,y_2), …, (x_n,y_n) (in this order).

First solution.

Let's first develop a formula for triangle.

If the vertices of triangle are (0,0) , (x_1,y_1) , (x_2,y_2) then the area is half the area of the parallelogram, which is determinant, so the answer is $(x_1y_2 - x_2y_1)/2$.

Actually, this is ***oriented area*** – it is usual area when the vertices go clockwise and minus usual area when vertices go counterclockwise. We can get usual are by taking the absolute value, but I prefer area to be oriented for now.

For triangle (x_1,y_1) , (x_2,y_2) , (x_3,y_3): if we shift the triangle by a vector $-(x_1,y_1)$ area won't change, so substitute $(x_2 - x_1, y_2 - y_1)$, $(x_3 - x_1, y_3 - y_1)$ and get:

$((x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1))/2 = ((x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1))/2$

In other words, if You denote S_{ABC} to be the oriented area of triangle ABC then

$$S_{ABC} = S_{OAB} + S_{OBC} + S_{OCA}$$

Here $O = (0,0)$ (For oriented area, unlike the usual area, this formula is always true!).

The formula for quadrilateral is

$S_{ABCD} = S_{ABC} + S_{CDA} = S_{OAB}+S_{OBC}+S_{OCA} + S_{OAC}+S_{OCD}+S_{ODA} = S_{OAB} + S_{OBC} + S_{OCD} +S_{ODA}$

(Since in oriented areas, $S_{OAC} = - S_{OCA}$). This can is generalized for each n by induction, internal terms cancel out, and we get a specific formula:

$$S_{A_1A_2\ldots A_n} = S_{OA_1A_2} + S_{OA_2A_3} + \ldots + S_{OA_nA_1} = \frac{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \ldots + x_ny_1 - x_1y_n}{2}$$

Remark. One could also find an area of triangle (x_1,y_1) , (x_2,y_2) , (x_3,y_3) directly, by a determinant. Consider pyramid whose vertices are $(x_1,y_1,1)$, $(x_2,y_2,1)$, $(x_3,y_3,1)$, $(0,0,0)$. It is 1/6 of parallelepiped formed by vectors $(x_1,y_1,1)$, $(x_2,y_2,1)$, $(x_3,y_3,1)$.
On the other hand, it is 1/3 of the triangle's area times the height, which is 1.

So the area is $\frac{1}{2}\left|\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}\right|$.

Second solution. Simply take integral of $-ydx$
You will get sum of oriented areas of trapezoids:

$$S_{A_1A_2\ldots A_n} = \frac{(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+\ldots+(x_n-x_1)(y_n+y_1)}{2}$$

Which is equivalent to the first solution.

Remark. There is also the third way to write the formula:

$$S_{A_1A_2\ldots A_n} = \frac{x_1(y_2-y_n)+x_2(y_3-y_1)+x_3(y_4-y_2)+\ldots+x_n(y_1-y_{n-1})}{2}$$

Does anybody have a geometric explanation for this one?

4. Two convex polygons P, Q are given, such that Q is strictly inside P.
A point X on the boundary of Q is called good, if there are two points A, B on the boundary of polygon P such that line AB doesn't cut the polygon Q in two parts, and X is a middle point of the interval AB.
Show that there are at least 2 good points.

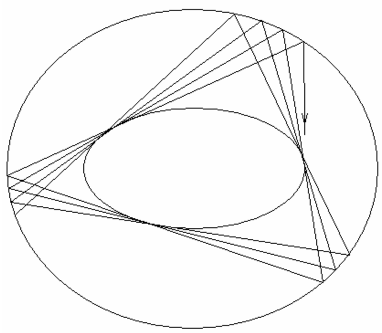
Solution. P is divided by the chord into 2 parts: one contains Q, another doesn't.
The area of the part that doesn't will be called S. It can be thought of as a function of the point on the direction of the chord.
When the chord of P is being rotated, while it is still tangent to Q, it sweeps certain area: one part of it sweeps area into S, another sweeps area out of S.
If a point on Q divides the chord into 2 parts, x and y, then the growth of S is
$dS = \frac{x^2-y^2}{2}d\theta$. But then, there should be at least one direction which gives local maximum for S and at least one which gives local minimum. For those two direction, derivative is 0, so $x=y$. QED.

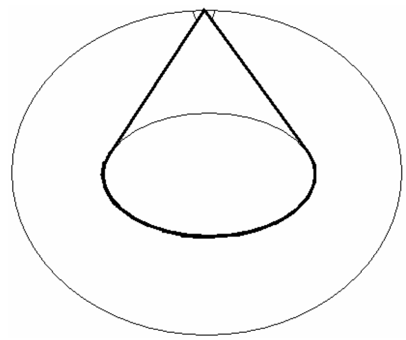
5. Consider a billiard ball on a table bounded by a convex smooth wall without pockets. The ball is so small that we consider it to be a point. The trajectory of the ball is a straight line, until it hits the wall, at the wall it is reflected so that the angle between the wall and the trajectory is preserved. So the trajectory is a broken line, consisting on intervals.

A convex line inside the table is called *caustics*, if the following property holds:

If the trajectory of the billiard ball touches this line once, then it touches it on with interval.
We are given a smooth convex line inside the billiard table which is caustics.
For each point A of the edge of the table, let p(A) be the perimeter of the convex hull of A and the given caustics.
Show that p(A) doesn't depend on A.

Here are pictures demonstrating both equivalent properties.
On the first picture there is a trajectory of the billiard ball, bouncing from the walls and touching the caustics each time.





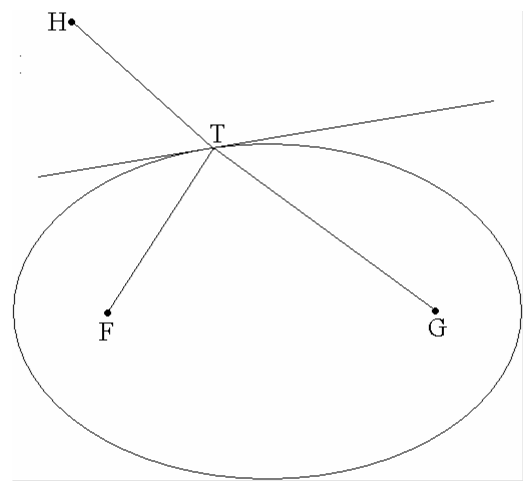
On the second picture, a rope of constant length is thrown around the inner curve, and we draw the outer curve by pulling this rope around with the pen.
The equivalence of those two pictures was noticed in 1972 by Minasian, a student of Yerevan university (and I learned this fact a few mothes ago from a book "Introduction to Ergodic Theory" by a famous mathematician, Jacob Sinai).

I shall be very sketchy, and I am sorry about it, but I am delaying those solutions for a day already.

First proof. Let us pull the rope aside by the pen, and its length will be preserved.
Let us derive this length to see what condition do we get.
The largest part of the rope which winds around the inner curve wont move. The part of it will be release into the straight part of the rope, and taken from another part of the rope, but it will move only slightly (small of order 2) since a tangent is very close to the curve.
The two straight parts will shift, one will become shorter and another longer.
Projection of the straight part of the rope after time *dt* on the same straight part before time *dt* will be of almost the same length as the rope itself (up to order 2) since derivative of cos at 0 is 0. So, the derivative of length will be the same as the distance from the projection of new position of point on the boundary to the straight parts of the rope, from the new position. To make those two distances cancel, the trajectory should be outer bisector for the two straight parts of the rope.
But the trajectory is the outer bisector, if and only if a billiard ball could be going along this rope.

Second solution. First proof it for a simple special case: when the inner part is an interval. This thing is called "the optical property of the ellipse".

The ellipse is a curve, which is a locus of points such that the sum of two distances from the point to two given points, called foci (plural of focus), is constant.
The optical property of the ellipse is that the billiard ball, arriving from the first focus, returns to the second focus after it hits the wall of elliptical billiard table.
This is precisely the special case of our problem.
After we have proven it, we can consider a more general special case, precisely when the inner curve is a convex polygon. It follows directly from the optical property of the ellipse (when you stretch the rope with a pen around the polygon, you get that an outer curve consists of elliptical arcs, and everything follows).
After we prove it for polygons, we simply approximate any convex by a polygon, and obtain the general theorem from the specific case.
The only nontrivial part of this proof is the optical property of the ellipse, other parts are easy. So, for those of you who never learned it, here is the proof of the optical property:
Denote the foci F, G, a point on the ellipse T, draw a tangent to the ellipse at T.
The reflection of F with respect to that tangent line is H. All we need to prove is that points G, T, H are on one line.
Assume the opposite: segment GH doesn't contain T, it cuts the tangent line in a different point, S. Then TH+TG > SH+SG. But, due to symmetry, TF=TH, SH=SF, so TF+TG > SF+SG. But it cant be, since ellipse is convex (why?), so S is outside the ellipse, so TF+TG < SF+SG.



This time problems are about factorials.

1. (a) Prove that $\int\limits_0^{\infty} e^{-x} x^N dx = N!$ for any natural *N*.

(b) Prove that $\int\limits_0^1 x^K (1-x)^M \, dx = \dfrac{K!M!}{(K+M+1)!}$ for natural *K*, *M*.

2. Define (using 1a) and compute ½!

3. Compute the volume of the N-dimensional ball of radius 1 and the area of its surface.

4*. (a) Let $I_N = \int\limits_0^{2\pi} \sin^N(x)dx$. Compute I_N for each integer *N*.

(b) Show that $\sqrt{N\pi} < 4^N \Big/ \binom{2N}{N} < \sqrt{\left(N+\frac{1}{2}\right)\pi}$

5. Compute $\lim\limits_{N\to\infty}\left(\dfrac{\sqrt[N]{N!}}{N}\right)$.

During the discussion of 3rd targil (factorials) many unexpected ideas were mentioned, mostly by Alexey (Gladkikh), Gal and Oded, several nice new solutions and remarks, so I have decided to write an appendix to the solutions.

1(b). Third solution (Alexey). Choose at random (with uniform probability distribution) a point ***A***, and independently, M+K more points. What is the probability that the point ***A*** will hold place M+1, meaning, there will be M points before and K points after?
We shall compute it in two ways. First way: well, there are
$\binom{M+K}{K} = \frac{(M+K)!}{M!K!}$ ways to choose divide M+K other points into M specific points and K specific points, and now multiply this number if ways by the probability that M specific points will be before and K specific points will be after. If point ***A*** is at place x then the probability is $x^M(1-x)^K$, so in general we get integral over probability measure of this dx. In the end we get

$$\frac{(M+K)!}{M!K!}\int_0^1 x^M(1-x)^K\,dx$$

The second way to compute the same thing: all animals are equal, so the probability for point ***A*** to be on each place, from 1 to M+K+1 is the same:

$$\frac{1}{M+K+1}$$

Hence

$$\frac{(M+K)!}{M!K!}\int_0^1 x^M(1-x)^K\,dx = \frac{1}{M+K+1}$$

So:
$$\int_0^1 x^M(1-x)^K\,dx = \frac{M!K!}{(M+K+1)!}$$

Of course, this solution, as well as integration by parts, works for integer values only.

2. Second solution (Gal & Dan) We shall compute $\left(-\frac{1}{2}\right)!$, and

$\left(\frac{1}{2}\right)! = \left(-\frac{1}{2}\right)! \cdot \frac{1}{2}$. Let $u = \sqrt{x}$, then $du = \frac{dx}{2\sqrt{x}}$, hence $2udu = dx$, so:

$$\left(-\frac{1}{2}\right)! = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx = \int_0^\infty \frac{e^{-u^2}}{u} 2udu = 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^\infty e^{-u^2} du$$

This integral is classical and equal to $\sqrt{\pi}$. So the answer is

$$\left(\frac{1}{2}\right)! = \left(-\frac{1}{2}\right)! \cdot \frac{1}{2} = \frac{\sqrt{\pi}}{2}$$

Remark. How to compute $\int_{-\infty}^{\infty} e^{-u^2} du$ was explained in the second solution for problem 3. This function, e^{-u^2}, is called ***Gaussian***, ***bell-shaped (פעמון)***, or ***normal distribution*** function, and it plays important role in probability theory, partly because of central limit theorem.
Though we know the total integral under it, we can’t get its indefinite integral as an elementary function (it is not arithmetical combination of powers, exponents, logs and trigo), hence a special notation was invented, it (up to some coefficients) is called **erf(*x*)**.

3. Third solution (Oded). Like in the first solution, for the volume of the ball we wrote, $V_N = \int_{-1}^{1} V_{N-1} \cdot \left(\sqrt{1-z^2}\right)^{N-1} dz$. So all we need is

$$\int_{-1}^{1} \left(\sqrt{1-z^2}\right)^{N-1} dz = \int_{-1}^{1} \left(\sqrt{1-z}\right)^{N-1} \left(\sqrt{1+z}\right)^{N-1} dz$$

Use a linear substitution which brings interval [-1,1] to interval [0,1]
$u = (z+1)/2$, $du = dz/2$, we get

$$\int_{-1}^{1} \left(\sqrt{1-z^2}\right)^{N-1} dz = \int_{-1}^{1} \left(\sqrt{1-z}\right)^{N-1} \left(\sqrt{1+z}\right)^{N-1} dz = \int_0^1 (2u)^{\frac{N-1}{2}} \left(2(1-u)\right)^{\frac{N-1}{2}} 2du =$$

$$= 2^N \int_0^1 u^{\frac{N-1}{2}} (1-u)^{\frac{N-1}{2}} du = \frac{N!2^N}{\left(\left(\frac{N-1}{2}\right)!\right)^2}$$

4. (a) $I_N = \int_0^\pi \sin^N(x)\,dx$

Second solution (Alexey)**.** $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, when we raise it to the power N we get sum of expressions of the kind $\binom{N}{K} e^{(N-K)ix} e^{-Kix}$ all divided by $(2i)^N$.

So, we have to compute $\int_0^\pi e^{(N-2K)ix}\,dx$.

If $K=2N$ then we get integral of 1, which is π. In other cases we get

$$\int_0^\pi e^{(N-2K)ix}\,dx = \left.\frac{e^{(N-2K)ix}}{(N-2K)i}\right|_0^\pi$$

this is 0 for even $N-2K$, that is, for even *N*, and for odd *N* we get.

$$\int_0^\pi e^{(N-2K)ix}\,dx = \left.\frac{e^{(N-2K)ix}}{(N-2K)i}\right|_0^\pi = \frac{-2}{(N-2K)i}$$

Hence the answer is, for even N:

$$I_N = \int_0^\pi \sin^N(x)\,dx = \frac{\sum_{K=0}^{N} \int_0^\pi \binom{N}{K} (-1)^K e^{(N-K)ix} e^{-Kix}\,dx}{(2i)^N} = \frac{\binom{N}{N/2}\pi}{2^N}$$

For odd *N*

$$I_N = \int_0^\pi \sin^N(x)\,dx = \frac{\sum_{K=0}^{N} \int_0^\pi \binom{N}{K} (-1)^K e^{(N-2K)ix}\,dx}{(2i)^N} = \frac{\sum_{K=0}^{N} \binom{N}{K} (-1)^K \int_0^\pi e^{(N-2K)ix}\,dx}{(2i)^N} =$$

$$= \sum_{K=0}^{N} \frac{-2(-1)^K \binom{N}{K}}{(2i)^N (N-2K)i} = \sum_{K=0}^{N} \frac{-2(-1)^{\frac{N+1}{2}-K} \binom{N}{K}}{2^N (N-2K)} = \sum_{K=0}^{N} \frac{(-1)^{\frac{N-1}{2}-K} \binom{N}{K}}{2^{N-1}(N-2K)}$$

Second solution (Alexey). Let $t=\sin^2 x$, then $1-t=\cos^2 x$, $\dfrac{dt}{\sqrt{t(1-t)}}=2dx$

$$I_N=\int_0^{\pi}\sin^N(x)\,dx=2\int_0^{\pi/2}\sin^N(x)\,dx=\int_0^1 t^{N/2}\frac{dt}{\sqrt{t(1-t)}}=\frac{\left(\frac{N}{2}\right)!}{\left(\frac{N}{2}-1\right)!\sqrt{\pi}}$$

Because of 1(b).

4(b). $\sqrt{N\pi}<4^N\Big/\binom{2N}{N}<\sqrt{\left(N+\frac{1}{2}\right)\pi}$

An elementary aproach, which gives a less good but similar inequality based on a classical riddle, I recalled it two days after the meeting.

$$4^N\Big/\binom{2N}{N}=\frac{2^{2N}(N!)^2}{(2N)!}=\frac{\left((2N)!!\right)^2}{(2N)!}=\frac{(2N)!!}{(2N-1)!!}$$

Denote $A_N=\dfrac{(2N)!!}{(2N-1)!!}=\dfrac{2}{1}\cdot\dfrac{4}{3}\cdot\ldots\cdot\dfrac{2N}{2N-1}$,

$B_N=\dfrac{(2N+1)!!}{(2N)!!}=\dfrac{3}{2}\cdot\dfrac{5}{4}\cdot\ldots\cdot\dfrac{2N+1}{2N}$.

Since $\dfrac{2}{1}>\dfrac{3}{2}>\dfrac{4}{3}>\dfrac{5}{4}>\dfrac{6}{5}>\ldots>1$ we get, when we multiply such inequalities

$2B_{N-1}>A_N>B_N$.

On the other hand, $B_{N-1}A_N=2N$, $B_NA_N=2N+1$ hence $4N>A_N^2>2N+1$.

A more precise inequality was found by Alexey.

$$\boxed{\sqrt{\left(N+\frac{1}{4}\right)\pi}<4^N\Big/\binom{2N}{N}<\sqrt{N\pi+1}}$$

Start by taking the ln:

$$\frac{\ln\pi+\ln\left(N+\frac{1}{4}\right)}{2}<N\ln 4+2\ln(N!)-\ln\left((2N)!\right)<\frac{\ln\pi+\ln\left(N+\frac{1}{\pi}\right)}{2}$$

Let us denote the left L_N, the middle M_N, the right R_N. Lets compute

$$L_{N+1}-L_N \qquad M_{N+1}-M_N \qquad R_{N+1}-R_N$$

We get

$$\frac{\ln\left(N+1+\frac{1}{4}\right)-\ln\left(N+\frac{1}{4}\right)}{2},$$

$$\ln 4+2\ln(N+1)-\ln(2N+1)-\ln(2N+2),$$

$$\frac{\ln\left(N+1+\frac{1}{\pi}\right)-\ln\left(N+\frac{1}{\pi}\right)}{2}$$

Simplify it and multiply by 2

$$\ln\left(1+\frac{1}{N+1/4}\right), \quad 2\left(\ln\left(1+\frac{1}{2N+1}\right)\right), \quad \ln\left(1+\frac{1}{N+1/\pi}\right)$$

If we would prove $L_{N+1}-L_N < M_{N+1}-M_N < R_{N+1}-R_N$ we would get the statement by trivial induction. Unfortunately, the inequalities are vice versa.

Let's prove $\ln\left(1+\frac{1}{N+1/4}\right)>2\left(\ln\left(1+\frac{1}{2N+1}\right)\right)>\ln\left(1+\frac{1}{N+1/\pi}\right)$

$$1+\frac{1}{N+1/4}>\left(1+\frac{1}{2N+1}\right)^2>1+\frac{1}{N+1/\pi}$$

$$\frac{1}{N+1/4}>\frac{2}{2N+1}+\frac{1}{(2N+1)^2}>\frac{1}{N+1/\pi}$$

$$\frac{1}{N+1/4}-\frac{1}{N+1/2}>\frac{1}{(2N+1)^2}>\frac{1}{N+1/\pi}-\frac{1}{N+1/2}$$

$$\frac{1/4}{(N+1/4)(N+1/2)}>\frac{1}{(2N+1)^2}>\frac{\frac{1}{2}-\frac{1}{\pi}}{(N+1/\pi)(N+1/2)}$$

$$\frac{1}{2N+1/2}>\frac{1}{2N+1}>\frac{2-\frac{4}{\pi}}{2N+2/\pi}$$

This is easy.

So, we can't apply standard induction from 1 to infinity, we can only hope to do it vice versa – from infinity to 1. By the easy version of inequality,

$$\lim_{N\to\infty}\sqrt{N}\binom{2N}{N}\Big/4^N=1$$

Hence L_H-M_H, M_H-R_H for huge H are very close to 0, closer than any given positive epsilon, and we have proved that

$(L_{N+1}-L_N)-(M_{N+1}-M_N),(M_{N+1}-M_N)-(R_{N+1}-R_N)$ is always negative, hence

$$L_{K+1}-M_{K+1}=L_H-M_H+\sum_{N=K+1}^{H-1}\left((L_{N+1}-L_N)-(M_{N+1}-M_N)\right)<\varepsilon,$$

$$M_{K+1}-R_{K+1}=M_H-R_H+\sum_{N=K+1}^{H-1}\left((M_{N+1}-M_N)-(R_{N+1}-R_N)\right)<\varepsilon$$

for each positive ε hence $L_{K+1}-M_{K+1},M_{K+1}-R_{K+1}\le 0$ and so $L_K-M_K,M_K-R_K<0$,
QED.

Especially for those who think that was not cool enough, Alexey had improved this inequality even further three days later:

$$\boxed{\sqrt{\left(N+\frac{1}{4}\right)\pi}<4^N\Big/\binom{2N}{N}<\sqrt{\left(N+\frac{1}{4}+\frac{1}{4N+3}\right)\pi}}$$

The idea is the following. Suppose we want to prove a good upper bound for our expression. In the spirit of what was done before, we have to choose a smaller number *t*, such that an inequality

$$2\left(\ln\left(1+\frac{1}{2M+1}\right)\right)>\ln\left(1+\frac{1}{M+t+1/4}\right)$$ will hold for all $M\ge N$.

He found out that when you choose any t, the expression

$$D_M=2\left(\ln\left(1+\frac{1}{2M+1}\right)\right)-\ln\left(1+\frac{1}{M+t+1/4}\right)$$

is negative for small *M* and positive after it passes certain point.
This point is *N* for which the original inequality can be proven.
To find this point he sets the derivative D_M as a function of *M* to be 0, and gets an expression for t.

5. $\lim_{N\to\infty}\left(\frac{\sqrt[N]{N!}}{N}\right)=\frac{1}{e}$

Gal asked, whether it is possible to prove it using Stirling's formula. I didn't accept it, since this equality is a first step in proving or guessing Stirling's formula.
For the sake of those who couldn't come to the meeting, here we shall give the proof of Stirling's formula.

Stirling's formula is about approximating $N!$
The problem 5 gives a (false) hope, that

$$\frac{N!}{N^N} \approx \frac{1}{e^N} \text{ that is, } \lim_{N\to\infty}\left(\frac{e}{N}\right)^N N! = 1.$$

Let us check how far from truth is that.

Denote $A_N = \ln\left(\left(\frac{e}{N}\right)^N N!\right) = N - N\ln N + \ln(N!)$, let

$$B_N = 1 + N\ln N - (N+1)\ln(N+1) + \ln(N+1) = 1 - N\ln\left(\frac{N+1}{N}\right) = 1 - N\ln\left(1+\frac{1}{N}\right)$$

We want A_N to have a limit, it is equivalent to convergence of the series $\sum_{N=1}^{\infty} B_N$. From the expression $B_N = 1 - N\ln\left(1+\frac{1}{N}\right) \approx 0$ we have a hope, but if we look closer

$$B_N = 1 - N\ln\left(1+\frac{1}{N}\right) = 1 - N\left(\frac{1}{N} - \frac{1}{2N^2} + O\left(N^{-3}\right)\right) = \frac{1}{2N} + O\left(N^{-2}\right)$$

So $\sum_{N=1}^{\infty} B_N$ diverges even though B_N goes to 0.

However, would we have

$$B_N = 1 - \left(N+\frac{1}{2}\right)\ln\left(1+\frac{1}{N}\right) = 1 - \left(N+\frac{1}{2}\right)\left(\frac{1}{N} - \frac{1}{2N^2} + O\left(N^{-3}\right)\right) = O\left(N^{-2}\right)$$

Then $\sum_{N=1}^{\infty} B_N$ would converge and A_N would have a limit.

This would happen if we'd have

$$B_N = 1 + \left(N+\frac{1}{2}\right)\ln N - \left(N+1+\frac{1}{2}\right)\ln(N+1) + \ln(N+1)$$

and $A_N = \ln\left(\left(\frac{e}{N}\right)^N \frac{N!}{\sqrt{N}}\right) = N - \left(N+\frac{1}{2}\right)\ln N + \ln(N!)$.

So, the expression $\left(\frac{e}{N}\right)^N \frac{N!}{\sqrt{N}}$ has a finite positive limit α, in other words

$$N! \approx \alpha\sqrt{N}\left(\frac{N}{e}\right)^N$$

To find α take Wallis formula, of exercise 4(b) which is a more precise version of Wallis formula.

$$\sqrt{N\pi} \approx \frac{4^N\left((N)!\right)^2}{(2N)!} \approx \frac{4^N\left(\alpha\sqrt{N}\left(\frac{N}{e}\right)^N\right)^2}{\alpha\sqrt{2N}\left(\frac{2N}{e}\right)^N}$$

Surprisingly, all powers of N and e and most powers of 2 cancel out and we get $\alpha = \sqrt{2\pi}$ and Stirling's formula is

$$\boxed{N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N}$$

It is on of the most beautiful and important formulas about factorials.
The most natural prove, which I showed above, I have read it in a book on probability theory, by a famous French mathematician, Henri Poincare, who lived about a 100 years ago. Stirling's formula has also several other classical proofs.

Gal has remarked, that Wallis formula $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots$ follows from Euler's formula for sine: $\boxed{\frac{\sin(\pi x)}{\pi} = x\cdot\prod_{N=1}^{\infty}\left(1-\frac{x^2}{N^2}\right)}$.

If you revert Wallis formula you get

$$\frac{2}{\pi} = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\ldots = \frac{1\cdot 3}{2^2}\cdot\frac{3\cdot 5}{4^2}\cdot\frac{5\cdot 7}{6^2}\cdot\frac{7\cdot 9}{8^2}\cdot\ldots = \frac{2^2-1}{2^2}\cdot\frac{4^2-1}{4^2}\cdot\frac{6^2-1}{6^2}\cdot\ldots =$$

$$= \left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{4^2}\right)\cdot\left(1-\frac{1}{6^2}\right)\cdot\ldots$$

This is precisely what You get when You substitute ½ into Euler's formula.

Another nice conclusion from Euler's formula turns out when You compute the coefficient of x^3, on both sides: $-\frac{\pi^2}{6} = -\sum_{N=1}^{\infty}\frac{1}{N^2}$.

Intuitively, Euler's formula presents $\frac{\sin(\pi x)}{\pi}$ as a polynomial, whose roots are all integer numbers, and hence it is a product over all integer numbers.

Gal has asked for the proof of Euler's formula. Unfortunately, the only proof I know uses complex functions (merukavot) so some of You may not understand it. But anyway, here it comes.

Firstly, let us discuss convergence. In general when you have a product $\prod_{N=1}^{\infty}(1+a_N)$ and the sequence a_N tends to 0. We say that this product ***converges well*** if it converges to a number, which is neither 0 nor infinity.

Lemma. $\prod_{N=1}^{\infty}(1+a_N)$ converges well iff $\sum_{N=1}^{\infty}a_n$ converges.
(iff is short for "if and only if")

Proof of lemma. $\prod_{N=1}^{\infty}(1+a_N)$ converges well iff its log, which is

$\sum_{N=1}^{\infty}\log(1+a_n)$ converges. But for small numbers $\log(1+a_n)\approx a_n$. QED.

From this lemma, we see Euler's product converges.

For every function *f*(*z*) there is a nice construction, called logarithmic derivative *f'*(*z*)/ *f*(*z*). It is defined well and nicely even when definition of logarithm is messy. A logarithmic derivative of a product is a sum of logarithmic derivatives (this is precisely Leibniz rule).
Logarithmic derivative of Euler's formula (of both sides) gives:

$$\boxed{\pi\operatorname{ctg}(\pi x)=\frac{1}{x}+\sum_{N=1}^{\infty}\left(\frac{1/N}{1+x/N}+\frac{-1/N}{1-x/N}\right)=\frac{1}{x}+\sum_{N=1}^{\infty}\left(\frac{1}{x+N}+\frac{1}{x-N}\right)}$$

It is as beautiful as the original formula itself, and is equivalent to it.
Take derivative of both parts, and change the signs, You will get a third beautiful equivalent formula,

$$\boxed{\frac{\pi^2}{\sin^2(\pi x)}=\sum_{N\in\mathbb{Z}}\frac{1}{(x+N)^2}}$$

It converges even better so the order of summation is not important.

We shall prove the third formula. Both left hand and right hand sides are defined in all complex plane, periodic with period 1, and analytic everywhere except integer points. At integer points they have precisely the same singularity $\frac{1}{z^2}$ at all integer points, and far from real line both are bounded and tend to 0.

So the difference of the 2 is an analytic bounded function on complex plane, and it is constant by Liouville theorem, 0 in our case since it goes to 0 far from real axis. QED.

Factorials – solutions.

1. **(a)** Prove that $\int_0^\infty e^{-x}x^N dx = N!$ for any natural *N*.

(b) Prove that $\int_0^1 x^K(1-x)^M\, dx = \dfrac{K!M!}{(K+M+1)!}$ for natural *K*, *M*.

Solution. **(a)** Induction. For *N*=0 You get integral = 0.

The step of induction: proving that $\int_0^\infty e^{-x}x^N dx = N\int_0^\infty e^{-x}x^{N-1}dx$.

Apply integration by parts. First function is x^N, the derivative of the second is e^{-x}. Minus (of deriving e^{-x}) times minus (of integration by parts) is plus.

(b) I think that the first solution (I saw it in Euler's book) was integration by parts (integrate x^K, differentiate $(1-x)^M$).

However, there is another classical solution I like better.

Consider double integral over quadrant (¼ plane):

$$I(K,M) = \int_0^\infty\int_0^\infty x^K y^M e^{-x-y}dxdy$$

We shall compute it in two ways.

First way: split it into product of two integrals.

$$I(K,M) = \int_0^\infty\int_0^\infty x^K y^M e^{-x-y}dxdy = \int_0^\infty x^K e^{-x}dx\int_0^\infty y^M e^{-y}dy = K!M!$$

Second way: substitute $z = x+y, t = \dfrac{x}{x+y}$ which is $x = zt, y = z-zt$

$$I(K,M) = \int_0^\infty\int_0^\infty \left(\frac{x}{z}\right)^K\left(\frac{y}{z}\right)^M z^{K+M}e^{-x-y}dxdy = \int_0^\infty\int_0^\infty t^K(1-t)^M z^{K+M}e^{-z}dxdy$$

To continue this computation we need to convert *dxdy* into *dzdt* which is done by the Jacobian.

$$J = \left|\begin{pmatrix} \partial x/\partial z & \partial x/\partial t \\ \partial y/\partial z & \partial y/\partial t \end{pmatrix}\right| = \left|\begin{pmatrix} t & z \\ 1-t & -z \end{pmatrix}\right| = \left|\begin{pmatrix} t & z \\ 1 & 0 \end{pmatrix}\right| = -z$$

So *dxdy* = |*J*|*dzdt* = *zdzdt*, hence

$$I(K,M)=\int_0^\infty\int_0^\infty t^K(1-t)^M z^{K+M}e^{-z}dxdy=\int_0^1\int_0^\infty t^K(1-t)^M z^{K+M+1}e^{-z}dzdt=$$

$$=\int_0^\infty z^{K+M+1}e^{-z}dz\int_0^1 t^K(1-t)^M dt=(K+M+1)!\cdot\int_0^1 t^K(1-t)^M dt$$

The two answers for the same questions that we have computed in two ways should be equal, hence $(K+M+1)!\cdot\int_0^1 t^K(1-t)^M dt=K!M!$

QED.

Remark. First part of the question gives a natural extension of *N*! to all real non-negative numbers, or in fact to all complex numbers for which Re$N\geq 0$. In fact, it can be extended to all complex plane, and it will have poles (infinite values) at all negative integers. The standard notation is

$$\Gamma(N)=\int_0^\infty e^{-x}x^{N-1}dx=(N-1)!$$

and it is called "gamma function" or "Euler's gamma function".
Different mathematicians have found some nice properties of Γ and proved several theorems of the type "why Γ is the most natural extension of $(N-1)!$ to the positive/complex numbers".
Here are some colored pictures http://mathworld.wolfram.com/GammaFunction.html

As for the second part, it allows to build the natural extension of binomial coefficients to non-integer number. The integral we asked about is usually denoted B(*K*+1,*M*+1) and called "beta function".

The formula we proved in standard notation looks so: $\mathrm{B}(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

2. Define (using 1a) and compute ½!

Solution. As it was explained in the above remark, the natural definition is $\left(\frac{1}{2}\right)!=\int_0^\infty e^{-x}\sqrt{x}dx$. I don't believe there is an elementary function, whose derivative is $e^{-x}\sqrt{x}$. So, we need a trick.
Substitute *K* = *M* = ½ in the formula 1(b):

$$2! \cdot \int_0^1 \sqrt{t}\sqrt{1-t}\,dt = \left(\left(\tfrac{1}{2}\right)!\right)^2$$

So. It is enough to compute $\int_0^1 \sqrt{t(1-t)}dt$. This integral is related to a circle.

Really, let $u = t - \frac{1}{2}$. Then $t(t-1) = \left(\frac{1}{2} - u\right)\left(\frac{1}{2} + u\right) = \frac{1}{4} - u^2$ and

$$\int_0^1 \sqrt{t(1-t)}dt = \int_{-1/2}^{1/2} \sqrt{\frac{1}{4} - u^2}\,dt.$$

This is precisely the are between diameter of a circle of radius ½ and its arc, which is half a circle of radius ½ so it is $\pi/8$.

So, we have $2! \cdot \pi/8 = \left(\left(\tfrac{1}{2}\right)!\right)^2$ and hence $\left(\tfrac{1}{2}\right)! = \sqrt{2! \cdot \pi/8} = \sqrt{\pi/4} = \sqrt{\pi}/2$

Remark. The calculator in Windows knows it ☺

3. Compute the volume of the N-dimensional ball of radius 1 and the area of its surface.

Solution. Denote S_N the area of its surface, V_N the volume. Firstly,

$$S_N = NV_N$$

This easy fact can be explained in 2 ways.

First explanation. Divide the surface into small (or infinitesimal) countries. Divide the volume into conic parts, whose vertex is the center of the ball, and bases are countries on the surface. Each N-dimensional cone can be computed as Sh/N. In our case h=1, so each volume part is N times smaller than corresponding area part. This constant ratio will be preserved after integration.

Second explanation. To compute volume of the ball, we shall split it into thin concentric spherical layers (like cabbages). At radius R we have layer of width dR and area $S_N R^{N-1}$, and integration of this gives

$$V_N = \int_0^1 S_N R^{N-1} dR = \frac{S_N}{N}.$$

So, it is sufficient to compute the area or the volume and not both.

First solution. Let z be one of the coordinates. The hyperplane at level z intersects the ball along a thin layer of radius $\sqrt{1-z^2}$, hence its $N-1$-dimensional volume is $V_{N-1}\cdot\left(\sqrt{1-z^2}\right)^{N-1}$. The total volume is given by the integral of this expression times dz.

$$V_N=\int_{-1}^{1}V_{N-1}\cdot\left(\sqrt{1-z^2}\right)^{N-1}dz$$

This integral begs for trigonometric substitution:

$$z=\cos x$$
$$dz=-\sin x dx$$
$$\sqrt{1-z^2}=\sin x$$

Then we get

$$V_N=V_{N-1}\int_{-\pi}^{\pi}(\sin x)^N dx$$

So now we have to solve 4(a) and multiply the answers.
The details are left to the reader (if he or she will really want to finish the first solution after they see the second one).

Second solution. Consider N-dimensional integral:

$$G_N=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}e^{-\left(x_1^2+x_2^2+\ldots+x_n^2\right)}dx_1dx_2\ldots dx_N$$

We shall compute it in 2 ways. First way is splitting it into product:

$$G_N=\int_{-\infty}^{\infty}e^{-x_1^2}dx_1\cdot\int_{-\infty}^{\infty}e^{-x_2^2}dx_2\cdot\ldots\cdot\int_{-\infty}^{\infty}e^{-x_n^2}dx_N=G_1^N$$

Second way is integrating it thing cabbage-like spherical slices of radius R centered at zero. The volume of each slice is $S_nR^{N-1}dR$ so we get

$G_N=\int_{0}^{\infty}e^{-R^2}S_nR^{N-1}dR$, now substitute $u=R^2, du=2RdR$:

$$G_N=\frac{S_n}{2}\int_{0}^{\infty}e^{-R^2}R^{N-2}2RdR=\frac{S_n}{2}\int_{0}^{\infty}e^{-u}u^{\frac{N}{2}-1}du=\frac{S_n}{2}\Gamma\left(N/2\right)=\left(\frac{N}{2}-1\right)!\frac{S_n}{2}$$

So, $G_1^N=\left(\frac{N}{2}-1\right)!\frac{S_n}{2}$ and $G_1^N=\left(\frac{N}{2}\right)!\cdot V_n$. To conclude this solution, we have yet to find G_1.

Instead of computing it directly, use the fact that we know the area of the unit circle in plane. $G_1^2 = \pi$. **The answer:** $\boxed{V_N = \dfrac{\pi^{N/2}}{\left(N/2\right)!} \quad , \quad S_N = N\dfrac{\pi^{N/2}}{\left(N/2\right)!}}$.

Remark. And if someone asks what is $\left(1/2\right)!$ or $\left(3/2\right)!$ we say that $\left(-1/2\right)! = \sqrt{\pi}$ and $(N+1)! = N!(N+1)$

hence $\left(1/2\right)! = \dfrac{\sqrt{\pi}}{2}$, $\left(3/2\right)! = \sqrt{\pi}\cdot\dfrac{1}{2}\cdot\dfrac{3}{2}$, $\left(5/2\right)! = \sqrt{\pi}\cdot\dfrac{1}{2}\cdot\dfrac{3}{2}\cdot\dfrac{5}{2}$.

Of course, for $N=2k+1$ we can write the formula in usual notations

$$V_N = 2\frac{(2\pi)^{(N-1)/2}}{1\cdot 3\cdot\ldots\cdot N} \quad , \quad S_N = 2\frac{(2\pi)^{(N-1)/2}}{1\cdot 3\cdot\ldots\cdot (N-2)} \quad ,$$

where the denominators contain products of first odd numbers.

4*. (a) Let $I_N = \int_0^{\pi} \sin^N(x)\,dx$. Compute I_N for each integer N.

(b) Show that $\sqrt{N\pi} < 4^N \Big/ \binom{2N}{N} < \sqrt{\left(N+\frac{1}{2}\right)\pi}$

Solution. (a) Use integration by parts. $\sin^N(x)$ is a product of 2 functions: first , $\sin^{N-1}(x)$, second $\sin(x)$. We will differentiate the first and integrate the second. The integral of sin is - cos, but minus will cancel out with minus of integration by parts. The value at the limits is 0, hence:

$$I_N = \int_0^{\pi} \sin^{N-1}(x)\sin(x)\,dx = (N-1)\int_0^{\pi} \sin^{N-2}(x)\cos(x)\cos(x)\,dx =$$

$$= (N-1)\int_0^{\pi} \sin^{N-2}(x)\left(1-\sin^2(x)\right)dx = (N-1)\left(I_{N-2} - I_N\right)$$

Hence $I_N = (N-1)\left(I_{N-2} - I_N\right)$ so $NI_N = (N-1)I_{N-2}$ or $I_N = \dfrac{N-1}{N} I_{N-2}$.

It is easily computed that $I_0 = \pi$ and $I_1 = 2$.

So, $I_{2K} = \dfrac{2k-1}{2k}\cdot\dfrac{2k-3}{2k-2}\cdot\ldots\cdot\dfrac{1}{2}\cdot\pi$, and $I_{2K+1} = \dfrac{2k}{2k+1}\cdot\dfrac{2k-2}{2k-1}\cdot\ldots\cdot\dfrac{2}{3}\cdot 2$.

Remark. To write those formulas shortly, special notations were invented. The product of all natural numbers up to N having the same parity as N is denoted $\boldsymbol{N!!}$. In this notation, $I_{2K}=\frac{(2k-1)!!}{(2k)!!}\cdot\pi$ and $I_{2K+1}=\frac{(2k)!!\cdot 2}{(2k+1)!!}$.

(b) The value of sin is not bigger (and usually smaller) than 1. Hence the value of $\sin^N(x)$ decreases as N grows, at least on the interval $[0,\pi]$. That is why I_N is decreasing sequence. So, we can write

$$I_{2N}>I_{2N+1}>I_{2N+2}.$$

One obvious consequence of this is so-called Wallis formula.
Since $I_{2N}\approx I_{2N+2}$, and is I_{2N+1} between, $I_{2N}\approx I_{2N+1}$ hence

$$\frac{\pi}{2}\approx\frac{((2N)!!)^2}{(2N-1)!!(2N+1)!!}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots$$

But we shall pay more attention and obtain more precise inequalities.
First we shall reduce !! to ! .

$$(2N)!!=2\cdot4\cdot6\cdot\ldots\cdot2N=2^N\cdot1\cdot2\cdot3\cdot\ldots\cdot N=2^N\cdot N!$$

$$(2N+1)!!=\frac{(2N+1)!}{(2N)!!}=\frac{(2N+1)!}{2^N\cdot N!}$$

Now lets develop $I_{2N}>I_{2N+1}>I_{2N+2}$:

$$\frac{(2N-1)!!}{(2N)!!}\cdot2\pi>\frac{(2N)!!\cdot2}{(2N+1)!!}>\frac{(2N+1)!!}{(2N+2)!!}\cdot2\pi$$

$$(2N+1)\cdot\frac{\pi}{2}>\frac{((2N)!!)^2}{((2N-1)!!)^2}>\frac{(2N+1)^2}{2N+2}\cdot\frac{\pi}{2}$$

$$\left(N+\frac{1}{2}\right)\cdot\pi>\frac{((2N)!!)^4}{((2N)!)^2}>\frac{4N^2+4N}{2N+2}\cdot\frac{\pi}{2}$$

$$\left(N+\frac{1}{2}\right)\cdot\pi>\left(\frac{(N!\cdot2^N)^2}{(2N)!}\right)^2>N\cdot\pi$$

$$\sqrt{\left(N+\frac{1}{2}\right)\cdot\pi}>\frac{4^N}{(2N)!\big/(N!)^2}>\sqrt{N\cdot\pi}$$

5. Compute $\lim_{N\to\infty}\left(\frac{\sqrt[N]{N!}}{N}\right)$.

First solution.

Lemma 1. If sequence $\{a_n\}$ has a limit, then the sequence of its averages $\left\{\frac{a_1+a_2+...+a_n}{n}\right\}$ has the same limit.

Proof is direct, with epsilons and deltas. All element of the second sequence after some index are near the limit, and others give finite (and reducing) contribution.

Lemma 2. To find a limit of a sequence $\{b_n\}$ it is enough to find the limit of the sequence $\{(n+1)b_{n+1} - nb_n\}$, and if we do, the limit of $\{b_n\}$ is the same.

Proof : it follows directly from lemma 1, since sequence $\{b_n\}$ are averages of sequence $\{(n+1)b_{n+1} - nb_n\}$.

Lemma 1'. If sequence of positive numbers $\{a_n\}$ has a limit, then the sequence of its geometric means $\left\{\sqrt[n]{a_1a_2\cdot...\cdot a_n}\right\}$ has the same limit.

Lemma 2'. To find a limit of a sequence of positive numbers $\{b_n\}$ it is enough to find the limit of the sequence $\left\{\frac{b_{n+1}^{n+1}}{b_n^n}\right\}$ has the same limit.

The last two lemmas follow from the former two by taking exp of all sequences. We shall use lemma 2', but I wanted to show you other things which are useful.

So, for limit of $\frac{\sqrt[N]{N!}}{N}$ it would be enough to find the limit of

$$\frac{(N+1)!}{(N+1)^{N+1}}\bigg/\frac{N!}{N^N}=\frac{N^N}{(N+1)^N}=\frac{1}{\left(1+\frac{1}{N}\right)^N}$$

The denominator is a famous expression for $\boldsymbol{e}$, so this sequence and the original one two converge to $1/e$.

Second solution (I learned it recently from a 13-year old who invented it himself, Gil Gitik). Take the log. $\ln\left(\frac{\sqrt[N]{N!}}{N}\right)=\frac{\ln\frac{1}{N}+\ln\frac{2}{N}+...+\ln\frac{N}{N}}{N}$

This is Riemann sum for an integral $\int_0^1 \ln x dx=\left(x\ln x-x\right)_0^1=-1$.

So, before taking ln, the limit was e^{-1}.

Targil 4. This time – combinatorics.

1. Six singers came to a festival. Each day several singers make a performance at the scene, while others listen from the audience.
How to organize the shortest possible festival, so that each singer will listen to every other singer?

2. Once upon a time, 17 cannibals have gathered together for a feast. During the feast, some cannibals were killed and eaten by some other cannibals.
It is known however, that there were no 5 cannibals such that neither of them has eaten any of the other 4.
Prove that there is a chain of 5 cannibals, such that each (except the 5^{th}) was eaten by the next member of the chain.

3. (a) A graph of 10 vertices doesn't contain 3 vertices which are all adjacent to each other. How many edges (at most) can it have?
(b) A graph of 10 vertices doesn't contain 4 vertices which are all adjacent to each other. How many edges (at most) can it have?

4. Yoav, when he was at school, once participated in a test in Arabic language (mevhan bearavit). During the test, he should have connected 10 Hebrew words to their 10 Arabic translations. He didn't know Arabic at all, so he has chosen the translation at random. His mark was 0, which means he had no correct guesses.
Many years later he asked me a question, and it is also my question to you. What was a probability of getting 0 at that test?

5.** There is a long line of $m+n$ people, which all want to buy an ice-cream. The price is 5₪, and out of those people m have a coin of 5₪, and n have a coin of 10₪, and the stand in random order. In the beginning, the cashier has k coins of 5₪. What is the probability that they won't get stuck?

6.** A rectangle is called ***good***, if one of its sides is integer. A big rectangle is divided into a finite number of smaller rectangles, which are all good. Show that the big rectangle is also good.

Targil 4. This time – combinatorics.
1. Six singers came to a festival. Each day several singers make a performance at the scene, while others listen from the audience.
How to organize the shortest possible festival, so that each singer will listen to every other singer?

Solution. Well, the question for 6 is too simple, the answer is 4, so of course we want to formulate the solution in a generalizable way.
We shall apply "duality". Speaking vaguely, we shall not see the days of the festival as subsets singers, but vice versa, the singers as subsets of days. Strictly speaking, each singer defines a subset of the days of the festival, those are days when he sings. A forbidden situation is when singer A was singing on all days when singer B was singing, that is, one subset is contained in another.
So, the question can be reformulated as follows: how big should be the set, so that it has a family of 6 subsets, none of which contains another?
One way to form such a family of subsets is to take subsets of given size. The number of such subsets is a binomial coefficient. The greatest number in each line of Pascal's triangle is the middle one (or the closest to the middle). For example, a set of 4 has a family of 6 subsets of 2 elements each.

So, to prove the general claim (for any number of singers/days) we must find the number of maximal family on mutually not-containing subsets in a set of days. We shall prove that the example we gave is a maximal.
Assume we have a family of mutually not-containing subsets in the set of size N, and they are of different size. Of course, this family can be replaced by another family, of the compliments to the original family, and it will have the same properties. So we may assume that some subsets have more than N/2 elements. Suppose the largest subsets in the family have M elements. ***Let us replace*** all the subsets of M elements by all the subsets of M – 1 elements, which are subsets of those subsets of M elements. The new subsets will be mutually non-containing between themselves and with the other members of the families. The question is, how this replacement affected the number of the subsets?
Each set of order M contains precisely M subsets of M – 1 elements. Each subset is counted no more than N – M + 1 times, but M > N/2, Hence $N - M + 1 \le M$. Hence the number of subset was not decreased by such a replacement. Hence, we can assume no subset is bigger than N/2. By symmetric procedure (or taking the compliment and repeating the same procedure) we conclude that all subsets should have the same size, [N/2].

2. Once upon a time, 17 cannibals have gathered together for a feast. During the feast, some cannibals were killed and eaten by some other cannibals.
It is known however, that there were no 5 cannibals such that neither of them has eaten any of the other 4.
Prove that there is a chain of 5 cannibals, such that each (except the 5^{th}) was eaten by the next member of the chain.

Solution. A cannibal will be called ***kind*** if he didn't eat anybody.
The cannibal is of the ***second kind***, if he ate only kind cannibals at the feast.
The cannibal is of the ***third kind***, if he ate cannibals of the second kind, and maybe also some who were kind, but no others.
Similar we define cannibal of ***k^{th} kind***, those who ate some cannibals of ***k-1^{st}*** kind, maybe of lower kind also, but no others.
If there is a cannibal of the 5^{th} kind, then he ate someone of the 4^{th} kind, who ate someone of the 3^{rd} kind, and so on, which is a chain of 5 cannibals who ate each other. If there is no cannibal of the 5^{th} kind, then there are only 4 kinds of cannibals, so there is a kind of cannibals for there are at least 5 cannibals of that kind. Those 5 cannibals didn't eat each other.

Remark. This fact is known in combinatorics as ***Dilworth's lemma.*** Here is a more scientific way to formulate it: a set of AB+1 elements with partial order must have a chain of length A+1 or an anti-chain of length B+1.

Remark. A proof might be alternatively formulated as follows:
A cannibal is considered cool if he survived the feast. He is considered almost cool if he was eaten only by the cool people, and so on…

3. (a) A graph of 10 vertices doesn't contain 3 vertices which are all adjacent to each other. How many edges (at most) can it have?
(b) A graph of 10 vertices doesn't contain 4 vertices which are all adjacent to each other. How many edges (at most) can it have?

Solution

(a) The answer is 25. It is achieved when by a bipartite graph (גרף דו-צדדי), 5 vertices on each side, all the 5 left vertices are connected to all 5 right vertices, but left vertices, as well as right vertices, are not connected among themselves. The hard part is the proof of maximality.

It is also true that the best result for any number of vertices for a graph which doesn't contain triangle is a bipartite graph, whose sides are as equal as possible: for 2N vertices there are N vertices on each side, for 2N+1 vertices there are N vertices on one side and N+1 on the other.

We shall prove that the bipartite graph is the best by induction.
Suppose we have proved it for graphs with fewer vertices than N. We are given a graph **G** with N vertices, many edges and no triangles, we have to prove that the number of edges is not bigger than it would be for a bipartite graph **B**.
The degree of each vertex of **B** is at least [N/2]. If we find a vertex of whose degree not greater than [N/2], we win. Indeed, we can erase two vertices, one from **G** and another from **B**. After that, bipartite graph will have not less edges by the assumption of induction, but it lost not less edges.

So, it is enough to show that for a graph **G** of N vertices with no triangles, there is a vertex of degree no more than N/2. If not, than the degree of each vertex is (N+1)/2 at least. It means that each vertex has at least (N+1)/2 neighbors, and at most (N – 3)/2 non-neighbors. So each two adjacent vertices have at least one common neighbor, hence we get a triangle.

(b) The same can be proven for a general case. In graph theory, a complete subgraph is called ***a clique***. So, the general question is: given a graph of N vertices doesn't contain K-cliques, how many edges can it have, at most? The answer is: split N vertices in K – 1 subsets of almost equal size, and connect all pairs of vertices from different subsets, and that is the maximum.

The proof is by induction, same as (a). Suppose a graph with no N-cliques surpasses our construction. If we have a vertex of degree no more than $N\frac{K-2}{K-1}$, erase it and apply the induction assumption. If all vertices have greater degree, then each vertex has less than $\frac{N}{K-1}-1$ non-neighbors.
Than choose the vertices, one-by-one, so that each vertex is a neighbor of all previously chosen vertices. Even after we have chosen K–1 vertices, all their non-neighbors a fewer than $N-(K-1)$, so we can choose a clique of K.

4. Yoav, when he was at school, once participated in a test in Arabic language (mevhan bearavit). During the test, he should have connected 10 Hebrew words to their 10 Arabic translations. He didn't know Arabic at all, so he has chosen the translation at random. His mark was 0, which means he had no correct guesses.
Many years later he asked me a question, and it is also my question to you. What was a probability of getting 0 at that test?

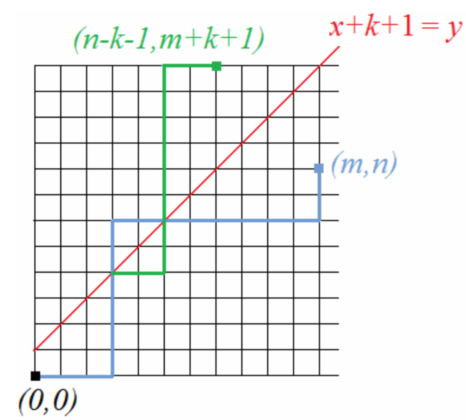
Proof. There are 10! permutation from which Yoav could choose. We shall use inclusion/exclusion principle to count how many permutations will be marked by 0. The answer is number of zero-mark permutations / 10!
Take all 10! permutation and subtract the permutation with one correct guess. There are 10 words that can be guessed correctly, foe each word there are 9! ways to be correct. So 10! – 9!10, now add double intersections, subtract triple intersections, and so on.
There are $\binom{10}{2}$ double intersections, each having 8! elements, $\binom{10}{3}$ triple intersections, each having 7! Elements, and so on, in general $\binom{10}{k}$ each having (10 – k)! elements. So we have $10! - 9!10 + 8!\binom{10}{2} - 7!\binom{10}{3} + \ldots$
And if we divide it by 10! we get the probability:

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!} \approx \frac{1}{e}$$

Well, it is a rational number of course, and 1/e is irrational (why?), but it is a very good approximation, the best we could get with denominator 10!, which is 8 digits long, so it has precisely the same first 8 digits as 1/e: 0.3678794.

5.** There is a long line of m+n people, which all want to buy an ice-cream. The price is 5₪, and out of those people m have a coin of 5₪, and n have a coin of 10₪, and they stand in random order. In the beginning, the cashier has k coins of 5₪. What is the probability that they won't get stuck?

Solution. The solution is, surprisingly, geometrical. There are $\binom{m+n}{m}=\frac{(m+n)!}{m!\cdot n!}$ ways to arrange the people in the line, we should count the number of all good ways and divide it by the number of all ways.
The key idea is to count the number of bad ways, instead of counting the number of good ways, and then we can subtract.
The ways to arrange people can be seen as ways to have a shortest walk in a city where all streets are parallel or orthogonal to each other, when you have to pass *m* blocks from west to east and *n* blocks from south to north, suppose between point (0,0) and a point (*m*,*n*).
Getting stuck means arriving to a point (*x*,*y*) which is on certain diagonal, on which "the cashier would have spent *k*+*1* coins of change", that is *x*+*k*+1=*y*.
The bad ways are ways which pass through that "red" diagonal.
Let's reflect the part of the way after its first time it touches the red diagonal with respect to the red diagonal.
We shall have a way which goes from (0,0) to $(n-k-1,m+k+1)$, which is the point symmetric to (*m*,*n*) with respect to the red diagonal.
This reflection gives one-to-one correspondence between all bad ways to (*m*,*n*) and all ways to $(n-k-1,m+k+1)$, so their number is $\binom{m+n}{m+k+1}$.



Hence the number of good ways is

$$good(k,m,n)=\binom{m+n}{m}-\binom{m+n}{m+k+1}.$$

So the chance is $1-\binom{m+n}{m+k+1}\Big/\binom{m+n}{m}$.

Remark. The special case of the ice cream problem, the number of good ways when *n*=*m* , *k*=0 has a name: *Catalan numbers*,

$$c_n=\binom{2n}{n}-\binom{2n}{n+1}=\frac{(2n)!}{n!n!}-\frac{(2n)!}{(n-1)!(n+1)!}=(2n)!\frac{n+1-n}{n!(n+1)!}=\frac{(2n)!}{n!(n+1)!}$$

Catalan numbers have a lot of combinatorial descriptions: number of legal sequences of n left and n right brackets, number of minimal triangulations of a convex polygon of n+2 vertices, several description with trees, number of ways to build a tower of base length n from cylindrical sticks, ant others.

6.** A rectangle is called ***good***, if one of its sides is integer. A big rectangle is divided into a finite number of smaller rectangles, which are all good. Show that the big rectangle is also good.

First solution. Firstly, it is obvious that all sides of small rectangles are parallel to the sides of the large rectangles (since if rectangle is parallel, then all its neighbors are parallel).
Consider integral $\iint e^{2\pi i x} e^{2\pi i y} dxdy$ over each rectangle. It is easy to verify that it is 0 iff the rectangle is good. The integral over the big rectangle is a sum of integrals over small rectangles, so it is 0. Hence the big rectangle is good.

Second solution. We shall translate the first solution into a more elementary language. Let $f(x)$ be a function such that $f(x) = f(y)$ iff $x - y$ is integer.
It can be $e^{2\pi i x}$ or, to keep it more elementary, $\{x\}$, the fractional part of x.
Define a function of rectangle

$$F(rectangle) = (f(\max x) - f(\min x)) \times (f(\max y) - f(\min y))$$

F is 0 iff a rectangle is good. F is an additive function, that is, if a rectangle R is divided into two rectangles, R_1 and R_2, then $F(R) = F(R_1) + F(R_2)$
Hence, is small rectangles are good, F each of them is 0, then F of big rectangle is also 0, so the big rectangle is good.
(There is a subtle point here: for some partitions you can not unite any two rectangles into one, but then you can split them in parts and unite them afterwards).

Third solution. Paint the big rectangle in checker-board coloring, starting from the bottom-left corner, such that the sides of checker squares will be ½.
It is easy to see that for each good rectangle its black area is equal to its white area. It is also easy to see that vice versa is true if one corner of rectangle is also a corner of a checker-square (why?).
In the big rectangle black area = white area, since it is divided into parts, each having black area = white area. So the big rectangle is good.

Fourth solution. Lets prolong all the sides of all rectangles to intersect with the sides of big rectangle. Introduce coordinate axis along the sides of big rectangle (left-bottom corner is (0,0)).
Take all lines, which have non-integer coordinates (both vertical and horizontal), and move them by the same distance d and in the same direction. All integer sides of small rectangles will remain the same. But

non-integer sides might change by d, so their area will change by ad. So the change of area, of each small rectangle and total, is a linear function of d. But if it would have all non-integer sides, it would be a quadratic function, that would give a contradiction.

The next solution is in the based on Euler's path method, which was developed by Euler to solve the riddle of Koenigsberg bridges:
http://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg

Fifth solution. Consider a graph: its vertices are vertices of small rectangles, its sides are integer sides of small rectangles. If a rectangle has 4 integer sides, only the horizontal sides will be considered to be edges of the graph. If two rectangles have a common integer side, we should take it twice.
It is easy to see that degrees of all corners of the big rectangle are 1, while the degrees of all other vertices are even.
Start traveling from a left-bottom corner along the edges of the graph, burning the bridges behind you (never repeat the edge which was taken before). At some moment, you will have to stop. It can happen only at a vertex of odd degree, which can only be another corner. But the increment of each coordinate is integer on each step. And one of coordinates of total increment is either width or height.

Sixth solution. From double-counting, we see that each graph has even number of vertices of odd degree. Since sum of all degrees of all vertices is twice the number of edges. Consider the graph that was constructed in the previous solution, and consider in it the connected component of the left-bottom corner. It should contain another vertex of odd degree, so one more corner. But all elements of its connected component have integer coordinates, so either width or height is integer.

Next solution will be based on a topological fact:
Lemma. Suppose a rectangle is painted in 2 colors, black and white. Then you can either find a white path from left to right, or a path from top to bottom.

Proof of lemma. Paint the area to the right of the rectangle white. Consider the white connected component of the white spot near the white side. If it reaches the left side – cool, we have a white path from left to right. Suppose it doesn't. Then the border of this white spot is all black. And that is the black path from top to bottom.

Seventh solution. Divide all good small rectangles into W-type (those have integer width) and H-type (all the others). Paint the interior of W-type rectangles white and the interior of H-type rectangles black.
Paint the interior part of all horizontal sides of all small rectangles black, and the interior part of all vertical sides of all small rectangles white.
The only unpainted points now are cross-intersections, paint them black, it doesn't matter.
If we have a white path from left to right, it goes along W-type rectangles between vertical sides, so it gives an integer increment of x. So, in this case, width is integer.
If not, by the lemma, we get a black path from top to bottom. In this case, for the same reason, the height is integer. QED.

The last solution is in the spirit of algebraic topology. In algebraic topology, there is a nice notion of homology. It has many definition, one is based on chains. A ***k-chain***, roughly speaking, is a k-dimensional oriented piece of a space, or k-dimensional polytope, or a formal linear combination of such. There is a notion of a ***boundary*** (sum of boundary pieces) which is k-1-chain and of a ***cycle*** – a chain whose boundary is 0.
If there is a mapping from one space to another, the chains can be pushed from the first space to the second.

Eighth solution. Consider each rectangle as a 2-chain. We can consider it on the plane or on the torus, since there is a natural mapping from $\mathbb{R}^2$ to the torus, which is factorization over $\mathbb{Z}^2$.
For rectangular 2-chains and their formal sums, define operator ***horizontal boundary***, which is upper side minus lower side. It is obvious that the rectangle is good if and only if its horizontal boundary is a cycle on the torus. But horizontal boundary is additive, and sum of cycles is a cycle
The horizontal boundary of the big rectangle is sum of horizontal boundaries of the small rectangles, so it is cycle, so the big rectangle is good.

Targil 5. Combinatorics again, but now with infinite sets.

1. Show that each sequence $\{a_n\}$ of real numbers has either infinite non-decreasing subsequence or infinite non-increasing subsequence.

2. Consider a set of distinct points in space $\{(x_i, y_i, z_i)\}$ such that all their coordinates are natural (positive integers). (Not all points of that kind, just some of them.)
Point (x_i, y_i, z_i) of this set is called minimal if for every other point (x_k, y_k, z_k) in this set, point has a smaller coordinate $x_i < x_k$, or $y_k < y_i$, or $z_i < z_k$.
Can a number of minimal points be infinite?

3. Show that there is a point in the plane, such that distances from it to all integer points are different. (A point (x,y) is called integer if x and y are integer).

4. (a) Is it possible to find 1000000 points in the plane, not all of them on one line, so that the distance between each two is integer?
(b) Is it possible to choose an infinite set of points in the plane, not all of them on one line, so that the distance between each 2 is integer?

5*. (a) We have a family of subsets of a countable set (say, natural numbers).
Each two members of the family have no more than 100 common elements.
Prove that the family is of countable size (at most).
(b) We have a family of subsets of a countable set, such that intersection of each 2 is finite. Can this family have more than a countable number of elements?

Targil 5. Combinatorics again, but now with infinite sets.

1. Show that each sequence $\{a_n\}$ of real numbers has either infinite non-decreasing subsequence or infinite non-increasing subsequence.

Solution. Denote $A_m = \{a_n \mid n > m\}$. If any A_m has maximal element, then building non-increasing subsequence is easy: choose maximal element in A_1, then choose maximal element of all elements after it and so on, each time choose maximal element of all elements that have bigger index then all chosen ones, and we have non-increasing sequence.
If some A_k has no maximal element, then for any $m > k$, A_m has no maximal element either. Then for any element there is a bigger element with bigger index, so we can choose a strictly increasing subsequence.

2. Consider a set of distinct points in space $\{(x_i, y_i, z_i)\}$ such that all their coordinates are natural (positive integers). (Not all points of that kind, just some of them.)
Point (x_i, y_i, z_i) of this set is called minimal if for every other point (x_k, y_k, z_k) in this set, point has a smaller coordinate $x_i < x_k$, or $y_k < y_i$, or $z_i < z_k$.
Can a number of minimal points be infinite?

Solution. This is done by induction over dimension.
For 1-dimensional case it is trivial – in each subset of natural numbers there is one minimal element.
For 2-dimensional case choose one minimal point (x_1, y_1). For each x between 1 and x_1 there can be no more than 1 minimal point with that x. For each y between 1 and y_1 there can be no more than 1 minimal point with that y. There can be no minimal points such that $x > x_1$ and $y > y_1$, so we have no more than finite number $(x_1 + y_1)$ of minimal points in plane.

Now for 3 dimensional case. Choose one minimal point (x_1, y_1, z_1).
For each x between 1 and x_1 there is only finite number of minimal points with that x. For each y between 1 and y_1 there is only finite number of minimal points with that y. For each z between 1 and z_1 there is only finite number of minimal points with that z.
There can be no minimal points such that $x > x_1$ and $y > y_1$ and $z > z_1$, so we have no more than finite number of minimal points. QED.

Remark. Of course this induction can continue to higher dimensions.

3. Show that there is a point in the plane, such that distances from it to all integer points are different. (A point (*x,y*) is called integer if *x* and *y* are integer).

First solution. Take $\left(\sqrt{2},\sqrt{3}\right)$, and show it works. Suppose two points $\left(x_1,y_1\right)$and $\left(x_2,y_2\right)$ give the same distance, then

$$\left(x_1-\sqrt{2}\right)^2+\left(y_1-\sqrt{3}\right)^2=\left(x_2-\sqrt{2}\right)^2+\left(y_2-\sqrt{3}\right)^2$$
$$x_1^2+y_1^2-2x_1\sqrt{2}-2y_1\sqrt{3}=x_2^2+y_2^2-2x_2\sqrt{2}-2y_2\sqrt{3}$$
$$\left(x_1^2+y_1^2-x_2^2-y_2^2\right)+\left(2x_2-2x_1\right)\sqrt{2}+\left(2y_2-2y_1\right)\sqrt{3}=0$$

Lemma 1. If $a+b\sqrt{2}+c\sqrt{3}=0$for some rational numbers *a, b, c* then *a*=*b*=*c*=0.

Lemma 2. $\sqrt{2},\sqrt{3},\sqrt{\frac{3}{2}}$ are irrational numbers.

From lemma 1 it follows that in our case $x_2=x_1, y_2=y_1$ so the points are actually the same. So it remains to prove lemma 1, but first we shall prove lemma 2.

Proof of lemma 2. If $\sqrt{2}=\frac{m}{n}$ then square it and multiply by denominator: $n^2=2m^2$, then in the left hand side you get even power of 2, and on the right hand side the odd power of 2 (in prime decomposition), it contradicts unique decomposition theorem.

The same will happen with $\sqrt{\frac{3}{2}}$ and $\sqrt{3}$, but in the last case you should count powers of 3 (odd on one side, even on the other).

Proof of lemma 1. $a+b\sqrt{2}+c\sqrt{3}=0$. If all 3 numbers *a, b, c* are 0, that's it, if 1 of them is nonzero its nonsense, if two of them are nonzero, we contradict lemma 2 which was proved already. The only remaining case, that we have yet to exclude, is when all 3 are nonzero.

$a+b\sqrt{2}=-c\sqrt{3}$, square both sides, you get $a^2+2b^2+2ab\sqrt{2}=3c^2$.

But $2ab\neq 0$, hence $\sqrt{2}=\frac{3c^2-a^2-2b^2}{2ab}$ is a rational number, contradiction.

Second solution. A very natural solution to a more general question belongs to a French mathematician Baire (1874-1932).

A set is called ***dense*** if any ball (say, in $\mathbf{R}^n$) contains its point.
A set is called ***nowhere dense*** if any ball contains a smaller ball which is disjoint to given set.
A set is called ***meager*** or of ***first (Baire's) category*** if it is a countable union of nowhere dense sets.
A set is called of ***second (Baire's) category*** if it is not of the first category.

Examples:
A set of rational points is dense, though it is countable.
A sphere (set of points on given distance from the given point), or a line in the plane or a hyperplane in $\mathbf{R}^n$ are nowhere dense.
Any countable set is meager. A set of points, on rational distance from at least one rational point, is meager. A set of perpendicular bisectors to all intervals with rational ends is meager.
A union of 2 meager sets is meager.

A ball, a full cube, and a complement to any meager set are of second category. Why? Because of

Baire's theorem. A complement of meager set is dense.

So, we can choose a point which is on different distance from all rational points, since union of (countable number of) all perpendicular bisectors to intervals with rational ends is of first category. You can also demand that all those distances would be irrational, transcendental, and add any other countable number of other "nowhere dense demands".

Proof of Baire's theorem. A meager set M is a union of countable number of nowhere dense sets A_1, A_2, A_3, ...Take any ball B. It has a sub-ball B_1 disjoint from A_1. B_1 has a sub-ball B_2 disjoint from A_2. B_2 has a sub-ball B_3 disjoint from A_3, and so forth. The intersection of all those balls has a point which is not in M. Since we found it in arbitrary ball B, complement of M is dense. QED.

4. (a) Is it possible to find 1000000 points in the plane, not all of them on one line, so that the distance between each two is integer?
(b) Is it possible to choose an infinite set of points in the plane, not all of them on one line, so that the distance between each 2 is integer?

(a) First solution.
Yes. There are infinitely many Pythagorean triples $a^2 + b^2 = c^2$, such as $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$.
So the distance between points $B(0,1)$ and $A_k = ((k^2 - 1)/2k, 0)$ is rational, for any k between 1 and 1000000.
If we multiply all things by common denominator, or 1000000!, all coordinates and distances will now be integer. Those are 1000000 points, and not all of them are on one line.

Second solution. (Agnis Andjans) Construct a small angle α with rational sine and cosine. To do this, take an angle in Pythagorean triangle with $m \sim n$.
Now take points on unit circle at angles 0, 2α, 4α, 6α, 8α, …
The distances between those points are $2\sin(k\alpha)$, and those are rational, since it is easy to prove by induction over k, that $\sin(k\alpha)$ and $\cos(k\alpha)$ are polynomials with integer coefficients in $\sin(\alpha)$ and $\cos(\alpha)$ hence they are rational numbers. By the way, coordinates of those points are also rational and for the same reason. If we multiply by common denominator, we get integer points, and all distances are rational, and no 3 points are on one line, since all are on one circle.

(b) No. Consider 3 non-collinear chosen points A, B, C. Every other chosen point D is on integer distance from A and B, so $|AD - BD| = k$ where k is a natural number not exceeding AB, by triangle inequality.
A locus of points D which satisfy $|AD - BD| = k$ is a hyperbola (or a line), so D belongs to a finite family of (less than AB) hyperbolas (with foci A and B), or to line AB, or to perpendicular bisector of AB.
For the same reason D belongs to a finite family of different hyperbolas whose foci are A and C, or to line AC, or to perpendicular bisector of AC.
Intersection of two different hyperbolas have no more than 4 common points (see targil 1), two different lines no more than one, and a straight line intersects a hyperbola in 2 points at most, so we have only a finite number of points.

5*. (a) We have a family of subsets of a countable set (say, natural numbers).
Each two members of the family have no more than 100 common elements. Prove that the family is of countable size (at most).
(b) We have a family of subsets of a countable set, such that intersection of each 2 is finite. Can this family have more than a countable number of elements?

Solution. (a) Suppose not. Take an uncountable family of subsets.
Firstly, we can assume that all of them are infinite, because there are only $\aleph_0$ finite subsets in a countable sets, so if there are finite sets, it won't hurt to exclude them.
Consider all pairs (subset, element) so that subset contains element. Since there are only $\aleph_0$ elements, and more than $\aleph_0$ subsets, one element should belong to more than $\aleph_0$ subsets. Denote this element a_1, and consider subsets that contain it. For the same reasons, more than $\aleph_0$ subsets should have yet another common element, a_2. For the same reason, more than $\aleph_0$ subsets containing a_1, a_2 should have another common element, a_3, and so on. After you keep on this induction long enough you conclude that more than $\aleph_0$ subsets contain 1000001 common points.

(b) First solution. Yes. Consider a countable set **P** of all points in plane with positive integer coordinate. For each positive real number a define a subset of this set of all points that satisfy $[ax] = y$.
It is easy to see that the intersection of two sets defined by two different numbers is finite, and there are continuum subsets.

Second solution. For any real number, choose a sequence of rational numbers which converges to it. It is a subset of countable set of rational numbers, and each two have final intersection.

Remark. It seemed to me at first that those two solutions are completely different, but they are actually the same. Rational numbers can be visualized as points of integer lattice, y is nominator and x is denominator, and a sequences of points we constructed gives a sequence of ratios converging to the slope.

Problems 6 – polynomials

1. Prove that a polynomial with real coefficient p(x) is nonnegative for all real values iff it is a sum of two squares (of polynomials with real coefficients).

2. A polynomial with real coefficients of 2 variables $p(x,y)$ is always positive. Is it true that it is always bigger then some positive ε?

3. (a) Suppose that a polynomial with integer coefficients is can be decomposed into a product of two polynomials with rational coefficients. Show that it is decomposed into a product of two polynomials with integer coefficients.

(b) Suppose p is a prime number and a polynomial with integer coefficients

$$a_nx^n+a_{n-1}x^{n-1}+ a_{n-2}x^{n-2}+\ldots+a_2x^2+a_1x+a_0$$

has the following properties:

a_n is not divisible by p,

$a_{n-1}, a_{n-2},\ldots ,a_2,a_1,a_0$ are divisible by p,

but a_0 is not divisible by p^2.

In this case the polynomial is not decomposable into the product of two polynomials with integer coefficients.

Definition. A polynomial is called ***irreducible*** if it can't be presented as a product of two polynomials of degree>0. Of course, this definition depends on the field (for example, it can be irreducible over **Q** and split over **R**).

4. Determine whether the following polynomials are irreducible over **Q**:

(a) $2x^3+3x^2+5x+7$

(b) $x^4+19x^2+2x+99$

(c) x^8+x+1

*(d) $x^n+x^{n-1}+\ldots+x^2+x+1$ (here answer depends on n)

5. Proof that if two polynomials of degree less then N have the same values at N different points, then they coincide.

6. Given two polynomials $p(z)$, $q(z)$ with complex coefficients. It is also given that for any complex number z ,

$$p(z) = 0 \text{ iff } q(z) = 0$$
$$p(z) = 1 \text{ iff } q(z) = 1$$

Prove that the polynomials are equal.

Problems 6 – polynomials

1. Prove that a polynomial with real coefficient $p(x)$ is nonnegative for all real values iff it is a sum of two squares (of polynomials with real coefficients).

Solution. A sum of squares is obviously nonnegative. The other part is more interesting.
Complex conjugation keeps the polynomial, so for each root above the complex line a+ib there is corresponding complex conjugate root below the complex line a–ib. Also, each real root is of odd multiplicity, otherwise function to both side of this root wouldn't be positive.
Thus we can divide all the complex roots into pairs: $\alpha_1, \overline{\alpha_1}, \alpha_2, \overline{\alpha_2}, \dots, \alpha_k, \overline{\alpha_k}$.
Hence the polynomial can be written as

$$p(x) = A(x-\alpha_1)(x-\alpha_2)\cdot\ldots\cdot(x-\alpha_k)\cdot(x-\overline{\alpha_1})(x-\overline{\alpha_2})\cdot\ldots\cdot(x-\overline{\alpha_k}),$$

where A is the highest coefficient. $(x-\alpha_1)(x-\alpha_2)\cdot\ldots\cdot(x-\alpha_k)$ is a polynomial with complex coefficients, it can be written as $r(x)+\mathrm{i}q(x)$ where $r(x), q(x)$ are polynomials with real coefficients. So

$$p(x) = A\big(r(x)+\mathrm{i}q(x)\big)\Big(\overline{r(x)+\mathrm{i}q(x)}\Big) = A\big(r(x)+\mathrm{i}q(x)\big)\big(r(x)-\mathrm{i}q(x)\big) =$$

$$= A\Big(\big(r(x)\big)^2 + \big(q(x)\big)^2\Big)$$

Since $p(x)$ is nonnegative so A should be nonnegative, so

$$p(x) = A\Big(\big(r(x)\big)^2 + \big(q(x)\big)^2\Big) = \big(\sqrt{A}\cdot r(x)\big)^2 + \big(\sqrt{A}\cdot q(x)\big)^2.$$

2. A polynomial with real coefficients of 2 variables $p(x,y)$ is always positive. Is it true that it is always bigger then some positive ε?

Solution. No. $p(x,y) = (1 - xy)^2 + x^2$ is always positive, since both squares can't be 0 – if $x = 0$ then $1 - xy = 1$.
But if $1 - xy = 0$ and y is very large then x can be very small. Hence the polynomial accepts all positive values.

3. (a) Suppose that a polynomial with integer coefficients is can be decomposed into a product of two polynomials with rational coefficients. Show that it is decomposed into a product of two polynomials with integer coefficients.

(b) Suppose p is a prime number and a polynomial with integer coefficients

$$a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_2x^2+a_1x+a_0$$

has the following properties:

a_n is not divisible by p,

$a_{n-1}, a_{n-2},\ldots ,a_2,a_1,a_0$ are divisible by p,

but a_0 is not divisible by p^2.

In this case the polynomial is not decomposable into the product of two polynomials with integer coefficients.

Remark. (a) is called Gauss lemma, (b) – Eisenstein criterion.

Solution. (a) We can multiply our rational factors by an integer numbers so that they will become integer. So product of two integer polynomials is a given polynomial times integer number: $N\cdot s(x) = q(x)\cdot r(x)$.
We want to prove that we can get rid off that integer number. Suppose N has a prime factor p. We shall prove either all coefficients of $q(x)$ or all coefficients of $r(x)$ are divisible by p. So p can be cancelled out, and in this way N can be gradually reduced to 1.
Suppose not all coefficients of $q(x)$ and not all coefficients of $r(x)$ are divisible by p. Let $s(x) = a_nx^n +\ldots+a_2x^2+a_1x+a_0$

$$q(x) = b_kx^k + \ldots +b_2x^2+b_1x+b_0$$
$$r(x) = c_mx^m + \ldots +c_2x^2+c_1x+c_0$$

Let b_v be the first coefficient of $q(x)$ which is not divisible by p, and c_w the first coefficient of $r(x)$ which is not divisible by p. Then b_vc_w is not divisible by p, and for each $j\neq 0$ the number $b_{v+j}c_{w+j}$ is not divisible by p, so na_{v+w} is not divisible by p, which is impossible. QED.

Another way to formulate the solution – look at all those polynomial mod p. You will get $0 = q(x)\cdot r(x) \pmod p$. So either $0 = q(x)$ or $0 = r(x) \pmod p$.

(b) First solution.
Suppose our polynomial $s(x) = a_nx^n +\ldots+a_2x^2+a_1x+a_0$ has a decomposition: $s(x) = q(x)\cdot r(x)$ where

$$q(x) = b_kx^k + \ldots +b_2x^2+b_1x+b_0$$
$$r(x) = c_mx^m + \ldots +c_2x^2+c_1x+c_0$$

So, $a_0 = b_0c_0$ hence one of the numbers b_0, c_0 is divisible by p but not both.
Without loss of generality assume that p divides b_0 and not c_0.
$a_1 = b_0c_1 + b_1c_0$ and p divides a_1 and b_0c_1, so p divides also b_1.
$a_2 = b_0c_2 + b_1c_1 + b_2c_0$ and p divides a_2 , b_0c_2 , b_1c_1, so p divides also b_2.
And so forth, we proof in k steps that p divides b_k.

But then p would divide $a_n = b_k c_m$, which is impossible. QED.

Second solution. Write everything mod p. You get $a_n x^n = q(x)\cdot r(x)$ (mod p). This means only the first coefficient of q and only the first coefficient of r are nonzero mod p. Hence both b_0 and c_0 are divisible by p, so $a_0 = b_0 c_0$ is divisible by p^2.

Definition. A polynomial is called ***irreducible*** if it can't be presented as a product of two polynomials of degree>0. Of course, this definition depends on the field (for example, it can be irreducible over **Q** and split over **R**).

4. Determine whether the following polynomials are irreducible over **Q**:
(a) $2x^3+3x^2+5x+7$
(b) $x^4+19x^2+2x+99$
(c) x^8+x+1
*(d) $x^n+x^{n-1}+\ldots+x^2+x+1$ (here answer depends on n)

Solution. (a) Yes.
If a polynomial of degree 3 is split, the factors are of degrees 1 and 2. Hence it has a root (in the same field over which it splits).
There is a way to find all rational roots of a polynomial. Every rational number has a representation m/n, where m and n are coprime, m integer and n natural. Substitute it into our polynomial:

$$2(m/n)^3+3(m/n)^2+5(m/n)+7 = 0$$

Multiply it by n^{degree}, in our case n^3:

$$2m^3+3m^2n +5\, mn^2 +7n^3 = 0$$

Now you have an equation in integer numbers. All terms in left hand side except the last one are divisible by m, and all terms the first one are divisible by n. So the last term $2m^3$ is divisible by n, and the first term $7n^3$ is divisible by m. But m and n are coprime, hence the 2 is divisible by n and 7 is divisible by m. So there is only finite number of numbers which can possibly be roots: 1, 7, -1, -7, ½ , -½, 7/2, -7/2.

Remark. That's the general principle: if polynomial with integer coefficients has a rational root, its nominator divides the free coefficient and its denominator divides the first coefficient, so the root can be found in finite number of verifications.

In our case, odd integer number is not an option, since value will be odd, and positive numbers are also out of question, so the two remaining possibilities are -1/2 and -7/2. Of course, first gives positive value and second negative, so this polynomial has no rational roots.

(b) $x^4+19x^2+2x+99 = x^4 + 20x^2 + 100 - x^2 + 2x - 1 = (x^2 + 10)^2 - (x - 1)^2 =$
$= (x^2 + x + 9)(x^2 - x + 11)$

(c) x^8+x+1 . The answer – yes.

Consider number $\omega = \dfrac{-1+i\sqrt{3}}{2} = "\sqrt[3]{1}"$ (cube of this number is 1).

It is root of polynomial x^3-1 and since it isn't one it is even a root of x^2+x+1. But $\omega^8+\omega+1=\omega^2+\omega+1=0$ so, our polynomial is not co-prime to x^2+x+1. Their greatest common divisor is a polynomial with coefficient in **Q**, since it is given by Euclidean algorithm, so it is of degree 2 and not 1, so it is x^2+x+1. Conclusion x^8+x+1 is divisible by x^2+x+1.
Perform long division to verify yourself, and you get a decomposition
$x^8+x+1=\left(x^2+x+1\right)\left(x^6-x^5+x^3-x^2+1\right)$.

(d) $x^n+x^{n-1}+\ldots+x^2+x+1$ is irreducible iff n+1 = p is prime.
If n+1 = k·m then
$x^n+x^{n-1}+\ldots+x^2+x+1 = (x^{(m-1)k}+\ldots+x^{2k}+x^k+1)(x^{k-1}+x^{k-2}+\ldots+x^2+x+1)$.
The hard part is to prove irreducibility for primes.
The simplest prove uses a trick – shifting by 1. Denote $x = y + 1$.
This transformation doesn't influence irreducibility property. But

$$x^{p-1}+x^{p-2}+\ldots+x+1=\frac{x^p-1}{x-1}=\frac{\left(y+1\right)^p-1}{y}=\frac{\sum_{j=1}^{p}\binom{p}{j}y^j}{y}=\sum_{j=1}^{p}\binom{p}{j}y^{j-1}$$

So, we have a polynomial with first coefficient 1, last coefficient p and all coefficients in the middle are divisible by p, so it is irreducible by Eisenstein's criterion (3b).

5. Proof that if two polynomials of degree less then N have the same values at N different points, then they coincide.

Solution. The difference of two such polynomials would be a polynomial with N roots, and a polynomial of degree less then N has less than N roots, unless it is constant 0.

Remark. Of course that problem was not a real challenge; it is a hint for the next problem. The inverse problem is bit more interesting: given N distinct points $x_1, x_2, \ldots x_N$, and n arbitrary values, $a_1, a_2, \ldots, a_N$, prove that there exists unique polynomial of degree < N such that $p(x_i) = a_i$. We have proven only uniqueness, there are different proofs for existance, from constructive ones to dimension counting.

6. Given two polynomials $p(z)$, $q(z)$ with complex coefficients. It is also given that for any complex number z ,

$$p(z) = 0 \text{ iff } q(z) = 0$$
$$p(z) = 1 \text{ iff } q(z) = 1$$

Prove that the polynomials are equal.

Solution. We may assume without loss of generality that deg p $\geq$ deg q.
A point z happens to be a root of multiplicity k of $p(z)$ iff it is a root of $p(z)$ and a root of degree $k - 1$ of polynomial $p'(z)$.
Total number of roots of $p(z)$ with multiplicities is its degree n.
Number of distinct roots of $p(z)$ is n minus total multiplicity of distinct roots of $p(z)$ as roots of $p'(z)$.
Number of distinct roots of $p(z) - 1$ is, for the same reason, n minus sum of multiplicities of roots of $p(z) - 1$ as roots of $p'(z)$. So, number of points in which $p(z)$ is 0 or 1 is at least $2n - \deg(p') = n + 1$.
So, p and q are both polynomials of degree less than $n + 1$, and they coincide in at least n points, hence they are equal.

Targil 7.
This targil is about groups and groupish ideas. Groups are popular in IMC.
For those who don't know: use wikipedia
http://en.wikipedia.org/wiki/Group_(mathematics)

1. Consider Rubik's cube (קוביה הונגרית). Is it possible to find a certain sequence of moves, such that you can solve the cube from any situation, if you repeat that specific combination of moves sufficiently many times?

2. Consider a table:

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

You are allowed to flip each two rows and to flip each two columns. How many different tables can You get by a sequence of such steps?

3**. We play the following game. On the circle, we have some red and blue points (always at least two). We can insert a red point into certain arc, and flip the colors of the ends of that arc (red changed to blue, blue changed to red). Conversely, if we have at least three points, we can erase a red point, simultaneously flipping the colors of its neighbors.
In the beginning of the game, we have only two points on the circle and both are blue. Can we perform a sequence of moves, so that in the end we shall have only two colored points, and both red?

4*. Let G be a finite group. For arbitrary subsets U,V,W of G, denote by N_{UVW} the number of triples (x,y,z) in $U \times V \times W$, for which xyz is the unity.
Suppose that G is partitioned into three sets A, B and C (i.e. sets A,B,C are pairwise disjoint and $G = A \cup B \cup C$).Prove that $N_{ABC} = N_{CBA}$.

5**. Let G be a group of $2^k(2m+1)$ elements and suppose it has an element of order 2^k. Prove that all the elements of odd order (together with the unit element) form a subgroup.

Targil 7.

1. Consider Rubik’s cube (קוביה הונגרית). Is it possible to find a certain sequence of moves, such that you can solve the cube from any situation, if you repeat that specific combination of moves sufficiently many times?

Solution. If we would, the group of rotations of Rubik’s cube would be cyclic. Then, it would be commutative. It isn’t. Rotations of two adjacent faces don’t commute.

2. Consider a table:

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

You are allowed to flip each two rows and to flip each two columns. How many different tables can You get by a sequence of such steps?

Answer. $7!\cdot 6!$

First solution. Consider the stabilizer of this table (subgroup of operations that keep it). It consists of at least 7 elements (any cyclic rotation of rows with the same rotation of columns). The set of possible tables is the (right) factor over the stabilizer.
We have $7!^2$ elements in group, so no more than $7!^2/7$ possible tables. But we can make any of 7! Permutation in upper row, and 6! permutations of the elements of left column disregarding the upper-left corner, so at least 7!6! different tables.

Second solution. For each 4 cells forming a rectangle ABCD, sides parallel to edges of the table, it is easy to proof that we have A+C = B+D (mod 7) for the values of the cells. This property (a) holds when we perform our operations (b) allows to reconstruct the table by first row and left column only. It is easy to that we can make 7!6! different situations in the first row and left corner, so…

3**. We play the following game. On the circle, we have some red and blue points (always at least two). We can insert a red point into certain arc, and flip the colors of the ends of that arc (red changed to blue, blue changed to red). Conversely, if we have at least three points, we can erase a red point, simultaneously flipping the colors of its neighbors.
In the beginning of the game, we have only two points on the circle and both are blue. Can we perform a sequence of moves, so that in the end we shall have only two colored points, and both red?

First solution. Let R be a clockwise 120^0-rotation of equilateral (regular) triangle, B its reflection with respect to a certain altitude. Then BRB = RR , BRR = RB , RRB = RB (all those equalities are obviously equivalent, first is obvious – conjugating rotation by

reflection gives you a rotation in opposite direction). Now, take your circle, go clockwise from certain point, and write down R for each red and B for each blue point. You get an element of group S_3 (group of 3-permutations or of triangle symmetries) when you write everything down. The element you get depend on the starting point, but its conjugation class doesn't. From the equalities we wrote, we see that the class of conjugacy won't change under those operations.
So, two red points correspond to RR, which is conjugate to rotation (element of order 3) and BB corresponds to identity, so they can't be transformed one into another.

Second solution. The parity of number of red points is an invariant, but it doesn't distinguish between two situations: it is even in both cases. So, assume we have an even number of red points and red points split the circle into even number of arcs. Count the number of blue points on odd arcs minus number of blue points on even arcs modulu 3. One can show with a routine verification it is an invariant, it is 2 in one case and 0 in another, QED.
Actually it is the same as the first solution, though the reason, why this particular invariant works, is unclear here.

4*. Let G be a finite group. For arbitrary subsets U,V,W of G, denote by N_{UVW} the number of triples (x,y,z) in $U \times V \times W$, for which xyz is the unity.
Suppose that G is partitioned into three sets A, B and C (i.e. sets A,B,C are pairwise disjoint and $G = A \cup B \cup C$).Prove that $N_{ABC} = N_{CBA}$.

Solution. Firstly, it is obvious that $N_{CBA} = N_{BAC}$ since zyx = 1 iff yxz = 1 because they are conjugated: $zyx = z \cdot yxz \cdot z^{-1}$. So, enough to prove $N_{ABC} = N_{BAC}$.
The elements (x,y) in $U \times V$ are of 3 kinds: $(xy)^{-1}$ can be in A, B, or C.
So $N_{UVA}+N_{UVB}+N_{UVC} = |U|\cdot|V|$. But $N_{ABA} = N_{BAA}$, $N_{ABB} = N_{BAB}$ because of conjugation, therefore $N_{ABC} = |A|\cdot|B| - (N_{ABA} + N_{ABB}) = |A|\cdot|B| - (N_{BAA} + N_{BAB}) = N_{BAC} = N_{CBA}$.

5**. Let G be a group of $2^k(2m+1)$ elements and suppose it has an element of order 2^k. Prove that all the elements of odd order (together with the unit element) form a subgroup.

Solution. We shall use induction on k. for k = 0 the statement is obvious.
Consider left action of group on itself. Each element of a group, when you multiply group elements by it, defines a permutation of group elements. Any element of odd order defines only odd cycles, so it defines an even permutation. An element of order 2^k defines 2m+1 cycles of even order (2^k), and that is odd permutation.
Consider a subgroup of those group elements that correspond to even permutations. Those elements contain all elements of odd order, but not all elements in the group.
It is a subgroup of order 2, so we reduced our problem to a problem on a smaller group, which follows from induction assumption.

Targil 8: polytops.

1. Compute: an angle between faces of a regular tetrahedron, an angle between the adjacent faces of a regular octahedron, and an angle between the two different long diagonals (connecting pairs of opposite vertices) in a cube.

2. (a) Tetrahedron A (not necessary regular) contains strictly tetrahedron B.
Can we claim that the sum of lengths of all edges of A is bigger then the sum of lengths of all edges of B?
(b) Can we prove a similar statement if we would take rectangular parallelepipeds (= cuboids = boxes) instead of tetrahedron?
(c) Can we prove a similar statement if we would take sums of areas of faces instead of sums of lengths of edges both in (a) and (b) ?

3. Is it possible to inscribe a regular octahedron in a cube, so that all octahedron's vertices will be inside cube's edges?

4. Given a unit cube, a line, and a plane orthogonal to the line. Prove that the length of the projection of the cube on the line is equal to the area of the projection of the cube on the plane (of course, line is not necessarily parallel to one of the cubes edges).

5. A dodecahedron and an icosahedron (both regular) have a common circumsphere (means they are inscribed in the same sphere). Prove that they have a common insphere (means there is a sphere tangent to all the faces of both polytops).

6. We are given 100 vectors $\{(x_k , y_k , z_k)\}$ such that $-1 \le x_k , y_k , z_k \le 1$ for all k.
A sum of all this vectors is considered, while we are allowed to change signs in certain vectors. We change the signs in such a way, that the result of this sum will be as short as possible.
(a) Show that we can choose the signs so that the length of the sum will be ≤ 3.
(b) Show that the previous statement would be wrong if we would take 2 instead of 3.
***(c) Find the minimal value, for which (a) will hold.

Targil 8: polytops.

1. Compute: an angle between faces of a regular tetrahedron, an angle between the adjacent faces of a regular octahedron, and an angle between the two different long diagonals (connecting pairs of opposite vertices) in a cube.

Solution. Consider octahedron, whose vertices are $(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$. Normal vectors to the faces are $(\pm1, \pm1, \pm1)$, and by coincidence, those are long diagonals of a unit cube, whose sides are parallel to the axes. So it is not hard to find the cosine of angle between two of those by the means of scalar product:

$$\cos(\alpha) = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix} \right\rangle \Bigg/ \left(\left\| \begin{pmatrix}1\\1\\1\end{pmatrix} \right\| \cdot \left\| \begin{pmatrix}-1\\1\\1\end{pmatrix} \right\| \right) = \frac{1}{3}.$$

So, angle between diagonals of cube and between normals of octahedron's adjacent edges are $\arccos(1/3)$, hence the angle between adjacent edges of octahedron is $180°-\arccos(1/3)$.

Now consider tetrahedron, and consider another tetrahedron which is symmetric to it with respect to the center. It is easy to see that the intersection between two tetrahedrons is octahedron. So, each of the two original tetrahedrons consists of an octahedron and 4 smaller tetrahedrons. Hence, an angle between the faces of octahedron and an angle between the faces of tetrahedron form 180° together, hence the angle for tetrahedron is $\arccos(1/3)$.

2. (a) Tetrahedron A (not necessary regular) contains strictly tetrahedron B.
Can we claim that the sum of lengths of all edges of A is bigger then the sum of lengths of all edges of B?
(b) Can we prove a similar statement if we would take rectangular parallelepipeds (= cuboids = boxes) instead of tetrahedron?
(c) Can we prove a similar statement if we would take sums of areas of faces instead of sums of lengths of edges both in (a) and (b) ?

Solution. (a) No. Let A be a long thin sharp spike, 3 vertices form a small triangle, and another is far away from those three, at distance d.
Sum of edges in A is approximately 3d.
Take inside A 4 points (not in one plane) – 2 close to the base, and 2 close to the tip of the spike. They form a tetrahedron, some of edges is approximately 4d.
(b) First solution. Let x be a unit vector, $x(A)$ length of projection of the box A to the line of vector x. There are three quadruples of edges in a box, choose a representative of each quadruple: intervals u, v, w. Denote $x(u)$, $x(v)$, $x(w)$ lengths of the projections of those intervals to the line of vector x. It is easy to see that $x(A) = x(u) + x(v) + x(w)$.
If we substitute all possible values of x and integrate over the unit sphere, we get an equality $mean(x(A)) = mean(x(u)) + mean(x(v)) + mean(x(w)) = C\cdot(|u| + |v| + |w|)$,
Here C is a constant, which doesn't depend on anything.
So, if a box B is inside the box A, then $x(B)< x(A)$ for each x, hence sum of edges of B is less than sum of edges of A. QED.

Second solution. Consider the locus of points, whose distance for given box is not bigger than R. Compute the volume of this form.
This form consists of the original box, 6 boxes of height R adjusted to its faces, 12 quarter-tubes of radius R and lengths = edges, and 8 equivalent parts which can be glued together to make a ball of radius R. So the volume is a polynomial in R:

$$\tfrac{4}{3}\pi \cdot R^3 + \pi(a+b+c) \cdot R^2 + S \cdot R + V$$

Here V is a volume of the original box, S area of its surface, and a,b,c its edges.
If the first box contains the second one, then its enhancement by R contains the enhancement of the second box, so we would have an inequality.

$$\tfrac{4}{3}\pi \cdot R^3 + \pi(a_1+b_1+c_1) \cdot R^2 + S_1 \cdot R + V_1 > \tfrac{4}{3}\pi \cdot R^3 + \pi(a_2+b_2+c_2) \cdot R^2 + S_2 \cdot R + V_2$$

Now cancel out the term of R^3, divide by R^2 and see what happens when R tends to infinity.

(c) We can prove it not only for box or tetrahedron, but also for arbitrary convex polytops. Project the polytop onto a plane, passing through zero. Any point of projection is covered by twice, from both sides of the polytop. So, twice the area of projection is sum of projections of all faces of the polytop.
Now, take all possible planes passing through zero, and consider all projections, of the polytop and of the faces. It can be considered as a function of a unit normal vector, to the plane, since a unit vector defines orthogonal plane. When we shall integrate over unit sphere, each face will contribute a quantity, proportional to its area, with the same coefficient.
Now consider this integral for two different polytops, the second inside the first. It is obvious that for the first will be bigger, but as we saw, those integrals are proportional to surface areas.

3. Is it possible to inscribe a regular octahedron in a cube, so that all octahedron's vertices will be inside cube's edges?

First solution. Let the cube be a unit cube. Let X, Y be two opposite vertices on the cube. Choose each of 6 edges, adjacent to those vertices points on distance d from X or from Y respectively. These 6 points form a non-regular octahedron, some of its edges (between two points close to X or 2 points close to y) are of length a, and others are of length b. If we take d close to zero, we see that $a < b$, if we take d close to 1, then $a > b$, so by continuity there is a value of d the octahedron will become regular.

Second solution. Consider 6 points: $\pm(-\tfrac{1}{2}, 1, 1)$, $\pm(1, -\tfrac{1}{2}, 1)$, $\pm(1, 1, -\tfrac{1}{2})$
The vectors $(-\tfrac{1}{2}, 1, 1)$, $(1, -\tfrac{1}{2}, 1)$, $(1, 1, -\tfrac{1}{2})$ form an orthogonal frame, since the scalar products are zeroes, they are of the same norm, so the 6 points form a regular octahedron. All 6 points are on the edges of the cube $[-1, 1]\times[-1, 1]\times[-1, 1]$.

4. Given a unit cube, a line, and a plane orthogonal to the line. Prove that the length of the projection of the cube on the line is equal to the area of the projection of the cube on the plane (of course, line is not necessarily parallel to one of the cubes edges).

Solution. A projection of a cube is a hexagon ABCDEF, such that each 2 opposite sides are parallel and equal (of course, it can be rectangular, but this is a degenerate case which can be considered as a limit of general case). Consider triangle ACE.
Let ABCK be parallelogram, then KCDE and KAFE are also parallelograms, these three parallelograms are halved by their diagonals: AC, CE, AE respectively, hence ACE is half of the area of ABCDEF. It is easy to see (draw the picture Yourself), that A, C, and E are projections of vertices K, L, M of the cube, which have a common neighbor vertex P on the cube, and the projection on the cube. Diagonal PQ of the cube is orthogonal to the plane KLM, hence the angle between the projection plane and KLM plane is the same as the angle between the line orthogonal to projection plane and the PQ line. It is obvious from the picture that P and Q are highest and lowest points of the cube w. r. t. the plane, so the projection of the cube to the line is the same as the projection of PQ to the line. Anyway, projection of the cube to the plane is twice the area of KLM, and projection of the cube to the line is the projection of PQ to the line, and both are proportional to the cosine of the same angle since the angle between the planes equals the angle between their normals (orthogonal lines).
So, the two things (projection length and orthogonal projection area) are proportional with constant coefficient, to prove they are the same we should either compute this coefficient or to check it holds in some specific, (for example degenerate) case.
We shall check it in the case, when the plane is parallel to a face of the cube, hence the line is parallel to the edge of the cube, then both length and area = 1.

5. A dodecahedron and an icosahedron (both regular) have a common circumsphere (means they are inscribed in the same sphere). Prove that they have a common insphere (means there is a sphere tangent to all the faces of both polytops).

Solution. It is well known (and obvious) that dodecahedron and icosahedron are dual, that is, if you take a convex hull of the centers of faces of dodecahedron you get icosahedron, and vice versa.
Let O be the center of dodecahedron, C center of one of its faces, V one of the vertices of that face. It is obvious that OCV is a right-angled triangle, C is the right angle.
OC is a radius of dodecahedrons insphere, OV of its circumsphere, so the ratio between two radii is the cosine of COV. Obviously, this ratio is the same for each dodecahedron.
Now, consider the icosahedron whose vertices are centers of that dodecahdron's faces.
Then on the line OV we shall have a center of certain face of this icosahedron, D.
For the same reason, the ratio between the radii of icosahedron's insphere and circumsphere is the cosine of COD, but COD = COV.
Hence the ratio between the inradius and the circumradius is the same for dodecahedron and icosahedron, and so if the two have a common circumsphere they inspheres have the same center and the same radius, so they are the same.

6. We are given 100 vectors $\{(x_k , y_k , z_k)\}$ such that $-1 \le x_k , y_k , z_k \le 1$ for all k.
A sum of all this vectors is considered, while we are allowed to change signs in certain vectors. We change the signs in such a way, that the result of this sum will be as short as possible.
(a) Show that we can choose the signs so that the length of the sum will be ≤ 3.

(b) Show that the previous statement would be wrong if we would take 2 instead of 3.
***(c) Find the minimal value, for which (a) will hold.

Solution. (a) We can assume without loss of generality that for each vector (x,y,z), the last coordinate z is nonnegative. Coordinate planes divide the 2×2×2 cube into eight 1×1×1 sub-cubes, out of those only the upper 4 contain vectors. If we have two vectors in the same sub-cube, replace them by their difference – it will still be inside the 2×2×2 cube, and the number of vectors will be reduced by one, making the problem less frightening. This reduction can be performed repeatedly, until there are no more than one vector in each sub-cube, so no more than 4 vectors. Suppose there are really 4 vectors. Denote these 4 vectors A, B, C, D, where

A	B
D	C

$$A_y, B_y > 0 > C_y, D_y\ ,$$
$$C_x, B_x > 0 > A_x, D_x\ .$$

If $B_x - A_x \le 1$, then B – A is still a vector inside the 2×2×2 cube so we can reduce the number of vectors even more, same if $C_x - D_x \le 1$, or $B_y - A_y \le 1$, or $A_y - D_x \le 1$. Also, if $A_z + C_z \le 1$ we could replace A and C by A + C, similarly if $B_z + D_z \le 1$, we could replace B and D by B + D. So, if none of these happens and we can't replace 4 vectors by 3 vectors, we have

$$1 < B_x - A_x \le 2, \quad 1 < C_x - D_x \le 2,$$
$$1 < B_y - A_y \le 2, \quad 1 < A_y - D_x \le 2,$$
$$1 < A_z + C_z \le 2, \quad 1 < B_z + D_z \le 2.$$

Consider then A – B + C – D, all 3 of its coordinates are between -1 and 1, so its norm is less then $\sqrt{3}$.
So, it remains to consider the case when we can reduce the problem to only 3 vectors. Norm of each $\le \sqrt{3}$. Before summing firs two, chose signs so that the angle between them will be not acute, so the length of the sum, by Pythagoras, isn't bigger than $\sqrt{6}$. Before adding the third, choose its sign so that the angle with the previous result won't be acute, the norm of the sum, again by Pythagoras, won't exceed 3. In order to make equalities of those inequalities, all vectors should have norm $\sqrt{3}$, so they should be vertices of the 2×2×2 cube, and they should be orthogonal to each other. Each of those conditions is achievable, but not both together at the same time, so it never comes to 3.

(b) Consider just two vectors: (1,1,1) and (1, –1, 0), and let all other vectors be 0. The two are orthogonal, so the choice of signs doesn't matter, the length of the sum will be $\sqrt{5}$ anyway.

(c) I don't know the solution, though I have thought a lot about it, if You solve it please tell me. The best lower bound (example) I know is described in problem 3, it was found by Alexey Gladkich (and that is how problem 3 was invented) and it gives $\sqrt{6.75}$.
I suspect it is the precise answer.
The solution, as we have shown above, should consist of the best example of 3 vectors, and a prove that this example is the best.

Targil 9 (following Alexey's story, about catastrophes and linear variations)

1. Given a polynomial of degree 3 with real coefficients: $x^3 + ax + b$.
Prove that it has 3 distinct real roots iff $(a/3)^3+(b/2)^2$ is negative;
and that it has only one real root of multiplicity one iff $(a/3)^3+(b/2)^2$ is positive.

2. In Moscow they have 9 sky-scrapers (assume Moscow is a plane, and sky-scrapers are points). A tourist, that stands at a certain point and looks around (counterclockwise), will see them in a certain cyclical order, at least if he doesn't stand on a line connecting two sky-scrapers. There might be 8! different cyclic orders.
(a) is it possible that each order will appear at some point?
(b) for which minimal number of buildings, will it be possible to see the buildings in every cyclic order?

3. A convex body C is contained by the unit cube. Projection of C to every face of the cube cover it completely. What is the minimal possible volume of C?

4. A rectangle is divided into disjoint union of the finite number of squares. Prove, that the aspect ratio (width / height) of the rectangle is a rational.

5. A square matrix is called ***bi-stochastic*** if all its numbers are nonnegative and sum of numbers in each column and in each row is one.
Prove that any bi-stochastic matrix is a linear combination of permutation matrix, with positive coefficients.

Targil 9 (following Alexey's story, about catastrophes and linear variations)

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Prove that it has 3 distinct real roots iff $(a/3)^3+(b/2)^2$ is negative;
and that it has only one real root of multiplicity one iff $(a/3)^3+(b/2)^2$ is positive.

First solution. The sum of roots is 0, the sum of product of pairs is a, the product of al 3 is $-b$. Consider different cases:
First case: we have 3 real roots, m , n , and $-m-n$ (cause the sum is equal 0), and WLOG we can assume that m, n are both negative or both positive (if we have a zero root, the statement is obvious).
Then $a = -m(m+n) - n(m+n) + mn = -(m^2 + mn + n^2)$
$b = mn(m+n)$
Hence it remains to verify that

$$-\left(\frac{m^2+mn+n^2}{3}\right)^3+\left(\frac{mn(m+n)}{2}\right)^2<0$$

$$\left(\frac{mn(m+n)}{2}\right)^2<\left(\frac{m^2+mn+n^2}{3}\right)^3$$

Divide by m^3n^3 and denote $x = m/n$, and it is positive.

$$\left(\frac{x^{1/2}+x^{-1/2}}{2}\right)^2<\left(\frac{x+1+x^{-1}}{3}\right)^3$$

Denote $y = x^{1/2} + x^{-1/2}$. Then $y > 2$.

Then $x+1+x^{-1} = y^2 - 1$ and our inequality becomes:

$$\frac{y^2}{4}<\frac{\left(y^2-1\right)^3}{27}$$

Denote $z = y^2 - 1$, then $z > 3$, and it remains to prove:

$$\frac{27}{4}(z+1)<z^3$$

For $z = 3$ we get equality, but the right hand side climbs faster, because its derivative is greater: $\frac{27}{4} < 3z^2$, hence the inequality holds for $z > 3$.

Second case. We have a double root, so the roots are k, k, and $-2k$.
Then it is easy to compute $(a/3)^3+(b/2)^2 = 0$.

Third case. We have two complex conjugated roots $m+in$, $m-in$, and a real root $-2m$.
Then $b = -2m(m^2+n^2)$ and $a = m^2+n^2 - 4m^2 = n^2 - 3m^2$. So, it remains to prove that in this case

$$\left(\frac{n^2-3m^2}{3}\right)^3+\left(\frac{-2m\left(m^2+n^2\right)}{2}\right)^2>0$$

$$\left(m\left(m^2+n^2\right)\right)^2>\left(m^2-\frac{n^2}{3}\right)^3$$

$$m^6+2m^4n^2+m^2n^4>m^6-m^4n^2+\frac{m^2n^4}{3}-\frac{n^6}{27}$$

$$3m^4n^2+\frac{2}{3}m^2n^4+\frac{n^6}{27}>0$$

And that is obvious.

That was a straightforward and messy solution.
Let us see a nicer, catastrophic one.

Solution 2. Consider the discriminant of the polynomial – the resultant of it with its derivative. (If you forgot or don't know about resultants, read the solution of problem 2 from targil 1)

$$\left|\begin{pmatrix}1&0&a&b&0\\0&1&0&a&b\\3&0&a&0&0\\0&3&0&a&0\\0&0&3&0&a\end{pmatrix}\right|=\left|\begin{pmatrix}1&0&a&b&0\\0&1&0&a&b\\0&0&-2a&-3b&0\\0&0&0&-2a&-3b\\0&0&3&0&a\end{pmatrix}\right|=\left|\begin{pmatrix}-2a&-3b&0\\0&-2a&-3b\\b&0&a\end{pmatrix}\right|=4a^3+27b^3$$

Hence $(a/3)^3+(b/2)^2=0$ iff there is a double root. The line $a=3\sqrt[3]{(b/2)^2}$ divides the a,b plane into two parts, one is $a>3\sqrt[3]{(b/2)^2}$ and another is $a<3\sqrt[3]{(b/2)^2}$.

If we move a couple of complex conjugated roots and the real root along the plane, without creating double root, we can continuously transform the polynomial with just one real root into a polynomial with just one real root, and the sign of the discriminant won't change since the discriminant won't go through 0.

Similarly, if we move all three real roots, keeping their order, we shall keep the sign of discriminant. Hence, each case, of 1 real root and of 3 real roots, correspond to a connected component of the plane with the zeroes of discriminant cut out.

It remains to verify, which of the 2 connected components (or alternatively, which sign of discriminant) corresponds to 1 real root and which to 3 real roots. To check it, substitute an example. For instance a polynomial with roots 1, –1, 0, which is x^3-x.

Or a polynomial with 1 real root 0, and two complex i, $-i$, whichi is x^3+x.

2. In Moscow they have 9 sky-scrapers (assume Moscow is a plane, and sky-scrapers are points). A tourist, that stands at a certain point and looks around (counterclockwise), will

see them in a certain cyclical order, at least if he doesn't stand on a line connecting two sky-scrapers. There might be 8! different cyclic orders.
(a) is it possible that each order will appear at some point?
(b) for which minimal number of buildings, will it be possible to see the buildings in every cyclic order?

Solution. (a) Let us count the number of areas we get, when we draw all the lines going through the couples of all the skyscraper points. This number will give us an upper bound for the possible number of observable cyclic orderings of the buildings, since all the points in the same part give you the same ordering, since no changes in the order of the buildings can happen if you don't cross any line.

For N skyscrapers we get $\binom{N}{2}=\frac{N(N-1)}{2}$ couples/lines. Each line is intersected by many lines at the 2 points that define it, and it may also be intersected at all lines defined by other N–2 points, if there are no trapezoids (no parallel lines). Also, we can complete (compactify) the plane by the infinite point, which is contained in all lines, and then we shall have a map (meaning countries and borders) on the sphere, and we shall be able two use Euler's formula. So, on each line we shall have $3+\binom{N-2}{2}$ points and the same number of edges (at most, if there are no trapezoids or coincidences), so total number of edges is at most $E=\binom{N}{2}\cdot\left(3+\binom{N-2}{2}\right)$.

Each trapezoid or coincidence reduces the number of faces (it is easy to see, that if you stir the points a little bit to avoid coincidence, you have all faces you had previously and usually more, so if we count the number of faces when there are no coincidences, we shall get the possible maximum.

The total number of vertices is: N original points, 1 at infinity, and number of pairs of couples (since each two lines give an intersection $\frac{N(N-1)(N-2)(N-3)}{8}$.

Since F – E + V = 2, we get F = 2 + E – V, hence maximal F is

$$2+\binom{N}{2}\cdot\left(3+\binom{N-2}{2}\right)-\left(N+1+\frac{N(N-1)(N-2)(N-3)}{8}\right)=$$

$$=1-N+\binom{N}{2}\left(3+\frac{(N-2)(N-3)}{2}\left(1-\frac{1}{2}\right)\right)=$$

$$=1-N+3\frac{N(N-1)}{2}+\frac{N(N-1)(N-2)(N-3)}{8}$$

Of course, it would be easier to substitute the numbers from the beginning, but we want to prove a stronger claim. Assume we have N=6 buildings in Moscow, then the plane is divided into only

$1-6+3\frac{6\cdot 5}{2}+\frac{6\cdot 5\cdot 4\cdot 3}{8}=1-6+45+45=85$ parts, and there are 5! = 120 possible cyclic orderings for 6 buildings, which is considerably more, so even if we choose a subset of 6 buildings, we won't be able to observe them in arbitrary order.

Now let us substitute N=5

$$1-5+3\frac{5\cdot 4}{2}+\frac{5\cdot 4\cdot 3\cdot 2}{8}=1-5+30+15=41.$$

It is more then 5! = 120, so we cannot extend the statement by this method for 5 buildings.

Of course, it cannot be considered a proof that for 5 we have a configuration of buildings either, since different parts of the plane, even those separated by lines might correspond to the same cyclic order.

(b) We have found some upper bound, lets start with lower. If we have 3 points in "generic situation" they form a triangle, and there are two possible cyclic orders, one is observed from each internal point, and another from almost every external point (w. r. t. that triangle), so 3 is still possible.

4 points in generic position can form 2 kinds of configuration – either a convex quadrilateral or a triangle and a point inside. In both cases, it is not too hard to check you have all orderings.

As for 5 points, I had a really nice proof that some orderings are not achieved when I gave this problem, but as I found out when I started typing it down, it was wrong. Right now I don't know neither the solution, nor the answer.

3. A convex body C is contained by the unit cube. Projection of C to every face of the cube covers it completely. What is the minimal possible volume of C?

Solution. Every edge meets this convex body, otherwise projection to the orthogonal face wouldn't cover one of the corners. Conversely, each edge meets the body, than each projection covers 4 corners, then, since it is convex, it covers the face. So we can take the condition of taking a point from each of 12 edges, and forget the original condition. After we choose 12 points, one on each edge, we should take their convex hull, and the volume of that body should be minimized y a smart choice of points on the edges. So, we have reduced the problem with infinite number of parameters to a problem with 12 parameters. Instead of computing/minimizing the volume of the convex hull, it is easier compute/maximize the volume of the complement. It consists of the "corners" – right-angled tetrahedrons, adjacent to each vertex of the cube. 4 vertices of such a tetrahedron are the vertex of the cube, and 3 points we have chosen on the edges, attached to that cube vertex.

Let us try moving one of those 12 points on edges, and look how it influences the volume of the complement. When we try to move it along the edge, only 2 corner tetrahedrons change, and their both volumes depend linearly on the position of that point. Hence, the maximum (and the minimum) will occur only when the point is on one of extreme positions. So, we may assume that in the best situation all points we choose on each of 12

edges are vertexes. Hence, the volume of each corner tetrahedron is either 0 (if these point was chosen as one of those 12 points) or 1/6 (if all neighbor vertices were chosen). Divide all vertices into 4 subsets of 2, members of the same subgroup are connected by vertical edges. Only one of the members of each subgroup can invest 1/6 to the volume of the complement, the other will give 0. So we can have at most 4/6 = 2/3 of volume in the complement, so the volume of the original body is at least 1/3.
From the above, it is easy to guess the construction for the 1/3.
Color the vertices of the cube in two colors, black and white, so that neighbors will be of different colors. Now take the convex hull of the white vertices.

4. A rectangle is divided into disjoint union of the finite number of squares. Prove, that the aspect ratio (width / height) of the rectangle is a rational.

Solution. Consider the certain sub-division of the certain rectangle into squares. Denote x_i the sides of the squares. The subdivision defines certain linear equations over the numbers x_i, such as: if a certain interval in the picture is presented as sum of sides of different subsets of squares, then those sums should be equal. Two more equations clame that sum of all sides of squares touching the lower rectangle's border is w, and sum of two rectangle's squares touching the right rectangle's border is h.
All coefficients in that system of equations are rational, except for w and h. After applying scaling we shall assume $h = 1$, and then w will be the aspect ratio.
The configuration of squares solves the problem for given w if and only if the system of linear equations has a solution in nonnegative real numbers. When we apply Gauss method to solve the system of all equations except the one containing w, we shall get either a single rational solutions (because coefficients are rational), or an infinite family of solutions, which depends linearly (with rational coefficients) upon a finite number of parameters. A solution we get from Gauss methods is limited by several inequalities, corresponding to non-negativity of all x_i. So, if we have an infinite family of solution (and that is the only way to get irrational w) then w moves in certain limits, between two rational limiting values, a and b.
In such a case, we might write $x_i = k_i t + m_i$, and for every t we shall get the same configuration of squares, but of different sizes. Say at $t = 0$ we shall get rectangle of width a, at $t = 1$ we shall get a rectangle of width b, and at some intermediate value we shall get width w. Sides of all squares change as a linear function of t. So their areas are quadratic functions, with corns up (all smiling), unless they are unchanged. But height is constantly 1, and w changes linearly, hence the area changes linearly. That's a contradiction.
Hence, there are no configurations which give infinite families of solutions, only those, that give a single rational solution.

5. A square matrix is called ***bi-stochastic*** if all its numbers are nonnegative and sum of numbers in each column and in each row is one.
Prove that any bi-stochastic matrix is a linear combination of permutation matrix, with positive coefficients.

Solution. When talking about elements of bi-stochastic numbers, all numbers in the open interval (0, 1) will be called ***fractional***, to distinguish them from 0 and 1 which will be called ***extreme***.
We shall prove the statement by induction over the number of fractional matrix entries. If we have 0 fractional elements in the matrix, all elements are 0s or 1s, it is easy to see that the matrix is a permutation matrix, and that is the base of induction.

Consider a matrix with K fractional elements. In the same column with any fractional element we obviously have a fractional element. In the same row with any fractional element we also have a fractional element.
Start with arbitrary fractional element of the matrix and build a sequence of fractional elements, passing with each odd time to a different fractional element of the same row and in each even time to a different fractional element in the same column. At a certain moment, You will have to repeat an element which occurred before in that sequence.
We can assume that the repeated element will be repeated after even number of moves (if for example, we want to make a vertical move from element Q to A, then either A was followed by a horizontal move, or A was followed by a vertical move to B, but then we can move directly from F to B). Hence, from this sequence, we can choose a cycle of fractional, each even element connected by vertical move to the next and by horizontal move to the previous.
If we add the same number x to all odd numbers in that cycle and subtract it from all even numbers of that cycle, then the sum in each column and the sum in each row is preserved. Choose maximal possible x and minimal possible x so that all elements of the matrix will be still in [0,1]. We shall get two matrices, C and D, both bi-stochastic, which both have less fractional elements than the original matrix. Hence, by induction, C and D are positive linear combinations of permutations.
It is obvious that the original matrix is positive linear combination of C and D.

Targil 10.
Some linear algebgra, from former IMC mostly.

1. Let $n \geq 2$ be an integer. What is the minimal and the maximal possible rank of a $n \times n$ matrix whose n^2 entries are precisely 1, 2, 3, … , n^2 ?

2. A polynomial $P(x_1, x_2, \ldots, x_k)$ is called good if there exist 2×2 real matrices A_1 , A_2, … , A_k such that $P(x_1, x_2, \ldots, x_k) = \det\left(\sum_{i=1}^{n} x_k A_k\right)$.
Find all values of k for which all homogeneous polynomials of degree 2 of k variables are good.

3. Let A be a real 4×2 matrix and B be a real 2×4 matrix such that
$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$. Find BA.

4. Let AB be real $n\times n$ matrices such that $AB + A + B = 0$.
Prove that they commute, i. e. $AB = BA$.

5. $A = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_1 & a_0 & a_1 & \ldots & a_{n-1} \\ a_2 & a_1 & a_0 & \ldots & a_{n-2} \\ . & . & . & \ldots & . \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \end{pmatrix}$

Compute $\det A$ if

a) $a_k = a_0 + d\cdot k$
b) $a_k = a_0 \cdot q^k$

Israeli Team for SEEMOUS
Second Stage Solutions.

1. A ***graph*** is, by definition, a collection of ***vertices*** and a collection of ***edges*** that connect pairs of vertices. Two vertices are called ***adjacent***, if they share an edge.
Given a graph, consider the function $c(n)$ – the number of ways to color each vertex with one of n given colors, so that no two adjacent vertices will have the same color. Show, that $c(n)$ is a polynomial of n.

First solution. Induction over number of vertices + number of edges.
The only graph of 1 vertex gives $c(n) = n$.
Of course, if graph is disconnected function $c(n)$ is a product of functions, corresponding to his connected components, and product of polynomials is a polynomial.
Take two adjacent vertices A, B in a graph. Let us erase the edge AB. Number of ways to color the new graph, $c_1(n)$ is a polynomial by induction (same vertices, less edges). Of those, there are $c(n)$ ways to color it so that A and B will be of different color, and $c_2(n)$ ways to color it so that so that A and B will having the same color. If we shall glue vertices A and B, the new graph will have less edges and less vertices than the original graph, and it can be colored in $c_2(n)$ ways. Hence $c(n) = c_1(n) - c_2(n)$, so it is a difference of two polynomials, hence it is itself polynomial.

Second solution. A way to split the vertices of given graph into certain equivalence classes will be called ***configuration***. Configuration is called good if no to vertices of the same class are adjacent. There is only finite number of configuration.
Each coloring corresponds to a specific configuration: vertices of the same color are declared equivalent. Let us count, how many colorings correspond to the same configuration. Take a configuration which has M classes.
First class can be colored in one of n colors, second in one of $n-1$ colors, and so on, hence if $M \geq n$ it corresponds to $n(n-1)(n-2)\ldots(n-M+1)$
If $M < n$ then the product we wrote, as well as the number of colorings, is 0.
So, number of colorings corresponding to certain configuration is a polynomial (which we wrote explicitly) and since we have finite number of configurations, the total number of colorings is a sum of finite number of polynomials, which is a polynomial.

2. A disc of radius $^1\!/_N$ is rolling inside the circular box of radius 1, where $N > 2$. (The friction between the edge of the disc and the wall of the box is very high so the disc doesn't slip with respect to the box at the point of tangency). A red point on the boundary of a small circle goes along a star-shaped closed trajectory.
Compute the area, bounded by this trajectory (as a function of N).

First solution. Let us start by building a parametrical equation of the star.
The center of the disc goes in circles of radius $1-\frac{1}{N}$ so it can be described as $v=\left(\left(1-\frac{1}{N}\right)\cos t,\left(1-\frac{1}{N}\right)\sin t\right)$. The vector which goes from the center of the disc to the red point goes around a circle of radius $\frac{1}{N}$ in the opposite direction, so it can be described as $u=\left(\frac{1}{N}\cos s,-\frac{1}{N}\sin s\right)$. Both parameters depend linearly on the length of the arc that we cover, t.
While the center goes around one time, the red point meets the boundary N times. This means the small discs rotates around itself $N-1$ times, hence $u=\left(\frac{1}{N}\cos\left((N-1)t\right),-\frac{1}{N}\sin\left((N-1)t\right)\right)$.
The point on the star can be described as $w = u + v$, which is also a vector function of t. A simple way to check we wrote it correctly – differentiating vectors u, v shows that their velocities are equal in their absolute value and that they cancel each other when the red point is near at the boundary (and then its velocity should be 0, because of the friction).
Of course, since $\dot{u}$ looks always directly clockwise and $\dot{v}$ is of the same absolute value the vector will always go clockwise so the star won't have self-intersections.
Integrating $-ydx$ around the star should, as usual, give the area inside.
Minus sign is because the trajectory, the way we have parameterized it, goes clockwise, so the upper boundary must be consider with plus and the lower with minus. So we get the following integral:

$$\int_0^{2\pi}-\left(\left(1-\frac{1}{N}\right)\sin t-\frac{1}{N}\sin\left((N-1)t\right)\right)\frac{d}{dt}\left(\left(1-\frac{1}{N}\right)\cos t+\frac{1}{N}\cos\left((N-1)t\right)\right)dt=$$

$$=\frac{1}{N^2}\int_0^{2\pi}\left((N-1)\sin t-\sin\left((N-1)t\right)\right)(N-1)\left(\sin t+\sin\left((N-1)t\right)\right)dt=$$

$$=\frac{N-1}{N^2}\int_0^{2\pi}(N-1)\sin^2 t-\sin^2\left((N-1)t\right)+(N-2)\sin t\sin\left((N-1)t\right)dt$$

To finish this, it is useful to know the following exercises:

Exercise 1. $\int_0^{2\pi}\sin^2 t dt=\int_0^{2\pi}\sin^2\left((N-1)t\right)dt=\pi$

(Hint: $\sin^2+\cos^2=1$)

Exercise 2. $\int_0^{2\pi}\sin t\sin\left((N-1)t\right)dt=0$

(Hint: $2\sin a\cdot\sin b=\cos(a-b)-\cos(a+b)$)

So, the answer is $\frac{(N-1)(N-2)}{N^2}\pi$.

Second solution. Like before, we describe the position of red point as the sum of two vectors $w = u + v$ where u goes clockwise in a circle of radius $1-\frac{1}{N}$ one time and v goes counter-clockwise in a circle of radius $\frac{1}{N}$, $N-1$ times, but we don't write the coordinates explicitly.

For any to vectors $k=(k_x,k_y), m=(m_x,m_y)$ denote the oriented area of the parallelogram they form $k=(k_x,k_y), m=(m_x,m_y)$. So, in time dt the vector w sweeps area $\frac{dw\times w}{2}$ (since its clockwise) which gives total area

$$\int_0^{2\pi}dw\times w=\int_0^{2\pi}\frac{1}{2}(du+dv)\times(u+v)=\frac{1}{2}\int_0^{2\pi}du\times u+dv\times v+du\times v+dv\times u$$

The integrals $\int_0^{2\pi}du\times v$, $\int_0^{2\pi}dv\times u$ are 0, since the angle between du and v, as well as dv and u rotates uniformly around 0 and makes several full circles.

$\frac{1}{2}\int_0^{2\pi}du\times u=\pi\left(1-\frac{1}{N}\right)^2$ since u sweeps one circle of radius $\left(1-\frac{1}{N}\right)$.

$\frac{1}{2}\int_0^{2\pi}dv\times v=-(N-1)\pi\left(\frac{1}{N}\right)^2$ since v sweeps $N-1$ circles of radius in the opposite direction. So the total integral is:

$$\frac{1}{2}\int_0^{2\pi} du\times u + dv\times v = \pi\left(1-\frac{1}{N}\right)^2 - (N-1)\pi\left(\frac{1}{N}\right)^2 = \pi\left(\left(\frac{N-1}{N}\right)^2 - (N-1)\left(\frac{1}{N}\right)^2\right) =$$
$$= \pi\frac{N-1}{N^2}(N-1-1) = \pi\frac{(N-1)(N-2)}{N^2}$$

3. A natural number k is considered ***good***, if for each N the number $1^k+2^k+\ldots+N^k$ is divisible by $1+2+\ldots+N$.
Describe the set of all good numbers.

Solution. If k is good, then it 1^k+2^k is divisible by 3. So $1+(-1)^k = 0 \pmod 3$ hence k can't be even.
Suppose now k is odd. $1^k+2^k+\ldots+N^k$ is divisible by $1+2+\ldots+N$ if and only if $2(1^k+2^k+\ldots+N^k)$ is divisible by $2(1^k+2^k+\ldots+N^k)=N(N+1)$.
N and $N+1$ are co-prime, so it is sufficient to verify separately that it is divisible by N and by $N+1$. It is enough to prove $2(1^k+2^k+\ldots+N^k)$ is divisible by $N+1$ for all N, then $2(1^k+2^k+\ldots+(N-1)^k)$ is divisible by N and $2(1^k+2^k+\ldots+N^k)$ also. We shall use "Gauss trick":
$2(1^k+2^k+\ldots+N^k) = 2((1^k+N^k) +(2^k+(N-1)^k) +\ldots+(N^k+1^k))$.
But this is definitely divisible by $N+1$ since a^k+b^k is always divisible by $a+b$ for odd k since $a^k+b^k =(a+b)(a^{k-1} - a^{k-2}b + a^{k-3}b^2 - \ldots + a^{k-1})$.

4. Let $\boldsymbol{A_1, A_2,\ldots,A_N}$ be nonzero matrices $M\times M$ (a matrix is called ***nonzero*** if at least one of its elements is nonzero). Prove that there exists a matrix B of the same size such that $\boldsymbol{BA_1BA_2B\ldots BA_NB}$ is a nonzero matrix.

Solution. The key is to consider the kernel and image spaces of matrices. We shall construct projection matrix B of rank 1, which satisfies the conditions. Projection matrix of rank 1 is defined by 2 linear subspaces: kernel of codimension 1 and image of dimension 1, which doesn't contain kernel. Each vector can be uniquely decomposed into sum of two vectors – one from the space of dimension 1 and second from the space of codimension 1. So, the projection can be described as taking the first vector of that decomposition.
For this product to be non-zero, all we need is that the image of B won't be sent into its kernel. So, we have to prove that we can choose a nonzero vector v (or the one-dimensional space) and a space W of codimension that neither A_i will send v into W.

To do this, we must achieve 2 things:
a) Find a vector v which don't belong to kernel of A_i for all i.
b) Find a hyperplane (containing 0) which doesn't contain $A_i v$ for all i.
So, it remains to prove 2 lemmas:
Lemma 1. There exists a vector which is not contained in all given linear subspaces, where number of subspaces is finite.
Lemma 2. There exists a hyperplane (containing 0), which doesn't intersect with a given finite set of points.

Since any subspace can be enlarged to hyperspace, lemma 1 is equivalent to its special case:
Lemma 3. There exists a vector which is not contained in all given hyperplanes, where number of hyperplanes is finite.

Lemma 2 is also follows from lemma 3, since if we replace a hyperplane $a_1x_1+a_2x_2+\ldots+a_nx_n = 0$ by a vector $(a_1, a_2, \ldots, a_n)$ and vice versa, the condition "a hyperplane contains the vector" turns into "a vector belongs to the hyperplane".
So, it is enough to prove lemma 3.

Remark. All this works only for infinite fields.

Proof of lemma 3. Apply the induction over dimension of the space.
The base of induction: space of dimension 1 can't be covered by finite number of points (field is infinite).
The step of induction: assume it is proven for spaces of dimension smaller than n. So, we have finite number of hyperplanes, and we try to prove they don't cover the space. There is infinite number of hyperplanes in the space, so we can choose a huperplane H which is different from all given hyperplanes. Intersection of H with other hyperplanes are sub-hyperplanes in H, so, by induction, they can't cover it.

5. An infinite sequence of real numbers $\{x_i\}$ will be called ***nice*** if $\sum x_i^2$ converges. Let $\{a_i\}$ be a sequence, such that for each nice sequence $\{x_i\}$ the series $\sum a_i x_i$ converges. Prove that the sequence $\{a_i\}$ is nice.

Solution. Assume $\{a_i\}$ isn't nice. So $\sum a_i^2$ diverges. We can cut the sequence $\{a_i\}$ into infinite number of segments, each of which is greater than 1. (That is done by induction, simply sum up the numbers from the end of segment number k until it exceeds 1, and that will be segment $k+1$.)
Let segment number k start at m_k and have n_k elements.

Then, by construction, $b_k = \sum\limits_{j=m_k-n_k+1}^{m_k} a_j^2 > 1$. We shall use a nice lemma:

Now, construct a sequence $x_j = \dfrac{a_j}{k \cdot b_k}$, for each j which belongs to segment number k. Then

$$\sum_{j=1}^{\infty} a_j x_j = \sum_{k=1}^{\infty} \sum_{j=m_k-n_k+1}^{m_k} a_j x_j = \sum_{k=1}^{\infty} \sum_{j=m_k-n_k+1}^{m_k} \frac{a_j^2}{k \cdot b_k} =$$

$$= \sum_{k=1}^{\infty} \frac{\sum\limits_{j=m_k-n_k+1}^{m_k} a_j^2}{k \cdot b_k} = \sum_{k=1}^{\infty} \frac{b_k}{k \cdot b_k} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

But sequence $\{x_i\}$ is good because

$$\sum_{j=1}^{\infty} x_j^2 = \sum_{k=1}^{\infty} \sum_{j=m_k-n_k+1}^{m_k} \frac{a_j^2}{k^2 \cdot b_k^2} = \sum_{k=1}^{\infty} \frac{b_k}{k^2 \cdot b_k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2 \cdot b_k} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

Targil 10.

Some linear algebgra, from former IMC mostly.

1. Let $n \geq 2$ be an integer. What is the minimal and the maximal possible rank of a $n \times n$ matrix whose n^2 entries are precisely 1, 2, 3, … , n^2 ?

Answer. Maximum n, minimum 2.

Solution. ***Minimum.*** Write all the numbers in their natural order.
Then all the lines are linear combinations of the first two. Because the difference of first two lines is (1 1 … 1) and by adding this line to the first line you can get all lines.
You can't achieve rank 1. If You can, lets permute rows and columns so that $a_{11}=1$, $a_{11} < a_{12} < \dots < a_{1n}$, $a_{11} < a_{21} < \dots < a_{n1}$.
Then, since every 2×2 minor is 1, we have $a_{km} = a_{k1}\, a_{1m}$.
From this follows, that the lower-right corner number is bigger than any other matrix entry by 2 at least.
Maximum. Take all entries on main diagonal to be odd, and all elements below it – even. Then det mod 2 is 1, so det is odd. So det isn't 0.
So rank is n.

2. A polynomial $P(x_1, x_2, \dots, x_k)$ is called good if there exist 2×2 real matrices A_1 , A_2, … , A_k such that $P(x_1, x_2, \dots, x_k) = \det\left(\sum_{i=1}^{n} x_k A_k\right)$.
Find all values of k for which all homogeneous polynomials of degree 2 of k variables are good.

Solution. By playing a bit with numbers, it is easy to construct an example for 2:

$$\left| x\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ -c & b \end{pmatrix} \right| = \left| \begin{pmatrix} x & y \\ -cy & ax+by \end{pmatrix} \right| = ax^2 + bxy + cy^2$$

It is always easy to reduce number of variables. So for 1 it works also.

Now the question is, which polynomial of degree 3 can we construct.
All possible polynomials are

$$\left|x\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + y\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} + z\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\right| =$$

$$= \left|\begin{pmatrix} a_1x+b_1y+c_1z & a_2x+b_2y+c_2z \\ a_3x+b_3y+c_3z & a_4x+b_4y+c_4z \end{pmatrix}\right| =$$

$$= (a_1x+b_1y+c_1z)(a_4x+b_4y+c_4z) - (a_2x+b_2y+c_2z)(a_3x+b_3y+c_3z)$$

But each homogenous polynomial of degree 2 in 3 variables can be seen as a bilinear form, in which you substitute twice the same vector, or

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The form $(a_1x+b_1y+c_1z)(a_4x+b_4y+c_4z)$ or $-(a_2x+b_2y+c_2z)(a_3x+b_3y+c_3z)$ corresponds to the matrix of rank 1. So, the question is, which forms can be presented as sum of two matrixes of rank 1, and those are forms that can be presented as matrixes of rank 2. Matrices of rank 2 always have nontrivial kernels, so their forms become 0 on non-zero vectors. That is why polynomial $x^2+y^2+z^2$ can't be represented by matrix of rank 2, since it is never 0 on real nonzero vector.

Remark. Notice, that quadratic form $x^2+y^2+z^2$ can be represented by many different matrices, precisely by any matrix of the form $\begin{pmatrix} 1 & -a & -c \\ a & 1 & -b \\ c & b & 1 \end{pmatrix}$.

3. Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$. Find BA.

Solution. Matrix A can be written as $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and matrix B as (B_1 B_2) where A_1, A_2, B_1, B_2 are 2×2 blocks.

$AB = \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{pmatrix}$, hence A1 is inverse to B1, and A_2 is inverse to B_2.

Hence $BA = B_1A_1 + B_2A_2 = \mathbf{1} + \mathbf{1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

4. Let AB be real n×n matrices such that $AB + A + B = 0$.
Prove that they commute, i. e. $AB = BA$.

Solution. Add unit matrix to both sides of the given identity. You get

$$(\mathbf{1}+A)(\mathbf{1}+B) = \mathbf{1}.$$

That means, matrices $\mathbf{1}+A$ and $\mathbf{1}+B$ are inverse to each other, hence

$$(\mathbf{1}+B)(\mathbf{1}+A) = \mathbf{1}$$

So
$$BA + A + B = 0$$
$$AB = -(A + B) = BA .$$

5. $A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_0 & a_1 & \cdots & a_{n-1} \\ a_2 & a_1 & a_0 & \cdots & a_{n-2} \\ . & . & . & \cdots & . \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}$

Compute det A if

a) $a_k = a_0 + d \cdot k$
b) $a_k = a_0 \cdot q^k$

Solution. a) Subtract second row from the first, then third from the second and so on. After n operations You get a matrix of same determinant:

$$\begin{pmatrix} -d & d & d & \cdots & d \\ -d & -d & d & \cdots & d \\ -d & -d & -d & \cdots & d \\ . & . & . & \cdots & . \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}$$

Now subtract row n-2 from row n-1, then row n-3 from n-2, and so on, in the end subtract the first row from the second. After n-1 operations You still get a matrix of the same determinant:

$$\begin{pmatrix} -d & d & d & \cdots & d \\ 0 & -2d & 0 & \cdots & 0 \\ 0 & 0 & -2d & \cdots & 0 \\ . & . & . & \cdots & . \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}$$

Now, there are just 2 permutation which we need to count, since in all the rows except the first and the last You have to choose the diagonal element, hence the determinant is

$$(-2d)^{2n-1}(-\mathrm{d}a_0 - \mathrm{d}a_n) = (-2d)^{2n}(a_0+\mathrm{dn}/2)$$

b) Subtract q times second row from the first, then q times third from the second and so on. After n operations You get a matrix of same determinant:

$$\begin{pmatrix} a_0 - qa_1 & 0 & 0 & \dots & 0 \\ * & a_0 - qa_1 & 0 & \dots & 0 \\ * & * & a_0 - qa_1 & \dots & 0 \\ . & . & . & \dots & . \\ * & * & * & \dots & a_0 \end{pmatrix}$$

I didn’t write the terms below the diagonal, because they don’t matter. The only nonzero permutation which remains here is diagonal, so determinant is $a_0^{n}(1-q^2)^{n-1}$.

Targil 11.

Periodicity and sequences.

1*. A necklace consists of R red and B blue beads. We say that it is good, if for any two substring of the same length a number of red beads in them differ by 1 at most.
a. Prove that a good necklace exists for each R and B.
b. Prove that it is unique up to rotation.

2. A sequence of natural numbers is defined by recursive formula $a_{n+1} = a_0^{a_n}$. Show that the a_n stabilizes modulo m, for each natural number m.
("Stabilizes" means that we start getting always the same number after some index).

3. (a) Winnie the Pooh and Piglet walk over the infinite street, tiled by blocks 1 feet long. They start at the first corner of the first block. They make constant strides, Pooh of length p and Piglet of length q, both p and q are irrational numbers of feet. After the first tile, each tile is stepped on by precisely one animal. (They walk on their toes.)
For which p and q can it happen? Find the precise condition.
(b) Is it possible to decompose the set of natural (integer positive) numbers into disjoint union of two strictly increasing sequences, $\{a_n\}$ and $\{b_n\}$, such that $b_n = n + a_n$, for each n ?

4. Two infinite sequences are given, $\{a_i\}$ of period n and $\{b_i\}$ of period m.
$a_k = b_k$ for $k < m+n$. Show that these two sequences coincide.

5**. The set of natural numbers is decomposed into a disjoint union of arithmetic progressions. Show that some two of those progressions have the same step.

Targil 11.
Periodicity and sequences.

1*. A necklace consists of R red and B blue beads. We say that it is good, if for any two substring of the same length a number of red beads in them differ by 1 at most.
a. Prove that a good necklace exists for each R and B.
b. Prove that it is unique up to rotation.

Solution. Suppose that R ≥ B. Then between each two blue beads there is at least one red bead. If not, and there are two adjacent blue beads, then there also are two adjacent red beads, so there are two substrings of 2 which differ by 2.
So, each necklace of the kind we consider has substrings of red, delimited by single blue beads. For each such necklace we can consider a reduced necklace of R – B red beads and B blue beads, by taking one red bead out of each red substring of that kind.
The original necklace can be reconstructed easily from the reduced one, by inserting one red bead between each pair of blue beads.

Lemma. The reduced necklace is good iff the original necklace is good.

The lemma is easily proved. If the reduced necklace is bad, there are two substrings of equal length, one having precisely k, another having at least k+2 blue beads. By taking minimal substrings of equal length which demonstrate badness of the reduces necklace, we can replace condition "at least k+2" by "precisely k+2". Now enhance it back, we have to add red bead to each red substring, and we can add red bead on both ends of substring or we can not, as we wish. So in the enhanced necklace we also have two substrings of the same length, one with k and another with k+2 blue beads.
The other direction is even easier, we have two different substrings, one with only k and another with at least k+2 blue beads. Now we shrink it, and the second substring shrinks more than the first, so we can append new beads to it, to keep it as long as the first substring. But the first substring has only k blue beads, and the second has at least k+2 and might get some new ones. QED of the lemma.

Hence, to construct a good circle for R and B, first construct a good circle for R – B and B, it is already done by induction (base of induction is obvious), and then enhance it.
To prove that two different good circles are actually the same, shrink both, use induction, and enhance them back again.

Remark. I didn't formulate things accurately here, but there are two ways to formulate proofs like that. First way: prove it for n = 1, and then show that if it is true for n < N then it is also true for n = N.
The second claim: choose minimal n for which the claim is wrong, and create a contradiction. The second way of proving such claims is shorter when You write it down.

2. A sequence of natural numbers is defined by recursive formula $a_{n+1} = a_0^{a_n}$. Show that the a_n stabilizes modulo m, for each natural number m.
("Stabilizes" means that we start getting always the same number after some index).

Solution. Consider the minimal number m for which there is such a_0 so that the claim is wrong. If m is decomposable into product of two relatively prime numbers, p and q, then the sequence will stabilize mod p and mod q so by the Chinese Remainder Theorem, it will stabilize mod m, which is impossible. Hence m is a power of a prime, $m = p^k$.
If a_0 is divisible by p, than its large powers are divisible by m, so it is not the case.
So, a_0 and m are co-prime. Hence $a_0^{\,n}$ mod m depend only on n mod $\varphi(m)$.
But a_n stabilizes mod $\varphi(m)$, because $\varphi(m) < m$, hence $a_0^{a_n}$ stabilizes mod m. QED.

3. (a) Winnie the Pooh and Piglet walk over the infinite street, tiled by blocks 1 feet long. They start at the first corner of the first block. They make constant strides, Pooh of length p and Piglet of length q, both p and q are irrational numbers of feet. After the first tile, each tile is stepped on by precisely one animal. (They walk on their toes.)
For which p and q can it happen? Find the precise condition.
(b) Is it possible to decompose the set of natural (integer positive) numbers into disjoint union of two strictly increasing sequences, $\{a_n\}$ and $\{b_n\}$, such that $b_n = n + a_n$, for each n ?

Solution. (a) The precise condition is $\frac{1}{p}+\frac{1}{q}=1$.
It is easy to understand why this condition is necessary. Because if we look at first N tiles, where N is large, Pooh will take $\sim\frac{N}{p}$ tiles and Piglet $\sim\frac{N}{q}$ tiles and together they should take ~ N tiles so $\frac{N}{p}+\frac{N}{q}\sim N$, when N goes to infinity, we see $\frac{1}{p}+\frac{1}{q}=1$.
Now we shall so this condition is also sufficient. Before the end of N'th tile Pooh will take precisely $\left[\frac{N}{p}\right]$ tiles and Piglet precisely $\left[\frac{N}{q}\right]$ tiles. Both $\frac{N}{p}$ and $\frac{N}{q}$ are not integer so since $\frac{N}{p}+\frac{N}{q}=N$ is integer we get $\left[\frac{N}{p}\right]+\left[\frac{N}{q}\right]=N-1$.
Hence $\left[\frac{N+1}{p}\right]+\left[\frac{N+1}{q}\right]=N$. The difference between two last expression is 1, so there is one more animal trace on N+1 first tiles than on N first tiles. So, precisely one animal stepped on tile N+1.

Remark. It is interesting what would happen for 3 animals, say Pooh, Piglet and Rabbit, or four animals, with Tigger for example. Of course $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ or $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{t}=1$ will be necessary condition, but I am not sure whether it is sufficient.

(b) We shall see two solutions. From first solution it will be clear that this decomposition is unique, but not how it looks like. From the second it will be clear how precisely does this decomposition behave, but not the uniqueness.

First solution. Paint all natural numbers in two colors, pink and brown, in the following way. In the beginning all numbers are colorless. At step number k choose the least colorless number, call it a_k and paint it pink. Then take number $a_k + k$, call it b_k and paint it brown. It is obvious that ak is a strictly increasing sequence, hence so is b_k, so b_k is bigger than all numbers which were painted earlier, so the numbers don't repeat themselves. It is also obvious that all numbers will be painted after $\aleph_0$ steps.

Second solution. Let φ be the positive root of quadratic equation $\varphi^2 = \varphi + 1$.
Then $\frac{1}{\varphi^2}+\frac{1}{\varphi}=1$. So if the bear makes φ^2 steps, and his friend φ steps (maybe because they are superstitious and think that will bring them gold), then each tile will be stepped upon once. So, natural numbers are nicely decomposed into 2 sequences : $a_N = \left[N\varphi\right]$ and $b_N = \left[N\varphi^2\right]$, as we have proven in (a).
But $a_N + N = \left[N\varphi\right] + N = \left[N(\varphi+1)\right] = \left[N\varphi^2\right] = b_N$.

4. Two infinite sequences are given, $\{a_i\}$ of period n and $\{b_i\}$ of period m.
$a_k = b_k$ for $k < m+n$. Show that these two sequences coincide.

Solution. If m and n have common divisor $r > 1$, we can consider r pairs of sub-sequences $\{a_{ir+k}\}$ and $\{b_{ir+k}\}$, where k is a constant number, $0 \le k < r$.
This way we reduce the problem to the same problem for sequences with lesser periods, $m' = m/r$ and $n' = n/r$. So, it is enough to prove the statement when m and n are coprime.
For any k < m we have $b_k = a_k = a_{k+n} = b_{k+n}$.
Since b_k depends only on k mod m, we can visualize all values of b_k as a vertices of regular m-gon, and at vertex number k we write number b_k.
Consider all lines, connecting vertex number k to vertex number $k+n$ (mod m).
The vertices of regular m-gon and those lines form a graph. This graph is a circle of length m, because m and n are coprime, so making jumps by n mod m will bring you to original point only after m moves. So, this graph is connected, and it will remain connected even when we erase one edge.
As we saw before, $b_k = b_{k+n}$ for any k < m, so if we take only m–1 of those edges, the connected vertices will all have the same value, but the graph is connected, so all values of b sequence are the same, so b is constant, so a is constant at its first $m + n - 1$ elements, so a and b are equal.

5.** The set of natural numbers is decomposed into a disjoint union of arithmetic progressions. Show that some two of those progressions have the same step.

Solution. Let P be common multiples of all periods. After certain moment, the only thing which influences the belonging of a number to any of those sequences is its remainder mod P. Remainders k mod P can be represented as vertexes of the regular polygon, $e^{2ki/P}$. Remainders corresponding to one arithmetic progression are vertexes of regular sub-polygon. The sequence of largest difference correspond to the sub-polygon with fewest number of vertexes, n. Consider function z^n. The sum of this function over all vertices of the polygon is 0. The sum of this function over vertices of regular polygon with more than n vertices is obviously 0. The sum over vertices of polygon with precisely n vertices is nonzero. So there should be more than one polygon of n vertices.

Targil 12 – some inequalities.

1. a) What is greater: e^{π} or π^{e} ?
b) The sum of several natural numbers is 2008.
What is largest possible value of their product?

2. Let $0 < x < \pi/2$. What is greater: $\mathrm{tg}(\sin(x))$ or $\sin(\mathrm{tg}(x))$?

3. We are given 5 positive real numbers: a, b, c, d, e such that

$$a^2 + b^2 + c^2 = d^2 + e^2$$
$$a^4 + b^4 + c^4 = d^4 + e^4$$

What is greater: $a^3 + b^3 + c^3$ or $d^3 + e^3$?

4. a) Let $\{x_{\mathrm{i}}\}$ be a decreasing sequence of positive numbers.
Prove that $\sqrt{\sum_{i=1}^{n} x_i^2} \le \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}$
b) Prove that there exists a universal constant C, such that for each decreasing sequence of positive numbers

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sqrt{\sum_{i=m}^{\infty} x_i^2} \le C \sum_{i=1}^{n} x_i$$

5**. Let $\{a_n\}$ be a sequence of positive numbers, such that $\sum_{n=1}^{\infty} a_n < \infty$.

a) Prove that $\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} < e \sum_{n=1}^{\infty} a_n$.
b) Prove that the constant in the inequality can't be improved, i. e. for any positive ε we can find a sequence of positive numbers s.t. $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} > (e - \varepsilon) \sum_{n=1}^{\infty} a_n$$

Targil 12 – some inequalities.

1. a) What is greater: e^{π} or π^{e} ?
b) The sum of several natural numbers is 2008.
What is largest possible value of their product?

a) First solution. e^x is a strictly convex function, so it is above the tangent line at 1. Tangent line at 1 is ex. Hence $e^x \geq ex$, and since convexity is strict the equality may happen only when $x = 1$.
Hence $e^{\pi/e} > \mathrm{e}\cdot\pi/e = \pi$.
So $e^{\pi} > \pi^{e}$.

Second solution. The question is, actually, what is greater, $\sqrt[e]{e}$ or $\sqrt[\pi]{\pi}$.
To find compute the derivative of $\sqrt[x]{x}$ function (or, if you don't like the hard work, the derivative of it logarithm, $x \ln x$). You will see that the derivative is positive before e, negative after e, and 0 at e.
So $\sqrt[e]{e}$ is the greatest of all $\sqrt[x]{x}$.

b) If k>4, then k(k–2) is bigger then k, so there should be no numbers bigger than 4. If k = 4, we can replace it by 2·2 and still have the same product. Hence WLOG, we can assume we have only 2's and 3's. But 3·3 is greater than 2·2·2, so we have less than three 2's.
2004 is divisible by 3, so we can have two 2's and 668 times 3, or one 4 and 668 times 3. The product is $3^{668}\cdot 4$.

Remark. The question is, morally, which numbers, k's or m's, are more useful, which is bigger, m^k or k^m, ore in other words, $\sqrt[k]{k}$ or $\sqrt[m]{m}$. The answer is, that the best number is e, but as long as we are bound to use integer numbers, the best is 3, the next two are 2 and 4, which are equivalent since $\sqrt[2]{2} = \sqrt[4]{4}$ and other number are worse since they are worse approximations of e. If the question would be about real, and not integer numbers, we would get equal numbers (by AM-GM) very close to e.

2. Let $0 < x < \pi/2$. What is greater: $\tan(\sin(x))$ or $\sin(\tan(x))$?

First solution. Let us start with a guess. On the domain sin goes upwards from 0 to 1 so tan(sin) goes monotonically upwards. While $\tan(x)$ goes upwards from 0 to infinity hence sin(tan) oscillates on the domain, dives to negative and returns to positive many times. So, at least sometimes the first

function is bigger, and there wouldn't be a point in this question unless it is always bigger. So, we shall prove that $\tan(\sin(x)) > \sin(\tan(x))$.
It is enough to prove the claim on the domain where both functions are still growing, i.e. when $\tan(x) \le \pi/4$. At $\arctan(\pi/4)$ the $\sin(\tan(x))$ reaches its global maximum, 1, while $\tan(\sin(x))$ keeps growing.

Denote $y = \tan(\sin(x))$. Then $\sin(x) = \arctan(y)$ $^{(*)}$.
Hence the main claim is $\arcsin(y) > \tan(x)$.
It is enough to prove that the rate of growing of LHS is higher, i. e.

$$\frac{dx}{\cos^2 x} < \frac{dy}{\sqrt{1-y^2}}$$

$$1 < \frac{\cos^2 x}{\sqrt{1-y^2}}\frac{dy}{dx}$$

While from $^{(*)}$ we have

$$\cos x \cdot dx = \frac{dy}{1+y^2}$$

Hence we can substitute $\frac{dy}{dx}$ into the claim:

$$1 < \cos^3 x \frac{1+y^2}{\sqrt{1-y^2}}$$

But from $^{(*)}$ we can express $\cos x = \sqrt{1-\arctan^2 y}$ so now we have to prove arrive to an innocently looking inequality, with no composed functions:

$$1 < \left(1-\arctan^2 y\right)^{3/2} \frac{1+y^2}{\sqrt{1-y^2}}$$

Where $0 < y < 1$. Here we have square roots twice, so let's take the square:

$$\frac{1-y^2}{\left(1+y^2\right)^2} < \left(1-\arctan^2 y\right)^3$$

But we can write $\qquad \frac{1}{1+y^2} = 1 - \frac{y^2}{1+y^2}$

Hence if $z = \arctan y$, then we can rewrite the claim as

$$\left(1-\tan^2 z\right)\left(1-\sin^2 z\right)\left(1-\sin^2 z\right) < \left(1-z^2\right)^3$$

$$\left(\frac{\cos^2 z - \sin^2 z}{\cos^2 z}\right)\left(\cos^2 z\right)^2 < \left(1-z^2\right)^3$$

$$\cos(2z)\cos^2 z < \left(1-z^2\right)^3$$

$$\frac{\cos(2z)+\cos^2(2z)}{2} = \cos(2z)\frac{1+\cos(2z)}{2} < \left(1-z^2\right)^3$$

$$\frac{\cos(4z)+2\cos(2z)+1}{4} < \left(1-z^2\right)^3$$

Here 0 < z < arctan(1) = $1/\sqrt{2}$ < 1.
So, we shall estimate the left hand side by its Taylor series. We know from Lagrange remainder that if we take 4k+1 terms we shall get upper bound for cosine. Hence:

$$\frac{\cos(4z)+2\cos(2z)+1}{4} < \frac{1}{4}\left(1+1-\frac{(4z)^2}{2}+\frac{(4z)^4}{24}-\frac{(4z)^6}{6!}+\frac{(4z)^8}{8!}\right)+$$

$$+\frac{1}{2}\left(1-\frac{(2z)^2}{2}+\frac{(2z)^4}{24}-\frac{(2z)^6}{6!}+\frac{(2z)^8}{8!}\right)=$$

$$=1-3z^2+\frac{4^3+2^3}{24}z^4-\frac{4^5+2^5}{6!}z^6+\frac{4^7+2^7}{8!}z^8 = 1-3z^2+3z^4-\frac{22}{15}z^6+\frac{43}{105}z^8$$

So, to achieve happiness, i. e. to prove that LHS < $(1 - z^2)^3$ it is enough to prove that

$$1-3z^2+3z^4-\frac{22}{15}z^6+\frac{43}{105}z^8 < 1-3z^2+3z^4-z^6$$

$$\frac{43}{105}z^8 < \frac{7}{15}z^6$$

$$z^2 < \frac{735}{645}$$

It is true in the domain which we consider. QED.
Sorry people, less Taylor terms are simply not enough. And yes, I did the computations by hands.

Second solution (the official one). Let $f(x)$ = tan(sin(x)) – sin(tan(x)). Then

$$f'(x) = \frac{\cos x}{\cos^2(\sin x)} - \frac{\cos(\tan(x))}{\cos^2 x} = \frac{\cos^3 x - \cos^2(\sin x)\cos(\tan(x))}{\cos^2(\sin x)\cos^2 x}$$

Let 0 < x < arctan(π/2). Since cos is concave (sad) on (0 , π/2) we get

$$\sqrt[3]{\cos^2(\sin x)\cos(\tan(x))} < \frac{\cos(\tan x)+2\cos(\sin x)}{3} \le \cos\frac{\tan x+2\sin x}{3} < \cos(x)$$

The last inequality follows from $\frac{\tan x + 2\sin x}{3} > x$, which is because

$$\left[\frac{\tan x + 2\sin x}{3}\right]' = \frac{1}{3}\left[\frac{1}{\cos^2 x} + 2\cos x\right] > \sqrt[3]{\frac{1}{\cos^2 x}\cdot\cos x\cdot\cos x} = 1$$

3. We are given 5 positive real numbers: a, b, c, d, e such that

$$a^2 + b^2 + c^2 = d^2 + e^2$$
$$a^4 + b^4 + c^4 = d^4 + e^4$$

What is greater: $a^3 + b^3 + c^3$ or $d^3 + e^3$?

First solution. Let $A=a^2$, $B=b^2$, $C=c^2$, $D=d^2$, $E=e^2$.
$P=A+B+C = D+E$, $Q = A^2+B^2+C^2 = D^2 + E^2$.
Consider in the $\{(x, y, z)\}$ space the set defined by

$$x + y + z = P$$
$$x^2 + y^2 + z^2 = Q$$

The first equation is a plane, the second is a sphere. So, the intersection is a circle. It cuts coordinate planes at points $(D, E, 0)$, $(E, D, 0)$, $(D, 0, E)$, etc… Hence it has three symmetric arcs in the positive domain $x, y, z > 0$ (one arc is where x is the greatest; another where y is greatest; the third is where z is greatest). So, let us assume that $A > B > C$ and $D > E$ so that points (A, B, C) and $(D, E, 0)$ will be one the same arc of the circle (actually, on the same half of the same arc, because $y > z$ in both cases).
Let $(X(t), Y(t), Z(t))$ be a parametric curve, going along that arc from the point $(D, E, 0)$ to the point (A, B, C), which means, for instance, that

$$(X(0), Y(0), Z(0)) = (D, E, 0)$$
$$(X(1), Y(1), Z(1)) = (A, B, C)$$

We shall also assume that the curve goes always forward along the arc with constant speed v. By upper dot we denote, as usual, derivative over t.
So, since $X + Y + Z$ and $X^2 + Y^2 + Z^2$ remain constant during the motion

$$\dot{X} + \dot{Y} + \dot{Z} = 0$$
$$2X\dot{X} + 2Y\dot{Y} + 2Z\dot{Z} = 0$$

So the vector $(\dot{X}, \dot{Y}, \dot{Z})$ is orthogonal to both $(1, 1, 1)$ and (X, Y, Z) and so it is proportional to their vector product $(Y - Z, Z - X, X - Y)$.
But we know that Z is growing along the way, and X is the greatest of 3, so the coefficient of proportion is positive.
So, we have computed $(\dot{X}, \dot{Y}, \dot{Z})$ up to a positive coefficient.
The question which really bothers us is whether the value of the function

$$F(t) = X^{3/2} + Y^{3/2} + Z^{3/2}$$

is greater at the beginning or at the end of the motion?
Let us derive it and find its sign.

$$\dot F(t)=\frac{3}{2}\left(\dot X\sqrt{X}+\dot Y\sqrt{Y}+\dot Z\sqrt{Z}\right)=a\left((Y-Z)\sqrt{X}+(Z-X)\sqrt{Y}+(X-Y)\sqrt{Z}\right)$$

Here a is a positive coefficient so it doesn't influence the sign.
Denote $x=\sqrt{X}, y=\sqrt{Y}, z=\sqrt{Z}$. Then, since $X > Y > Z > 0$ on our half-arc, so also $x > y > z > 0$. Then

$$(Y-Z)\sqrt{X}+(Z-X)\sqrt{Y}+(X-Y)\sqrt{Z}=\left(y^2-z^2\right)x+\left(z^2-x^2\right)y+\left(x^2-y^2\right)z=$$

$$=y^2x-x^2y-z^2(x-y)+(x-y)(x+y)z=(x-y)\left(-xy-z^2+(x+y)z\right)=$$

$$=-(x-y)(x-z)(y-z)<0$$

(Of course, this expression was Vandermonde of size 3).
So, $\dot F(t)<0$, and the value if

$$d^3+e^3=\mathrm{F}(0) > \mathrm{F}(1)=a^3+b^3+c^3.$$

Second solution (the official). WLOG, we may assume that $a\geq b\geq c$ and $d\geq e$.
Let $c^2 = e^2 + \Delta$. Then $d^2 = a^2 + b^2 + \Delta$.
Hence $a^4+b^4+\left(e^2+\Delta\right)^4=\left(a^2+b^2+\Delta\right)^2+e^4$

$$2e^2\Delta=2a^2b^2+2\left(a^2+b^2\right)\Delta$$

$$\Delta=-\frac{a^2b^2}{a^2+b^2-e^2}$$

But $a^2+b^2-e^2\geq\frac{2}{3}\left(a^2+b^2+c^2\right)-\frac{1}{2}\left(d^2+e^2\right)=\frac{1}{6}\left(d^2+e^2\right)>0$

So Δ is negative.

$$0<c^2=e^2-\frac{a^2b^2}{a^2+b^2-e^2}=\frac{a^2e^2+b^2e^2-e^4-a^2b^2}{a^2+b^2-e^2}=\frac{\left(a^2-e^2\right)\left(e^2-b^2\right)}{a^2+b^2-e^2}$$

So $a > e > b$.

Therefore $d^2=a^2+b^2-\frac{a^2b^2}{a^2+b^2-e^2}<a^2+b^2-\frac{a^2b^2}{a^2}=a^2$.

Hence $a > d\geq e > b\geq c$.
Consider the function

$$f(x) = a^x + b^x + c^x - d^x - e^x$$

WLOG $a = 1$(if not, divide everything by a^x).
We shall prove that this function has only 2 zeroes on the real line, 2 and 4, and that f changes its sign at those points. Suppose the contrary. Then, by

Rolle's theorem, there f' has at least two distinct zeroes, $x_1 < x_2$, $f'(x_1) = f'(x_2) = 0$.

$$f'(x) = \ln b \cdot b^x + \ln c \cdot c^x - \ln d \cdot d^x - \ln e \cdot e^x$$

Hence $\ln b \cdot b^{x_i} + \ln c \cdot c^{x_i} = \ln d \cdot d^{x_i} + \ln e \cdot e^{x_i}$ for $i = 1, 2$.
But $1 > d \geq e > b \geq c$. So

$$\frac{(-\ln b)b^{x_1} + (-\ln c)c^{x_1}}{(-\ln b)b^{x_2} + (-\ln c)c^{x_2}} \leq b^{x_1 - x_2} < e^{x_1 - x_2} \leq \frac{(-\ln d)d^{x_1} + (-\ln e)e^{x_1}}{(-\ln d)d^{x_2} + (-\ln e)e^{x_2}},$$

a contradiction.
Hence $f(x)$ has constant sign at intervals $(-\infty, 2), (2, 4), (4, \infty)$. It is positive at 0, so it is negative at 3.

4. a) Let $\{x_i\}$ be a decreasing sequence of positive numbers.
Prove that $\sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}$

b) Prove that there exists a universal constant C, such that for each decreasing sequence of positive numbers

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sqrt{\sum_{i=m}^{\infty} x_i^2} \leq C \sum_{i=1}^{n} x_i$$

Solution. a) Take the square: $\sum_{i=1}^{n} x_i^2 \leq \left(\sum_{i=1}^{n} \frac{x_i}{\sqrt{i}} \right)^2$

But if You replace all numbers by lower numbers with higher indices:

$$\left(\sum_{i=1}^{n} \frac{x_i}{\sqrt{i}} \right)^2 = x_1^2 + \frac{x_2}{\sqrt{2}} \left(2x_1 + \frac{1}{\sqrt{2}} x_2 \right) + \frac{x_3}{\sqrt{3}} \left(2x_1 + 2\frac{x_2}{\sqrt{2}} + \frac{x_3}{\sqrt{3}} \right) + \ldots >$$

$$> x_1^2 + \frac{x_2^2}{\sqrt{2}} \left(2 + \frac{1}{\sqrt{2}} \right) + \frac{x_3^2}{\sqrt{3}} \left(2 + 2\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \right) + \frac{x_4^2}{\sqrt{4}} \left(2 + 2\frac{1}{\sqrt{2}} + 2\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \right) + \ldots$$

It remains to estimate sum of inverse roots. It is easy to guess (for example, if you approximate series by integral) that it is approximately $\sqrt{n}$.
To make precise statement consider

$$\sqrt{n}-\sqrt{n-1}=\frac{\left(\sqrt{n}-\sqrt{n-1}\right)\left(\sqrt{n}+\sqrt{n-1}\right)}{\sqrt{n}+\sqrt{n-1}}=\frac{1}{\sqrt{n}+\sqrt{n-1}}<\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n-1}}\right)\Big/4$$

(Because always $\frac{1}{A+B}\le\left(\frac{1}{A}+\frac{1}{B}\right)\Big/4=\frac{A+B}{4AB}$.

Since $4AB\le(A+B)^2$, which is same as $0\le(A-B)^2$.)

Sum up such inequalities from 2 to m and You get

$$\sqrt{m}-1<\frac{1}{4}\left(\frac{1}{\sqrt{1}}+\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{3}}+\ldots+\frac{2}{\sqrt{m-1}}+\frac{1}{\sqrt{m}}\right)$$

So $\sqrt{m}<\frac{2}{\sqrt{1}}+\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{3}}+\ldots+\frac{2}{\sqrt{m-1}}+\frac{1}{\sqrt{m}}$

(The last formula could be obtained with less subtle technique, but technique is what we want to learn). Now we can finish the proof:

$$\left(\sum_{i=1}^{n}\frac{x_i}{\sqrt{i}}\right)^2>x_1^2+\frac{x_2^2}{\sqrt{2}}\left(2+\frac{1}{\sqrt{2}}\right)+\frac{x_3^2}{\sqrt{3}}\left(2+2\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)+\ldots>x_1^2+x_2^2+x_3^2+x_4^2\ldots$$

b)

$$\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\sqrt{\sum_{i=m}^{\infty}x_i^2}\le\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\sum_{i=m}^{\infty}\frac{x_i}{\sqrt{i+1-m}}=\sum_{i=1}^{\infty}x_i\sum_{m=1}^{i}\frac{1}{\sqrt{m}}\cdot\frac{1}{\sqrt{i+1-m}}$$

All we have to do is to estimate

$$\sum_{m=1}^{i}\frac{1}{\sqrt{m}}\cdot\frac{1}{\sqrt{i+1-m}}=\frac{1}{i}\sum_{m=1}^{i}\frac{1}{\sqrt{\frac{m}{i}}}\cdot\frac{1}{\sqrt{1+\frac{1}{i}-\frac{m}{i}}}\sim\int_0^1\frac{dx}{\sqrt{x}\sqrt{1-x}}$$

Because it is approximately Riemann's sum for that integral. Actually, if we divide the interval [0,1] into i equal parts, then in the part number m we have $x<\frac{m}{i},1-x<1+\frac{1}{i}-\frac{m}{i}$ hence $\frac{1}{\sqrt{\frac{m}{i}}}\cdot\frac{1}{\sqrt{1+\frac{1}{i}-\frac{m}{i}}}<\frac{1}{\sqrt{x}\sqrt{1-x}}$ and

$$\sum_{m=1}^{i}\frac{1}{\sqrt{m}}\cdot\frac{1}{\sqrt{i+1-m}}=\frac{1}{i}\sum_{m=1}^{i}\frac{1}{\sqrt{\frac{m}{i}}}\cdot\frac{1}{\sqrt{1+\frac{1}{i}-\frac{m}{i}}}<\int_0^1\frac{dx}{\sqrt{x}\sqrt{1-x}}$$

So, it remains to compute that integral, and it will serve as (a pretty tight) universal constant. It is done by a trigo substitution: $x=\sin^2 t$

$dx = 2\sin t\cos t\,dt$

$$\int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}} = \int_0^{\pi/2} 2dt = \pi .$$

There is a more elementary way to find a less tight bound:

$$\sum_{m=1}^{i} \frac{1}{\sqrt{m}} \cdot \frac{1}{\sqrt{i+1-m}} = 2\sum_{m=1}^{i/2} \frac{1}{\sqrt{m}} \cdot \frac{1}{\sqrt{i+1-m}} \le 2\frac{1}{\sqrt{i/2}} \sum_{m=1}^{i/2} \frac{1}{\sqrt{m}} \le 2\frac{1}{\sqrt{i/2}} 2\sqrt{i/2} = 4$$

5.** Let $\{a_n\}$ be a sequence of positive numbers, such that $\sum_{n=1}^{\infty} a_n < \infty$.

***a*)** Prove that $\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} < e \sum_{n=1}^{\infty} a_n$.

***b*)** Prove that the constant in the inequality can't be improved, i. e. for any positive ε we can find a sequence of positive numbers s.t. $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} > (e-\varepsilon) \sum_{n=1}^{\infty} a_n$$

This inequality is called ***Carleman inequality***. Carleman is a famous Swedish mathematician.
This inequality was given at IMC and it was the most expensive problem in its year. We shall start with b) part, since it is easier and gives a clue to a).

b) Consider the sequence $a_n = \frac{1}{n^u}$ where u is only slightly bigger then 1.
Then the sum of a_n is bounded, but very big, so the first few terms are only a small part of the sum. For large n's,

$$\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = \left(\frac{1}{\sqrt[n]{1 \cdot 2 \cdot \ldots \cdot n}} \right)^u \sim \left(\frac{e}{n} \right)^u = e^u a_n \sim e a_n$$

Hence for u sufficiently close to 1, the LHS and RHS of Carleman inequality are approximately equal. QED.

This gives a clue to a part. It is very natural to try and decompose Carleman inequality into sum of Cauchy (AM-GM) inequalities, say

$$\sqrt[n]{b_{n,1} \cdot b_{n,2} \cdot \ldots \cdot b_{n,n}} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \le \frac{b_{n,1}a_1 + b_{n,2}a_2 + \ldots + b_{n,n}a_n}{n}$$

Then, if you choose the coefficients smartly, so that $\sqrt[n]{b_{n,1} \cdot b_{n,2} \cdot \ldots \cdot b_{n,n}}$ is less than e times sum of all coefficients before each a_k, you will get Carleman. But how to guess those coefficients?
It is well known that Cauchy inequality becomes equality only if all the numbers in it are the same. Assume (though it is literally wrong, but morally almost true), that Carleman inequality becomes equality for the sequence $a_n = \frac{1}{n}$. So, the coefficients should be chosen so, that the numbers will become equal, so a_n should be multiplied by n before applying Cauchy.
That was the main philosophical idea, now we shall execute it.

a) We shall start with reminding classical calculus lemmas:

Lemma 0. $(1+a)^n \geq 1 + na$ if $1+a > 0$ and n is a natural number. (Bernoulli inequality). Equality holds only if $a = 0$ or $n = 1$.

Lemma 1. $b_n = \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n < e$.

Lemma 2. $\frac{n+1}{e} < \sqrt[n]{n!}$.

Proof of lemma 0. By induction, for n = 1 it is obvious, for n+1, supposing it was proven for n, it follows:
$(1+a)^{n+1} = (1+a)^n(1+a) \geq (1 + na)(1+a) = 1 + (n+1)a + na^2 \geq 1 + (n+1)a$

Proof of lemma 1. It is well known that $b_n \to e$. So, it would be enough to prove that b_n is monotonically increasing, i. e. $b_n < b_{n+1}$.

$$\left(\frac{n+1}{n}\right)^n < \left(\frac{n+2}{n+1}\right)^{n+1}$$

$$\frac{n}{(n+1)} < \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1}$$

$$1 - \frac{1}{n+1} < \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}$$

That is a direct consequence of Bernoulli inequality (lemma 0).

Proof of lemma 2. Since $b_1 < b_2 < \ldots < b_n < e$ we get

$$\frac{n+1}{\sqrt[n]{n!}}=\sqrt[n]{\frac{(n+1)^n}{n!}}=\sqrt[n]{b_1\cdot b_2\cdot\ldots\cdot b_n}<e$$

Finally, let's prove Carleman inequality. By Cauchy and lemma 2:

$$\frac{n+1}{e}\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}<\sqrt[n]{n!\cdot a_1\cdot a_2\cdot\ldots\cdot a_n}\le\frac{1\cdot a_1+2\cdot a_2+3\cdot a_3+\ldots+n\cdot a_n}{n}$$

Hence $\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}<e\dfrac{1\cdot a_1+2\cdot a_2+3\cdot a_3+\ldots+n\cdot a_n}{n(n+1)}$.

Sum all these inequalities. You get

$$\sum_{n=1}^{\infty}\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}<e\sum_{k=1}^{\infty}a_k\sum_{n=k}^{\infty}\frac{k}{n(n+1)}=e\sum_{k=1}^{\infty}ka_k\sum_{n=k}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=e\sum_{k=1}^{\infty}ka_k\frac{1}{k}=e\sum_{k=1}^{\infty}a_k$$

QED.

Targil 13 – limits.

1. Compute $\lim_{x\to 0+}\int_{x}^{2x}\frac{\sin^m(t)}{t^n}dt$, where m and n are natural numbers, (the answer may depend on m and n).

2. a) Does $\sum_{k=0}^{\infty}\frac{1}{k\cdot(\ln k)\cdot(\ln(\ln k))}$ converge?

b) Does $\sum_{k=0}^{\infty}\frac{1}{k\cdot(\ln k)\cdot(\ln(\ln k))^2}$ converge?

3. Let $\{a_{\mathrm{n}}\}$ be a sequence defined by $a_0 = 1$ and recursive formula

$$a_{n+1}=\frac{1}{n+1}\sum_{k=0}^{n}\frac{a_k}{n-k+2}$$

Find the limit $\sum_{k=0}^{\infty}\frac{a_k}{2^k}$.

4*. Compute $\lim_{x\to 0}\frac{\sin(\tan(x))-\tan(\sin(x))}{\arcsin(\arctan(x))-\arctan(\arcsin(x))}$.

5*. $\{a_n\}$ is a sequence of positive real numbers such that $a_n + a_m \geq a_{n+m}$ for all m, n.
Prove that the sequence a_n/n converges.

Targil 13 – limits.

1. Compute $\lim\limits_{x\to 0+}\int\limits_x^{2x}\frac{\sin^m(t)}{t^n}dt$, where m and n are natural numbers, (the answer may depend on m and n).

Solution. By mean value theorem of integration

$$\int\limits_x^{2x}\frac{\sin^m(t)}{t^n}dt=\int\limits_x^{2x}\left(\frac{\sin(t)}{t}\right)^m\frac{1}{t^{n-m}}dt=\left(\frac{\sin(y)}{y}\right)^m\int\limits_x^{2x}\frac{1}{t^{n-m}}dt\text{, where } x\le y\le 2x.$$

Hence $\frac{\sin(y)}{y}\xrightarrow[x\to 0]{}1$, so $\left(\frac{\sin(y)}{y}\right)^m\xrightarrow[x\to 0]{}1$. Therefore

$$\lim_{x\to 0+}\int\limits_x^{2x}\frac{\sin^m(t)}{t^n}dt=\lim_{x\to 0+}\int\limits_x^{2x}\frac{1}{t^{n-m}}dt=\left.\frac{m-n}{t^{n-m+1}}\right|_x^{2x}=\lim_{x\to 0+}(m-n)\left(\frac{1}{2^{n-m+1}}-1\right)\frac{1}{x^{n-m+1}}$$

if $n-m\neq -1$. Since the integral is positive anyway, it only matters whether $\lim\limits_{x\to 0+}\frac{1}{x^{n-m+1}}$ tends to zero or to infinity in these cases, so

if $n-m>-1$ then it tends to infinity,
if $n-m<-1$ then it tends to zero,
and if $n-m=-1$ then

$$\lim_{x\to 0+}\int\limits_x^{2x}\frac{\sin^m(t)}{t^n}dt=\lim_{x\to 0+}\int\limits_x^{2x}\frac{1}{t}dt=\lim_{x\to 0+}\left(\ln(2x)-\ln x\right)=\ln 2$$

Remark. This problem is pretty obvious, but it really did appear on IMC.

2. a) Does $\sum\limits_{k=100}^{\infty}\frac{1}{k\cdot(\ln k)\cdot(\ln(\ln k))}$ converge?

b) Does $\sum\limits_{k=100}^{\infty}\frac{1}{k\cdot(\ln k)\cdot(\ln(\ln k))^2}$ converge?

Solution. a) Apply the "harmonic trick" : split the sequence into sub-strips:
strip number n will contain numbers between 2^n+1 and 2n+1.
Then all numbers in the strip number n are (up to a bounded factor)
$\frac{1}{2^n\cdot n\cdot(\ln n)}$ and there are 2^n elements in this strip.

Therefore the sum of that strip is $\frac{1}{n\cdot(\ln n)}$.

So it is enough to investigate the convergence of the series $\sum_{n=?}^{\infty}\frac{1}{n\cdot(\ln n)}$.

Apply the harmonic trick again: cut the sequence into strips, strip number m will contain n's between 2^m+1 and $2m+1$.

The elements in strip number m are, up to a bounded factor $\frac{1}{2^m\cdot m}$, this strip has 2^m elements, so the sum in the strip is, up to a bounded factor $\frac{1}{m}$.

So the sum over all strips behaves like $\sum_{m=?}^{\infty}\frac{1}{m}$, and that one diverges (the easiest proof for that is applying harmonic trick again.

b) As in a), apply the harmonic trick twice. Each time you get the series with the same convergence properties. After firs time you get $\sum_{n=?}^{\infty}\frac{1}{n\cdot(\ln n)^2}$, and after second time you get $\sum_{m=?}^{\infty}\frac{1}{m^2}$, and that one converges.

3. Let $\{a_n\}$ be a sequence defined by $a_0 = 1$ and recursive formula

$$a_{n+1}=\frac{1}{n+1}\sum_{k=0}^{n}\frac{a_k}{n-k+2}$$

Find the limit $\sum_{k=0}^{\infty}\frac{a_k}{2^k}$.

Solution. Consider a generating function of that sequence $f(x)=\sum_{n=0}^{\infty}a_nx^n$.

Then the limit of the series that we should compute is simply $f\left(\frac{1}{2}\right)$.

The equation $(n+1)a_{n+1}=\sum_{k=0}^{n}\frac{a_k}{n-k+2}$ after we multiply both sides by xn and sum them up turn to $\sum_{n=0}^{\infty}(n+1)a_{n+1}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_kx^k\,\frac{x^{n-k}}{n-k+2}=\sum_{m=0}^{\infty}\frac{x^m}{m+2}\sum_{k=0}^{\infty}a_kx^k$

$$f'(x)=f(x)\sum_{m=0}^{\infty}\frac{x^m}{m+2}$$

So, we should try to compute $g(x)=\sum_{m=0}^{\infty}\frac{x^m}{m+2}=\frac{1}{2}+\frac{x}{3}+\frac{x^2}{4}+\frac{x^3}{5}+\dots$.

But we know that $-\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\dots$

Therefore $g(x)=-\frac{x+\ln(1-x)}{x^2}$.

Hence $\frac{df}{f}=-\frac{x+\ln(1-x)}{x^2}dx$

$$\ln f+const=-\int\frac{dx}{x}-\int\frac{\ln(1-x)}{x^2}dx=-\ln x+\int\left(\frac{\ln(1-x)}{x}\right)'+\frac{1}{x(1-x)}dx=$$

$$=-\ln x+\frac{\ln(1-x)}{x}+\int\frac{1}{x(1-x)}dx=-\ln x+\frac{\ln(1-x)}{x}+\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx=$$

$$=-\ln x+\frac{\ln(1-x)}{x}+\ln x-\ln(1-x)=\frac{1-x}{x}\ln(1-x)$$

When $x = 0$ the right hand side might be seen as a limit (since it comes from power series) so it is -1 ; $f(0) = 1$ and its ln is 0, hence left hand side is equal to *const*, so *const* $= -1$.

So $f(x)=\exp\left(1+\frac{1-x}{x}\ln(1-x)\right)$

Therefore $f\left(\frac{1}{2}\right)=\exp\left(1+\ln\left(\frac{1}{2}\right)\right)=e^{1-\ln 2}=\frac{e}{2}$

4*. Compute $\lim_{x\to 0}\frac{\sin(\tan(x))-\tan(\sin(x))}{\arcsin(\arctan(x))-\arctan(\arcsin(x))}$.

Simple lemma. Suppose we have a function $f(x)$ such that $f(0) = 0$ and f has a continuous derivative around 0.

Suppose also that $u,v\to 0$ and always $u\neq v$. Then $\frac{f(u)-f(u)}{u-v}\to f'(0)$.

Proof. By Lagrange theorem, $\frac{f(u)-f(u)}{u-v}=f'(w)$ where w is between u and v, so when $u,v\to 0$, w also tends to 0. QED of lemma.

Notice that all functions are analytic.
$\sin(\tan(x)) \neq \tan(\sin(x))$ at least sometimes, for example when
$x = \arctan\left(\frac{3}{2}\pi\right)$ because, then LHS is negative and RHS is positive.
Since functions are analytic, their points of coincidence are isolated, so at some neighborhood of 0, except for 0 itself, $\sin(\tan(x)) \neq \tan(\sin(x))$, so we learn from complex functions. So, at some neighborhood of 0 their inverse functions, $\arcsin(\arctan(x)), \arctan(\arcsin(x))$ also don't coincide, hence we can consider that fraction without problems.

From now on we shall say p(x) ~ q(x) to indicate that $\lim_{x\to 0} \frac{p(x)}{q(x)} = 1$.

It is known that $x \sim \sin x \sim \tan x$.
Denote $y = \tan(\sin(x))$ then y ~ x.

$$\lim_{x\to 0} \frac{\sin(\tan(x)) - \tan(\sin(x))}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))} =$$

$$= \lim_{x\to 0} \frac{\sin(\tan(\arcsin(\arctan(y)))) - y}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))} =$$

$$= \lim_{x\to 0} \frac{\sin(\tan(\arcsin(\arctan(y)))) - y}{x - \tan(\sin(\arctan(\arcsin(x))))}$$

The last equality follows from the above lemma, when you apply function $f(x) = \tan(\sin(x))$ to $u = \arcsin(\arctan(x)), v = \arctan(\arcsin(x))$, you get $\arcsin(\arctan(x)) - \arctan(\arcsin(x)) \sim x - \tan(\sin(\arctan(\arcsin(x))))$.

If we denote $g(y) = \sin(\tan(\arcsin(\arctan(y))))$, and its inverse function $g^{-1}(x) = \tan(\sin(\arctan(\arcsin(x))))$, then what we got is

$$\lim_{x\to 0} \frac{g(y) - y}{x - g^{-1}(x)}$$

From $x \sim \sin x \sim \tan x$ it follows that $x \sim g(x) \sim g^{-1}(x)$.
But we have seen before that both numerator and denominator are nonzero, so g has no stable points in the neighborhood of 0 other than 0.

Hence Taylor series of g has at least 2 nonzero terms, and the first is x: $g(x) = x + ax^n + \ldots$, where a is an nonzero number which I know nothing about, and n > 1 is a number which I know almost nothing about, and I don't want to (though they can be computed). Then $g^{-1}(x) = x - ax^n + \ldots$.

Now since $y \sim x$,

$$\lim_{x \to 0} \frac{g(y) - y}{x - g^{-1}(x)} = \lim_{x \to 0} \frac{ay^n + o(y^n)}{-(-ay^n + o(y^n))} = 1$$

So, the answer is 1.

5*. $\{a_n\}$ is a sequence of positive real numbers such that $a_n + a_m \geq a_{n+m}$ for all m, n.

Prove that the sequence a_n/n converges.

Proof. Firstly, $a_n \leq na_1$ hence the sequence a_n/n is bounded from above. It is also bounded from below by 0. So it is bounded. Hence it has liminf (lower limit) which will be denoted by L.

For each $\varepsilon > 0$, we can find an index k such that $a_k/k < L + \varepsilon$.

Then $a_k < k(L + \varepsilon)$.

Then $a_{mk} \leq ma_k < mk(L + \varepsilon)$.

For each natural N, we can write $N = mk + r$, where $r < k$ (division with remainder). Hence $a_N \leq a_{mk} + a_r \leq ma_k + ra_1 < mk(L + \varepsilon) + ma_1$.

Therefore $a_N / N < L + \varepsilon + ma_1 / N < L + 2\varepsilon$, for sufficiently large N.

So, for any $\varepsilon > 0$, $a_N / N < L + 2\varepsilon$, for sufficiently large N.

Since L is liminf, then for sufficiently large N also $a_N / N > L - 2\varepsilon$.

Hence the sequence converges to L.

Targil 14 – functions.

1. Let $f:\mathbb{R}\to\mathbb{R}$ be thrice differentiable.

Show that there exists $\xi\in(-1,1)$ s. t. $\dfrac{f'''(\xi)}{6}=\dfrac{f(1)-f(-1)}{2}-f'(0)$.

2. Does there exist a continuously differentiable function s. t. for every real x $f(\mathrm{x})>0$ and $f'(x)>f(f(x))$?

3. Consider continuous function $f:[0,1]\to\mathbb{R}$ such that

$$x\cdot f(y)+y\cdot f(x)\le 1.$$

a) Prove that $\int\limits_0^1 f(x)dx\le\dfrac{\pi}{4}$.

b) Find such a function f that the inequality of a) will become equality.

4. Let $f,g:[a,b]\to(0,\infty)$ be continuous, non-decreasing functions, such that for every $x\in[a,b]$ we have $\int\limits_a^x\sqrt{f(t)}dt\le\int\limits_a^x\sqrt{g(t)}dt$

and
$$\int\limits_a^b\sqrt{f(t)}dt=\int\limits_a^b\sqrt{g(t)}dt$$

Prove that
$$\int\limits_a^b\sqrt{1+f(t)}dt\ge\int\limits_a^b\sqrt{1+g(t)}dt$$

5*. Does there exist a

a) continuous
b) monotone
c) continuously differentiable

function $f:[0,1]\to[0,1]$ s. t. every $y\in[0,1]$ it has uncountable number of inverse images of y ?
Uncountable means strictly bigger than $\aleph_0$.
Inverse image of y is such an x that $f(x)=y$.

6.** For a continuous function $f:\mathbb{R}\to\mathbb{R}$ it is given that any positive

real x, y the sequence $f(x+ny)$, for $n\in\mathbb{N}$, tends to infinity.

Does it follow that $f(x)\to\infty$ as $\mathrm{x}\to\infty$?
(It is similar to the problem we had in the competition, but now we require continuity).

Targil 14 – functions.

1. Let $f:\mathbb{R}\to\mathbb{R}$ be thrice differentiable.

Show that there exists $\xi\in(-1,1)$ s. t. $\dfrac{f'''(\xi)}{6}=\dfrac{f(1)-f(-1)}{2}-f'(0)$.

Proof. One proof, which is not completely honest, but acceptable at IMC, is to say that the LHS is $f[-1\ ,0\ ,0\ ,1]$ (Newton's divided difference), and hence it is true.

$$\begin{array}{llll} f(-1) & f(0)-f(-1) & f'(0)-\left(f(0)-f(-1)\right) & \dfrac{f(1)-2f'(0)-f(-1)}{2} \\ f(0) & f'(0) & f(1)-f(0)-f'(0) & \\ f(0) & f(1)-f(0) & & \\ f(1) & & & \end{array}$$

A more honest way is to give a proof, at least for this special case. Construct a polynomial of degree 3 which coincides with f at points –1, 0, 1, and touches f at 0.

$$g(x)=-\frac{f(-1)}{2}x^2(x-1)-f(0)(x-1)(x+1)+\frac{f(1)}{2}x^2(x+1)-f'(0)x(x-1)(x+1)$$

So, the function $f-g$ has 3 root, one of those of multiplicity 2, on [–1, 1].

By Rolle's theorem, its third derivative has at least one root. So, at one point, the third derivative of f coincides with the third derivative of g. But the third derivative of g is constant and equal to LHS of the original expression times 6.

2. Does there exist a continuously differentiable function s. t. for every real x $f(x)>0$ and $f'(x)>f(f(x))$?

Solution. f is positive, hence f' is positive, hence f is growing monotonically. Hence $f'(x)>f(f(x))>f(f(0))=const>0$, hence the slope of f is bounded from below, hence when we go to negative direction the value of the function goes down by at least given rate, hence it reaches negative values, which was specifically forbidden.

A more algebraic way to write this down (and it would cause less argument between the judges at the Olympiad):

Denote $C = f(0) > 0$.

$f'(x) > f(f(x)) > 0$, hence f monotonically increases.

$f(x) > 0$, therefore $f(f(x)) > f(0) = C$.

So $f'(x) > f(f(x)) > C$.

By Lagrange for each $x < y$ there is z between them such that

$f(y) - f(x) = f'(z)(y-x) > C(y-x)$

$f(y) - C(y-x) > f(x)$

Choose any y, take $\Delta = \dfrac{2f(y)}{C}$ and take $x = y - \Delta$.

Then $0 > f(y) - C\cdot\Delta = f(y) - C(y-x) > f(x)$.

Contradiction.

3. Consider continuous function $f:[0,1]\to\mathbb{R}$ such that

$$x\cdot f(y) + y\cdot f(x) \le 1.$$

a) Prove that $\int\limits_0^1 f(x)\,dx \le \dfrac{\pi}{4}$.

b) Find such a function f that the inequality of a) will become equality.

Solution. a)

$$\int\limits_0^1 f(x)\,dx = \int\limits_0^{\pi/2} f(\sin t)\,d\sin t = \int\limits_0^{\pi/2} f(\sin t)\cos t\cdot dt = \int\limits_0^{\pi/2} f(\cos u)\sin u\cdot du$$

Here we did substitutions $x = \sin t$, $u = \pi/2 - t$.

Hence the integral can be computed as the mean of the two last expressions:

$$\int\limits_0^1 f(x)\,dx = \frac{1}{2}\left(\int\limits_0^{\pi/2} f(\sin t)\cos t\cdot dt + \int\limits_0^{\pi/2} f(\cos u)\sin u\cdot du\right) =$$

$$= \frac{1}{2}\int\limits_0^{\pi/2} \left(f(\sin t)\cos t + f(\cos t)\sin t\right)\cdot dt \le \frac{1}{2}\int\limits_0^{\pi/2} 1\cdot dt = \frac{\pi}{4}$$

b) For example: $f(x)=\sqrt{1-x^2}$. The integral is $\dfrac{\pi}{4}$ since the area bounded by the graph is a quarter of unit circle. The condition holds

$$x\cdot f(y)+y\cdot f(x)=x\cdot\sqrt{1-y^2}+\sqrt{1-x^2}\cdot y\le\left(x^2+1-x^2\right)\left(1-y^2+y^2\right)=1$$

by Cauchy-Schwartz inequality.

It can be also shown with trigonometric substitution: $f(x)=\cos(\arcsin x)$,

$x=\sin\alpha, y=\sin\beta, f(x)=\cos\alpha, f(y)=\cos\beta$ therefore:

$$x\cdot f(y)+y\cdot f(x)=\sin\alpha\cos\beta+\sin\beta\cos\alpha=\sin(\alpha+\beta)\le 1$$

4. Let $f,g:[a,b]\to(0,\infty)$ be continuous, non-decreasing functions, such that for every $x\in[a,b]$ we have $\int_a^x\sqrt{f(t)}dt\le\int_a^x\sqrt{g(t)}dt$

and
$$\int_a^b\sqrt{f(t)}dt=\int_a^b\sqrt{g(t)}dt$$

Prove that
$$\int_a^b\sqrt{1+f(t)}dt\ge\int_a^b\sqrt{1+g(t)}dt$$

Solution. $\sqrt{f(t)},\sqrt{g(t)}$, are non-decreasing, therefore

$F(x)=\int_a^x\sqrt{f(t)}dt$, $G(x)=\int_a^x\sqrt{g(t)}dt$ are convex. So, we have two convex graphs one above another, with common ends: $F(a)=0=G(a)$,

$$F(b)=\int_a^b\sqrt{f(t)}dt=\int_a^b\sqrt{g(t)}dt=G(b).$$

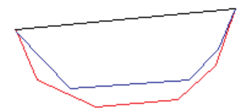
Let us rewrite the statement we have to prove in terms of *F*, *G*:

$$\int_a^b\sqrt{1+\left(F'(t)\right)^2}dt\ge\int_a^b\sqrt{1+\left(G'(t)\right)^2}dt.$$

That is precisely the expression for the length of the curve:

$$\int_a^b\sqrt{1+\left(F'(x)\right)^2}dx=\int_a^b\sqrt{1+\left(\frac{dF}{dx}\right)^2}dx=\int_a^b\sqrt{dx^2+dF^2}=\int_a^b\sqrt{dl^2}=\int_a^b dl=length.$$

So, what we have to prove is that the graph of F is longer than the graph of G. The length of arc of a convex function might be computed as a limit of inscribed broken lines. So all we need to prove is



Lemma. If one convex polygon is inside another, then its perimeter is smaller.

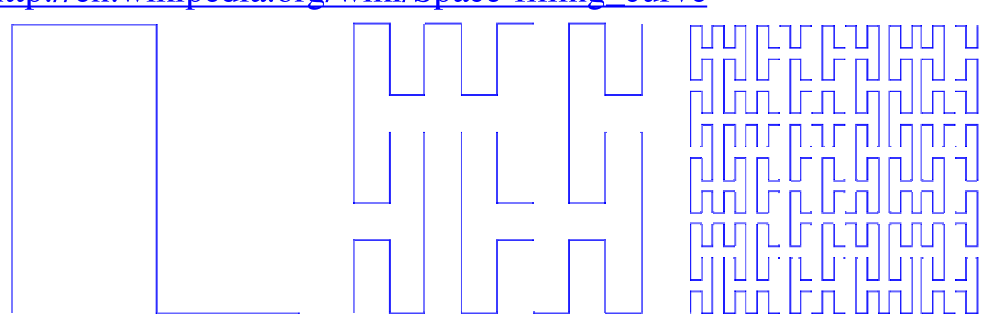
Proof. Extend one of the first interval of the internal line. It will cut the external polygon into 2 parts. If we drop the lower part, the perimeter of the external curve decreases. Now we may omit the common side of two curves which we have created, and reduce the number of the sides of the upper line, so upper line is shorter by induction.

5*. Does there exist a

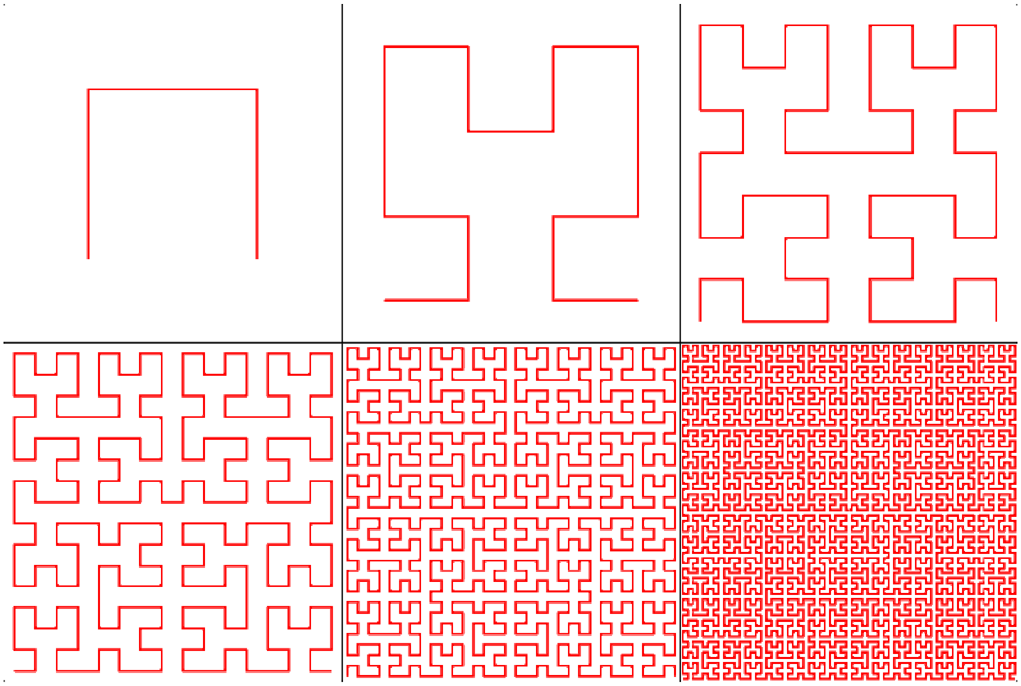
a) continuous
b) monotone
c) continuously differentiable

function $f:[0,1]\to[0,1]$ s. t. every $y\in[0,1]$ it has uncountable number of inverse images of y ?
Uncountable means strictly bigger than $\aleph_0$.
Inverse image of y is such an x that $f(x) = y$.

Solution. a) Consider Peano curve, which is a continuous curve covering the square, or (which is the same) an onto map from [0,1] to [0,1]×[0,1].
The first coordinate of Peano curve is a continuous function, and inverse image has at least one point for each possible value of second coordinate, which is $2^{\aleph_0}$.
Peano curve is constructed by limiting procedure like this:
http://en.wikipedia.org/wiki/Space-filling_curve



Here is another example by Hilbert:



b) No. Inverse image of a point under monotone function is either a point or an interval (if two points belong to inverse image, than all intermediate points also do). Each interval contains a rational point.
If we would have such a function, then inverse image of each point would have a rational point. That would give an injective (חח"ע) mapping from [0,1] to the countable set of rational numbers, which can't exist.

c) No. Let f be such a map. Then for each value y of this map there is an x such that $y = f(x)$ and $f'(x) = 0$, since inverse image of y contains an accumulation point and clearly derivative at the accumulation point should be 0.
Every x such that $f'(x) = 0$ belongs to an open interval on which $|f'(x)| \leq \varepsilon$. Union of those intervals for all y's can be represented as disjoint union of intervals. Total length of those intervals ≤ 1, so f will send union of those intervals to the union of intervals of total length $\leq \varepsilon$.
If $\varepsilon < 1$, then we see that image of these intervals can't cover everything, QED.
Remark. If we choose smaller and smaller ε, we see that the image of these points such that $f'(x) = 0$ is of measure 0. This is actually the 1-simensional case of the famous Sard's theorem.

Definitions. (1) A function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called ***differentiable***, if it can be locally, at each point, approximated by linear function $l(x) = Ax + b$,

where A and b are matrix and vector of appropriate size (function depends on the point at which we approximate).
(2) Linear transformation, corresponding to A matrix, which approximates differentiable function at a point, is called ***differential*** of a function.
(3) If rank(differential at point x) < dimension(target space) then x is called ***critical point***.
(4) If x is critical point, $f(x)$ is called ***critical value***.

Theorem (Sard). The set of critical values is of measure 0.

6.** For a continuous function $f:\mathbb{R}\to\mathbb{R}$ it is given that any positive

real x, y the sequence $f(x+ny)$, for $n\in\mathbb{N}$, tends to infinity.

Does it follow that $f(x)\to\infty$ as $x\to\infty$?

It is similar to the problem we had in the competition, but now we require continuity and the answer is different.

Solution. Yes, it does.
Suppose it doesn't. Then for some M there is a sequence x_k, converging to infinity, such that $f(x_k) < M$. Then, since f is continuous, for $|x - x_k| < \varepsilon_k$, we have $f(x_k) < 2M = N$, where ε_k are small numbers, chosen separately for different k. So, to get the contradiction we need to do one thing: build an arithmetic sequence which intersects infinite subset of these small intervals.
Assume we have built a sequence $\{ny\}$ which intersects K intervals:
$n_k y\in\left(x_{m_k}-\varepsilon_{m_k}, x_{m_k}+\varepsilon_{m_k}\right)$, for some indices n_k, m_k for $k\le K$.
We can move the y in certain interval so that the conditions
$n_k y\in\left(x_{m_k}-\varepsilon_{m_k}, x_{m_k}+\varepsilon_{m_k}\right)$ still hold for the same n_k, m_k, because intersection of open intervals is still an open interval, if it is nonempty.
We shall find such y that satisfies this condition for as large K as we want by induction over K. For $K = 1$ it is obvious.
Assume that $n_k y\in\left(x_{m_k}-\varepsilon_{m_k}, x_{m_k}+\varepsilon_{m_k}\right)$, for given n_k, m_k for $k\le K$, in the interval $y\in\left(\alpha Y, Y\right)$, where $\alpha < 1$.
Possible values of ny will cover the interval between ny and $(n+1)y$ if $\left(n+1\right)\alpha Y < nY$ i. e. $1+\frac{1}{n}=\frac{n+1}{n}<\frac{1}{\alpha}$, or $n>\frac{\alpha}{1-\alpha}$.

So, all numbers above $Y\frac{\alpha}{\alpha-1}$ will be covered by possible values of ny.
But x_k tends to infinity, so we can choose x_m such that it can be equal to ny for a certain value of y in the interval. This completes the induction.
By this inductive procedure we shall build an infinite set of indices m_k and a nested system of intervals $(\alpha_k Y_k, Y_k)$ such that if we choose y in interval number K then $\{ny\}$ intersects intervals $\left(x_{m_k}-\varepsilon_{m_k}, x_{m_k}+\varepsilon_{m_k}\right)$ for $k \le K$, the intersection of all those intervals has at least on point y, and for that y sequence $\{ny\}$ intersects infinite number of intervals. Hence $f(ny)$ doesn't tend to infinity, contradiction, QED.

Selection of Israeli Team for IMC 2008.

1. A group is generated by two elements a,b. The following relations hold:

$$a^2 = 1$$
$$b^2 = 1$$
$$(ab)^{10} = 1$$

Find the maximal possible size of this group.

2. For $f : \mathbb{R} \to \mathbb{R}$ it is given that any positive real x, y the sequence $f(x+ny)$,

for $n \in \mathbb{N}$, tends to infinity. Does it follow that $f(x) \to \infty$ as $x \to \infty$?

3. Denote $[X, Y] = XY - YX$ (it is called commutator). Assuming that A, B, C are 2×2 matrices, prove that $\left[[A, B]^2, C\right] = 0$.

4. For which N can we draw a full graph of N vertices on the plane, so that each arc will be intersected no more than once, and no three arcs would have a common inner point?
(A graph is called ***full***, if each two vertices are connected by an arc.)

5. For tetrahedron ABCD, an altitude (גובה) is a straight line passing trough one vertex and orthogonal to the plane containing 3 other vertexes.
It is given, that no two edges of ABCD are orthogonal.
Prove that there exists a straight line, passing through vertex A and having a common point with each tetrahedron's altitude.

Good luck!

Selection of Israeli Team for IMC 2008.

1. A group is generated by two elements a,b. The following relations hold:

$$a^2 = 1$$
$$b^2 = 1$$
$$(ab)^{10} = 1$$

Find the maximal possible size of this group.

Answer. 20.

Solution. $a^{-1} = a$, $b^{-1} = b$, hence every element in the group can be expressed as an a product of a's and b's .

But $ab\ ab\ ab\ ab\ ab\ ab\ ab\ ab\ ab\ a = b$, hence every element can be expressed as a word starting with a. Represent each element by the shortest word starting with a. It cannot have aa or bb in the middle, otherwise we could make it shorter. So, a's and b's alternate in this word. So the word is $abab$… and it is determined by its length. We cannot have more than 20 symbols, since they would cancel out. So there are no more, than 20 elements in the group.

Now let's see an example of such a group with precisely 20 elements. Consider the group of symmetries of regular 10-gon. There are exactly 20. Let a be a mirror reflection with respect to an orthogonal bisector of two opposite sides, and b be a mirror reflection with respect to a diagonal, passing through two opposite vertices of those sides. Then ab is a rotation by 36° hence $(ab)^{10} = 1$. Since a, b are reflections $a^2 = b^2 = 1$. It is easy to see that the group is generated by a and b, since ab generates all rotations, it also has 2 reflections, so we have more than 10 elements of the group generated by a and b. The order of subgroup generated by a, b, is a divisor of 20 which is bigger than 10, so it is 20.

2. For $f : \mathbb{R} \to \mathbb{R}$ it is given that any positive real x, y the sequence $f(x+ny)$,

for $n \in \mathbb{N}$, tends to infinity. Does it follow that $f(x) \to \infty$ as $x \to \infty$?

Answer. No.

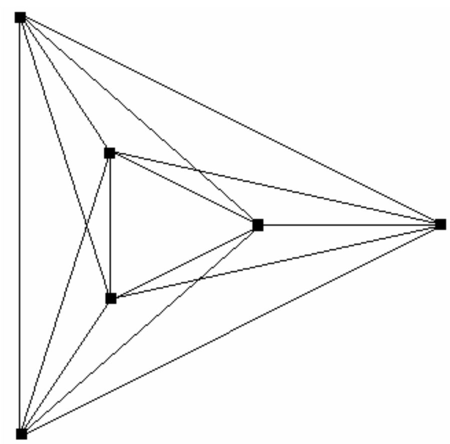
Solution. Let a > 1 be a transcendent number, and consider the following function: $f(a^n) = 0$ for every natural n, and $f(x) = x^2$ for all other points.

Any arithmetic progression can have no more than two common points with the sequence a^n, since if it would have 3 common points, a would be a root of a polynomial with rational coefficients. Hence any sequence $f(x+ny)$ tends to infinity, and $f(x)$ doesn't.

3. Denote $[X,Y] = XY - YX$ (it is called commutator). Assuming that A, B, C are 2×2 matrices, prove that $\left[[A,B]^2, C\right] = 0$.

Solution. tr XY = tr YX. Hence tr(XY – YX) = 0.
Consider matrix $[A,B]$ in its Jordan form. It has either one Jordan cell 2×2 or two Jordan cells 1×1. Trace = 0, so in the first case this matrix is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and in the second case $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$. In both cases the $[A,B]^2 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, so it commutes with any C.

4. For which N can we draw a full graph of N vertices on the plane, so that each arc will be intersected no more than once, and no three arcs would have a common inner point?



Solution. $N < 7$.
The picture shows a graph for $N = 6$, such pictures for smaller N can be obtained by erasing points from this picture.
Suppose we have a full graph for 7 vertices, which satisfies all conditions. It has 21 edges, and X crossings between edges.
Let O be a crossing point, between arcs AC and BD. Consider quadrangle defined by arcs AB, BC, CD, and DA. We can assume that its edges are uncrossed, because if an edge would enter triangle ABO by crossing AB, it will have to exit the triangle through AB, AO, or BO and make second crossing with an edge which is crossed already. Unless the edge enters ABO through AB and stops at A or at B, but then we can shift the edges a bit and eliminate that intersection.
So, if we draw a graph with minimal possible number of crossings, then around each crossing we shall have 4 uncrossed edges. Here each uncrossed edge can be counted at most twice, hence the number of uncrossed edges is at least 2X. The number of crossed edges is precisely 2X, so 4X < 21, hence X ≤ 5.

Consider a graph whose vertices are points of crossing together with the original vertices, and edges are original uncrossed edges and halves of crossed edges. This graph, unlike the original, is planar. It has V=7+X vertices, E = 21+2X edges, and F ≤ ⅔E faces hence by Euler's formula

$$2 = F - E + V \le V - E/3 = 7 + X - (7 + 2X/3) = X/3$$

Hence $6 \le X$.

But before we proved $X \le 5$, a contradiction.

Since we cannot build such a graph for 7, we can't do it for higher *N* either.

5. For tetrahedron ABCD, an altitude (גובה) is a straight line passing through one vertex and orthogonal to the plane containing 3 other vertexes.

Prove that there exists a straight line, passing through vertex A and having a common point with each tetrahedron's altitude.

Solution. Assume A is (0,0,0), denote vectors u = AB, v = AC, w = AD. Normal vector to the plane ABC is $[u,v]$, the vector product, each vector x in that plane is orthogonal to $[u,v]$. The plane, passing through A, and containing the altitude from D, is contains vectors w and $[u,v]$, so it's normal vector is $[[u,v],w]$, (vectors w and $[u,v]$ are parallel, since it is given, that w is not orthogonal to u and v). Hence any line, passing through A and orthogonal to $[[u,v],w]$, intersects the altitude from D, unless they are parallel. So, we need to have a line which is orthogonal to $[[u,v],w]$ and two similar vectors $[[w,u],v]$ and $[[v,w],u]$, but at the same time, that line should not be parallel to any of altitudes from B, C, D, which are parallel to the vectors $[u,v]$, $[w,u]$, $[v,w]$. Recall Jacobi identity:

$$[[u,v],w] + [[w,u],v] + [[v,w],u] = 0$$

If You don't, prove it by applying thrice a well known formula of analytic geometry:

$$[[u,v],w] = (u,w)v - (v,w)u$$

From Jacobi identity, it follows that vectors $[[u,v],w]$, $[[w,u],v]$, $[[v,w],u]$ really are coplanar and have a common orthogonal vector. It still remains to prove that this vector can't be, say, $[u,v]$.

Assume that $[u,v]$ is orthogonal to $[[v,w],u]$.

Then there are real numbers a, b such that

$$a[v,w]+ bu = [u,v]$$

Take scalar product with v, and we get

$$b(u,v) = 0$$

But u isn't orthogonal to v, hence $b = 0$, so $a[v,w]= [u,v]$, so normal vectors of two different faces are parallel, so two different faces are parallel, which is impossible.

אולימפיאדת סטודנטים-מבחן מיון.

תשס"ח (2007-2008) – שלב ראשון.

משך המבחן 4 שעות.

1. תהא $\mathbf{X}$ מטריצה 2×2 מעל הרציונאליים,
אשר מקיימת $\mathbf{X}^2 = 2\mathbf{I}$, כאשר $\mathbf{I}$ זו מטריצת יחידה.
א. חשב את $\operatorname{trace} \mathbf{X}$
ב. חשב את $\det \mathbf{X}$

2. תהא f פונקציה על הממשיים בעלת נגזרת רציפה מכל סדר אשר מקיימת $|f(x)| < A$ לכל x.
הוכח כי $f''(x)$ (נגזרת שנייה של f) מתאפסת בנקודה מסוימת.

3. בוחרים N נקודות מקריות באופן בלתי תלוי ובהתפלגות אחידה בתוך הריבוע $[0,1]\times[0,1]$.
נקודה (x,y) שנבחרה נקראת נקודה מיוחדת אם לא נבחרו נקודות אחרות במלבן $[0,x]\times[0,y]$.
נסמן ב M את מספר הנקודות המיוחדות מבין ה N שנבחרו.
חשב את התוחלת של M.

4. נתונים 6 מספרים ממשיים $x_1, y_1, z_1, x_2, y_2, z_2$.
נתבונן בקבוצת כל הנקודות שמקיימות את שני האי-שוויונים:

$$z_1 - z > \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

$$z - z_2 > \sqrt{(x-x_2)^2 + (y-y_2)^2}$$

חשב את הנפח של קבוצה זו.

5. נתונה שורת נורות אינסופית. כל נורה נמצאת במצב דלוק או כבוי.
כל דקה מכבים כל נורה ששתי הנורות שנמצאות לידה נמצאות במצב שונה,
ומדליקים כל נורה ששתי הנורות שלידה היו במצב זהה.
נמספר את הנורות באמצעות המספרים השלמים $\mathbf{Z}$.
נסמן ב $f(i,t)$ את מצב הנורה i בזמן t (1 אם הנורה דולקת, 0 אחרת).
התאורה נקראת מחזורית לפי זמן אם קיים A שלם שונה מ-0 כך ש $f(i, t+A) = f(i,t)$ לכל i,t.
התאורה נקראת מחזורית לפי מרחב אם קיים B שלם שונה מ-0 כך ש $f(i+B, t) = f(i, t)$ לכל i,t. יש להוכיח כי
א. אם התאורה מחזורית לפי מרחב אזי היא מחזורית לפי זמן.
ב. אם התאורה מחזורית לפי זמן אזי היא מחזורית לפי מרחב.

בהצלחה!

אולימפיאדת סטודנטים-מבחן מיון.

תשס"ח (2007-2008) – שלב ראשון.

פתרונות

1. תהא $\mathbf{X}$ מטריצה 2×2 מעל הרציונאליים,
אשר מקיימת $\mathbf{X}^2 = 2\mathbf{I}$, כאשר $\mathbf{I}$ זו מטריצת יחידה.
א. חשב את $\text{trace}\,\mathbf{X}$
ב. חשב את $\det \mathbf{X}$

קל לראות שמטריצות כאלה קיימות, למשל $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. לכן קיימת לפחות תשובה אחת.

פתרון ראשון.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & bc+d^2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(a+d)b = (a+d)c = 0$$

$$a^2+bc = bc+d^2 = 2$$

אם $b=0$ אז $a^2=2$, סתירה כי a רציונלי.

לכן $b\neq 0$ אבל $(a+d)b=0$ כלומר $\text{trace}\,X = a+d = 0$.

נציב $a=-d$ במשוואה $a^2+bc=2$ ונקבל $-ad+bc=2$ כלומר

$$\det X = ad-bc = -2$$

פתרון שני. ערכים עצמיים של מטריצה מקיימים $x^2-2=0$ כלומר $\pm\sqrt{2}$.
סכום של ערכים עצמיים זה trace והוא רציונלי, לכן ערך עצמי אחד $-\sqrt{2}$ והשני $\sqrt{2}$.
(אילו $-\sqrt{2}$ או $\sqrt{2}$ היה מופיע פעמיים, סכומם היה אי-רציונלי).
לכן trace = סכום ערכים עצמיים = 0,
det = מכפלת ערכים עצמיים = -2 .

2. תהא f פונקציה על הממשיים בעלת נגזרת רציפה מכל סדר אשר מקיימת $|f(x)| < A$
לכל x. הוכח כי $f''(x)$ (נגזרת שנייה של f) מתאפסת בנקודה מסוימת.

פתרון ראשון. נניח אחרת - $f''(x)$ לא מתאפסת באף נקודה.
אפשר להכפיל את f ב- 1- בלי לפגוע בתנאים, וזה יחליף סימן גם לנגזרת שנייה.
לכן אפשר להניח בלי הגבלת הכלליות כי $0 < f''(x)$ לכל x.
לכן $f'(x)$ עולה ממש.
אם פונקציה $f'(x)$ מקבלת ערך חיובי K בנקודה מסוימת אז החל מהנקודה הזאת היא גדולה
מ-K לכן $f(x)$ שהיא אינטגרל של $f'(x)$ עולה יותר מהר מאשר פונקציה ליניארית עולה

$Kx+L$ לכן היא לא חסומה, סתירה. באופן דומה, אם $f'(x)$ מקבלת ערך שלילי בנקודה מסוימת, אז לפני הנקודה הזאת $f(x)$ יורדת בקצב יותר מהר לכן גם במקרה זה היא יותר גדולה מפונקציה ליניארית שמקבלת ערכים גדולים מאוד עבור x ששואף למינוס אינסוף.

פתרון שני. אפשר לנסח את אותו הדבר בלשון גיאומטרי.

כמו קודם, אפשר להניח כי $f''(x) > 0$ לכל x.

זאת אומרת שפונקציה f קמורה. לכן הגרף שלה נמצא מעל כל משיק. אם פונקציה נמצאת מעל פונקציה ליניארית וחסומה, אז הפונקציה ליניארית בעלת שיפוע 0.

לכן $f'(x) = 0$ לכל x. כלומר $f''(x) = 0$, סתירה.

פתרון שלישי. כמו קודם, אפשר להניח כי $f''(x) > 0$ לכל x.

נרשום טור טיילור עם שארית בצורת לגרנז':

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_1)(x-x_0)^2$$

כאשר x_1 נמצאת בין x לבין x_0. אבל בכל מקרה $f''(x_1)(x-x_0)^2$ חיובי, לכן

$$f(x) > f(x_0) + f'(x_0)(x-x_0)$$

אם $f'(x_0)$ אינו מתאפס, אז אפשר לבחור x כך שהאגף הימני של האי-שוויון (ולכן גם השמאלי) יהיה גדול כרצוננו.

אם לא, ואי-אפשר לבחור x_0 כזה כך ש-$f'(x_0)$ אינו מתאפס אז f' הוא 0 זהותי וגם f''. סתירה.

3. בוחרים N נקודות מקריות באופן בלתי תלוי ובהתפלגות אחידה בתוך הריבוע $[0,1]\times[0,1]$.

נקודה (x,y) שנבחרה נקראת נקודה מיוחדת אם לא נבחרו נקודות אחרות במלבן $[0,x]\times[0,y]$.

נסמן ב M את מספר הנקודות המיוחדות מבין ה N שנבחרו.

חשב את התוחלת של M.

פתרון ראשון. את הנקודות אפשר למיין בסדר עולה לפי x או לפי y, הסדרים האלה יהיו בלתי תלויים כי קואורדינאטות שונות של כל נקודה לא תלויות אחת בשנייה.

הנקודה הראשונה לפי x, היא מיוחדת בסיכוי של 1.

הנקודה השנייה לפי x, היא מיוחדת בסיכוי של ½.

הנקודה השלישית לפי x, היא מיוחדת בסיכוי של $^1\!/_3$.

…

הנקודה מספר N לפי x, היא מיוחדת בסיכוי של $^1\!/_N$.

כמות של נקודות מיוחדות זה סכום של אינדיקאטורים של מיוחדות של נקודה מספר K לפי x (כלומר משתנים מקריים ששווים ל-1 אם נקודה מספר K לפי x ושווים ל-0 אחרת).

לכן התוחלת שאנו מחפשים היא סכום התוחלת של האינדיקאטורים האלה כלומר

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{N}\approx \ln N+\gamma$$

פתרון שני (ראש בקיר). ניקח נקודה, שנמצאת במיקום (x,y). היא מיוחדת אם כל נקודה אחרת נמצאת מחוץ למלבן ששטחו xy. הסתברות לנקודה אחרת להימצא מחוץ למלבן זה $1-xy$ והסתברות שכל הנקודות כך היא $\left(1-xy\right)^{N-1}$.

נגדיר משתנה מקרי ששווה 1 כאשר נקודה נתונה מיוחדת ושווה 0 אחרת. תוחלת של משתנה מקרי זה שווה $\int_0^1\int_0^1\left(1-xy\right)^{N-1}dxdy$ ותוחלת של כמות הנקודות המיוחדת גדולה פי N כי הכמות שווה לסכום של N מ"מ כאלה. לכן התשובה מתקבלת ע"י חישוב אינטגרל.

$$N\int_0^1\int_0^1\left(1-xy\right)^{N-1}dxdy=\int_0^1\left(\left.\frac{\left(1-xy\right)^N}{-y}\right|_0^1\right)dy=\int_0^1\left(\frac{\left(1-y\right)^N}{-y}-\frac{1}{-y}\right)dy=$$

$$=\int_0^1\left(\frac{1-\left(1-y\right)^N}{y}\right)dy=\int_0^1\sum_{k=1}^N\binom{N}{k}\left(-y\right)^{k-1}dy=\sum_{k=1}^N\binom{N}{k}\frac{\left(-1\right)^{k-1}}{k}$$

התשובה $\sum_{k=1}^N\binom{N}{k}\frac{\left(-1\right)^{k-1}}{k}$ היא נכונה אבל היא פחות יפה ואלגנטית מאשר $1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{\mathrm{N}}$, וכמובן, שתי התשובות זהות.

פתרון שלישי. נלך בשיטה השנייה (ראש בקיר) עד שנקבל את התשובה בתור אינטגרל

$$\int_0^1\left(\frac{1-\left(1-y\right)^N}{y}\right)dy$$

כרגע, במקום לפתוח סוגריים נבצע הצבה $z=1-y$ ונקבל

$$\int_0^1\left(\frac{1-\left(1-y\right)^N}{y}\right)dy=\int_0^1\left(\frac{1-z^N}{1-z}\right)dz=\int_0^1\left(1+z+z^2+...+z^{N-1}\right)dz=$$

$$=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{\mathrm{N}}$$

4. נתונים 6 מספרים ממשיים $x_1, y_1, z_1, x_2, y_2, z_2$.

נתבונן בקבוצת כל הנקודות שמקיימות את שני האי-שוויונים:

$$z_1-z>\sqrt{(x-x_1)^2+(y-y_1)^2}$$

$$z-z_2>\sqrt{(x-x_2)^2+(y-y_2)^2}$$

חשב את הנפח של קבוצה זו.

פתרון ראשון (גיאומטריה אנליטית).
קל לראות שכל אי-שוויון מגדיר חרוט. בשביל שיהיה לשני חרוטים האלה חיתוך לא ריק, צריך שאחד יכיל את הקודקוד של השני. אכן אם נחבר את אי-שוויונים נקבל

$$z_1 - z_2 > \sqrt{(x-x_1)^2 + (y-y_1)^2} + \sqrt{(x-x_2)^2 + (y-y_2)^2}$$

אבל לפי אי-שוויון המשולש

$$\sqrt{(x-x_1)^2 + (y-y_1)^2} + \sqrt{(x-x_2)^2 + (y-y_2)^2} \ge \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

הצורה תהיה חסומה בכל מקרה, הרי אפילו

$$\begin{cases} z_1 > z \\ z - z_2 > \sqrt{(x-x_2)^2 + (y-y_2)^2} \end{cases}$$

מגדיר צורה חסומה.
לצורה יש גבול שחלקו – המשטח של החרוט הראשון וחלקו – משטח של החרוט השני.
אז חייב להיות קו חסום שמפריד בין שני החלקים של הגבול.
חיתוך בין שני משטחי החרוטים הוא קו. הוא מקיים שתי משוואות

$$z_1 - z = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

$$z - z_2 = \sqrt{(x-x_2)^2 + (y-y_2)^2}$$

נעלה את שתי המשוואות בריבוע ונחסיר. כאשר מעלים משוואה כזאת בריבוע מקבלים משוואה מסדר 2 שבאגף ימני יש את $x^2 + y^2$ ובאגף שמאלי יש את z^2 וכאשר נחסיר איברים ריבועים יצטמצמו ונקבל משוואה ליניארית. לכן קו ההפרדה נמצא במישור.
חתך של חרוט כזה ע"י מישור הוא כידוע אליפסה, פרבולה או ענף של היפרבולה. אצלנו יש קו חסום, לכן זה אליפסה. המישור הזה חותך את צורה לשני חלקים, שכל אחד מהם הוא "כוס גלידה" (כלומר חרות שנחתך ע"י מישור לאורך אליפסה).
נפח של כזה חרות, כמו של פירמידה שווה ל Sh/3 כאשר S שטח האליפסה, h גובה החרות (כלומר מרחק מקודקוד למישור האליפסה).
נסמן $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$. התמונה סימטרית ביחס ל-M שהוא אמצע AB.
לכן בעצם שתי הגלידות חופפות.
נעביר מישור P שעובר דרך A, B, שעובר דרך (או מכיל את) ציר ה-z. למישור זה יש שתי נקודות שמשותפות לשני החרוטים E, F. כל אחד משני החרוטים סימטרי ביחס למישור P.
לכן EF אחד מצירי האליפסה. הציר השני של אליפסה מאונך ל-EF.
באמצעות גיאומטריה אנליטית אפשר לחשב את שני חצאי הצירים ואת הגובה של הגלידה, להכפיל את שלושתם ולהכפיל במקדם. זה פחות מעמוד של חישובים, אבל אנחנו נוותר על התענוג (ונשאיר אותו לקורא העקשן).

פתרון שני. (פתרון גיאומטרי, דומה לפתרון שכתב אלכסיי גלדקיך בתחרות).
נסמן $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$. התמונה סימטרית ביחס ל-M שהוא אמצע AB. נעביר מישור P שעובר דרך A, B, שעובר דרך (או מכיל את) ציר ה-z. למישור זה יש שתי נקודות שמשותפות לשני החרוטים E, F. כל אחד משני החרוטים סימטרי ביחס למישור P.
נעביר מישור Q שמאונך למישור P ומכיל את E, F. הוא חותך כל קונוס לאורך אליפסה. אבל אליפסה זו סימטרית לגבי מישור P, לכן EF הוא הקוטר שלה. לכן כל אליפסה כזאת סימטרית ביחס ל-M. לכן זו אותה אליפסה.
לכן הצורה שאנו מעוניינים בה מורכבת משני חלקים זהים, שכל אחד מהם מחרות שנחתך ע"י מישור לאורך אליפסה. נפח של כל חלק שווה ל Sh/3 כאשר S שטח האליפסה, h גובה החרות (כלומר מרחק מקודקוד למישור האליפסה). לכן נפח שצריך למצוא הוא 2Sh/3.
נשאר לחשב את S ואת h.
אפשר לבטא את השטח של אליפסה באמצעות πpq כאשר p,q הם חצאי צירים.
אבל אנו נזדקק לביטוי אחר.

טענה. נניח שמוקד של אליפסה מחלק את הציר הארוך לקטעים באורך a,b. אז שטח האליפסה $\pi\sqrt{ab}\dfrac{a+b}{2}$.

הוכחת טענה. ברור שחצי הציר הארוך של האליפסה הוא $\dfrac{a+b}{2}$.
חצי הציר הקצר q הוא ניצב במשולש ישר זווית שקודקודיו: מוקד האליפסה, מרכז האליפסה ונקודה על האליפסה שהכי קרובה למרכז. כל להבין שהיתר $\dfrac{a+b}{2}$ והניצב השני הוא $\dfrac{a-b}{2}$ ומכאן לפי משפט פיתגורס מתקבל q.

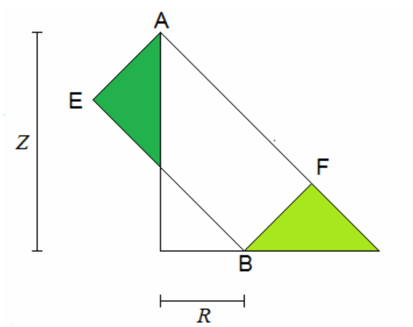
נצייר את הכדור שמשיק לחרות שקודקודו A ולמישור Q של אליפסת החיתוך. זהו כדור דנדלין (http://en.wikipedia.org/wiki/Dandelin_spheres) של האליפסה והוא, כידוע, משיק למישור של האליפסה במוקד שלה.
נתבונן במישור P. יש בוא את המלבן AEBF, והכדור שציירנו הוא מעגל חסום במשולש AEF.
הציר הארוך של האליפסה זה קטע EF. המעגל החסום של AEF משיק ל-EF בנקודה T.
אנו צריכים לחשב את $b=\mathrm{FT}$, $a=\mathrm{ET}$, ואת h שהוא גובה של המשולש EFA מהזווית הישרה A ואז, לפי הטענה שכבר הוכחנו , התשובה תהיה $\dfrac{2}{3}\pi h\sqrt{ab}\dfrac{a+b}{2}$.
נשאיר חלק קטן מהחישוב עבר קוראים בתור תרגיל:
תרגיל. נניח שבמשולש ישר זווית המעגל החסום מחלק את היתר לקטעים באורך a,b אז שטח המשולש היא ab.

אבל $S_{\text{AFE}}=\dfrac{h(a+b)}{2}$.

לכן התשובה מקבלת צורה: $\dfrac{2}{3}\pi h\sqrt{ab}\,\dfrac{a+b}{2}=\dfrac{2\pi}{3}\sqrt{S_{\text{AFE}}}\cdot S_{\text{AFE}}=\dfrac{2\pi}{3}\left(S_{\text{AFE}}\right)^{3/2}$

שטח של משולש AFE שווה למחצית שטח המלבן AEBF.

נסמן $Z=z_2-z_1\,,R=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$. בציור רואים כיצד לחשב את השטח של AEBF. חותכים ממנו משולש ומעבירים אותו לצד השני.



אז נוצר טרפז, ששטחו שווה להפרש של שני שטחי משולשים שווי-צלעות וישרי זוויות, לגדול יש ניצבים באורך Z ולקטן ניצבים באורך R.

$$2S_{\text{AFE}}=S_{\text{AFBE}}=\frac{Z^2-R^2}{2}$$

$$\sqrt{S_{\text{AFE}}}=\frac{\sqrt{Z^2-R^2}}{2}$$

ומכאן התשובה הסופית:

$$\frac{2\pi}{3}\left(S_{\text{AFE}}\right)^{3/2}=\frac{2\pi}{3}\left(\frac{\sqrt{Z^2-R^2}}{2}\right)^3=\frac{\pi}{12}\left(Z^2-R^2\right)^{3/2}$$

פתרון שלישי (מקורי). נבצע טרנספורמציית ששומרות על נפח ועל צורה של חרוטים. קודם כל, אפשר לבצע הזזה – מחסירים את הווקטור (x_2,y_2,z_2) מכל הווקטורים ואז השאלה מקבלת רישום פשוט יותר. בנוסף, אפשר לבצע סיבובים של מישור xy, זה שומר גם על נפח וגם על צורה של חרוטים. בצורה כזאת אפשר להגיע למצב שקודקוד של חרוט אחד הוא $(x_1,0,z_1)$ ושל השני $(0,0,0)$, כאשר בעצם x_1,z_1 השתנו.

יש עוד העתקה ששומרת על תבנית ריבועית $z^2-x^2-y^2$ ועל צורת החרוטים האלה.

נניח כי $S<C$ ומתקיים $C^2-S^2=1$. נתבונן בהעתקה ליניארית שמוגדרת ע"י מטריצה:

$$\begin{pmatrix} C & 0 & S \\ 0 & 1 & 0 \\ S & 0 & C \end{pmatrix}$$

ברור שדטרמיננטה =1, לכן היא שומרת נפח. קל לבדוק גם שהיא שומרת (על התבנית הריבועית $z^2-x^2-y^2$, באמצעות הצבה ישירה או קפל מטריצות:

$$\begin{pmatrix} C & 0 & S \\ 0 & 1 & 0 \\ S & 0 & C \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} C & 0 & S \\ 0 & 1 & 0 \\ S & 0 & C \end{pmatrix}=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

נבחר $S=\frac{-x_1}{\sqrt{z_1^2-x_1^2}}, C=\frac{z_1}{\sqrt{z_1^2-x_1^2}}$ (ברור כי $z_1 > x_1$ אחרת החרוטים לא נחתכים ונפח החיתוך הוא 0). אז העתקה ליניארית זו תעביר את הנקודה $(x_1, 0, z_1)$ לנקודה על ציר z.
ובכן הגענו למצב שקודקודי החרוטים הם $(0,0,0)$, $(0,0,H)$. הצורה שהתקבלה היא איחוד של שני חרוטים שגובה של כל אחד מהם R ורדיוס של הבסיס גם R כאשר $H/2 = R$.
לכן נפח של כל חלק הוא $\pi R^3/3$ ונפח הכולל $\frac{2}{3}\pi R^3$.
נשאר לבטא את R במושגים של קואורדינאטות שהיו בהתחלה. אבל כל הפעולות שעשינו שומרות על הביטוי $(z_1-z_2)^2-(x_1-x_2)^2-(y_1-y_2)^2$ ולכן

$$H=\sqrt{(z_1-z_2)^2-(x_1-x_2)^2-(y_1-y_2)^2}$$

$$R=\frac{\sqrt{(z_1-z_2)^2-(x_1-x_2)^2-(y_1-y_2)^2}}{2}$$

והתשובה הסופית היא

$$\frac{2}{3}\pi R^3=\frac{\pi}{12}\left((z_1-z_2)^2-(x_1-x_2)^2-(y_1-y_2)^2\right)^{3/2}$$

הערה. העתקות ששומרות על תבנית $x^2+y^2-z^2$ יוצרות חבורה שידוע בשם O(2,1) שמופיעה הרבה בגיאומטריה היפרבולית ובתורת היחסות הפרטית.

5. נתונה שורת נורות אינסופית. כל נורה נמצאת במצב דלוק או כבוי.
כל דקה מכבים כל נורה ששתי הנורות שנמצאות לידה נמצאות במצב שונה,
ומדליקים כל נורה ששתי הנורות שלידה היו במצב זהה.
נמספר את הנורות באמצעות המספרים השלמים **Z**.
נסמן ב $f(i,t)$ את מצב הנורה i בזמן t (1 אם הנורה דולקת, 0 אחרת).
התאורה נקראת מחזורית לפי זמן אם קיים A שלם שונה מ-0 כך ש $f(i, t+A) = f(i,t)$
לכל i,t.
התאורה נקראת מחזורית לפי מרחב אם קיים B שלם שונה מ-0 כך ש $f(i+B, t) = f(i, t)$.
לכל i,t. יש להוכיח כי

א. אם התאורה מחזורית לפי מרחב אזי היא מחזורית לפי זמן.
ב. אם התאורה מחזורית לפי זמן אזי היא מחזורית לפי מרחב.

פתרון. הבעיה מבוססת על עיקרון פשוט. אם נתונה קבוצה סופית (שמכילה N איברים), ונתון כלל שקובע לפי איבר נוכחי, איזה איבר צריך לקחת אחרי איבר נוכחי. נניח שאנחנו מתחילים באיבר מסוים ומפעילים את הפעולה הזאת הרבה פעמים. אז סדרת איברים שמתקבלת תתחיל להיות מחזורית (אחרי N פעולות לכל היותר).
באולימפיאדת SEEMOUS ב-2007 שתי בעיות מתוך 4 היו מבוססות על העיקרון הזה.

קל מאוד להוכיח את העיקרון הזה. אחרי צעד $N+1$ יופיע איבר שכבר היה קודם, אחרי צעד K למשל. בגלל שהמשך הסדרה נקבע ע"י איבר בודד אז המשך הסדרה אחרי איבר K זהה להמשך הסדרה אחרי איבר $N+1$ לכן מאיבר K והלאה יש מחזור $N + 1 – K$.

כמובן אם הסדרה היא אינסופית לשני הכיוונים (כלומר אם אינדקס הולך ממינוס אינסוף עד אינסוף), אז הסדרה היא מחזורית לגמרי ולא רק מחזורית החל ממקום מסוים הרי עבור כל אינדקס נתון M אפשר להתבונן באינדקס $M – N$ ולהסיק שמחזור התחיל כבר קודם.

הייתה בעיה בניסוח של סעיף א' שהרבה סטודנטים שמו לב אליה.
כוונת המשוררים הייתה שהזמן t הוא מספר שלם שהולך ממינוס אינסוף עד פלוס אינסוף.
זה לא צוין בניסוח ורוב האנשים הבינו באופן טבעי שזמן t הוא מספר טבעי.
אז יוצא שהמחזור עלול לא להתחיל ברגע הראשון אלה כעבור זמן מה.
למשל, אם מתחילים במצב ...10101010101... אז המצב הזה כבר לא יחזור על עצמו.
כל הסטודנטים שהבינו את השאלה בצורה הטבעית וציינו שהשאלה שגויה קיבלו ציון מלא על סעיף א'. בכל מקרה, מחברי השאלון מתנצלים על אי-דיוק בניסוח.

א. יש מספר סופי (2^B) של מצבי תאורה מחזוריים במרחב בעלי מחזור B. מצב נוכחי של תאורה קובע באופן חד-משמעי את המצב הבא. לכן, לפי העיקרון שהסברנו, החלפת מצבי תאורה היא מחזורית, כלומר התאורה מחזורית לפי זמן.

ב. נניח שהתאורה מחזורית לפי זמן. לכן להתנהגות של כל נורה ספציפית יש מחזור משלה באורך A. לכן יש רק 2^A דרכי התנהגות מסוימות אפשריות לנורה כלשהי.

לפי הגדרה, מצב נורות $i, i+2$ ברגע t קובע את מצב הנורה $i+1$ ברגע $t+1$. מהכלל שמקשר בין 3 מצבים אלה ברור שמצב של נורה $i+1$ ברגע $t+1$ ושל נורה i ברגע t קובעים את מצב של נורה $i+2$ ברגע t. לכן ברור שמי שיודע את ההתנהגות של נורות $i, i+1$ יכול לקבל גם את ההתנהגות של נורה $i+2$.

לכן יש כלל שמחשב לפי התנהגות של זוג נורות צמודות $i, i+1$ את התנהגות של זוג נורות צמודות $i+1, i+2$. מצד שני, התנהגות של זוג נורות צמודות שייכת לקבוצה סופית של 4^A איברים. לכן לפי העיקרון שהסברנו, זה מחזורי לפי מרחב.

Israeli Team for SEEMOUS
Final Selection Exam

Please write your solutions in English.

1. A ***graph*** is, by definition, a collection of ***vertices*** and a collection of ***edges*** that connect pairs of vertices. Two vertices are called ***adjacent***, if they share an edge.
Given a graph, consider the function $c(n)$ – the number of ways to color each vertex with one of n given colors, so that no two adjacent vertices will have the same color. Show, that $c(n)$ is a polynomial of n.

2. A disc of radius $^1/_N$ is rolling inside the circular box of radius 1, where $N > 2$. (The friction between the edge of the disc and the wall of the box is very high so the disc doesn't slip with respect to the box at the point of tangency). A red point on the boundary of a small circle goes along a star-shaped closed trajectory.
Compute the area, bounded by this trajectory (as a function of N).

3. A natural number k is considered ***good***, if for each N the number $1^k+2^k+\ldots N^k$ is divisible by $1+2+\ldots+N$.
Describe the set of all good numbers.

4. Let $\boldsymbol{A_1, A_2,\ldots,A_N}$ be nonzero matrices $M\times M$ (a matrix is called ***nonzero*** if at least one of its elements is nonzero). Prove that there exists a matrix B of the same size such that $\boldsymbol{BA_1BA_2B\ldots BA_NB}$ is a nonzero matrix.

5. An infinite sequence of real numbers $\{x_i\}$ will be called ***nice*** if $\sum x_i^2$ converges. Let $\{a_i\}$ be a sequence, such that for each nice sequence $\{x_i\}$ the series $\sum a_i x_i$ converges. Prove that the sequence $\{a_i\}$ is nice.

Good luck!

Israeli Team for SEEMOUS
Second Stage Solutions.

1. A ***graph*** is, by definition, a collection of ***vertices*** and a collection of ***edges*** that connect pairs of vertices. Two vertices are called ***adjacent***, if they share an edge.
Given a graph, consider the function $c(n)$ – the number of ways to color each vertex with one of n given colors, so that no two adjacent vertices will have the same color. Show, that $c(n)$ is a polynomial of n.

First solution. Induction over number of vertices + number of edges.
The only graph of 1 vertex gives $c(n) = n$.
Of course, if graph is disconnected function $c(n)$ is a product of functions, corresponding to his connected components, and product of polynomials is a polynomial.
Take two adjacent vertices A, B in a graph. Let us erase the edge AB. Number of ways to color the new graph, $c_1(n)$ is a polynomial by induction (same vertices, less edges). Of those, there are $c(n)$ ways to color it so that A and B will be of different color, and $c_2(n)$ ways to color it so that so that A and B will having the same color. If we shall glue vertices A and B, the new graph will have less edges and less vertices than the original graph, and it can be colored in $c_2(n)$ ways. Hence $c(n) = c_1(n) – c_2(n)$, so it is a difference of two polynomials, hence it is itself polynomial.

Second solution. A way to split the vertices of given graph into certain equivalence classes will be called ***configuration***. Configuration is called good if no to vertices of the same class are adjacent. There is only finite number of configuration.
Each coloring corresponds to a specific configuration: vertices of the same color are declared equivalent. Let us count, how many colorings correspond to the same configuration. Take a configuration which has M classes.
First class can be colored in one of n colors, second in one of n-1 colors, and so on, hence if $M \geq n$ it corresponds to $n(n\text{-}1)(n\text{-}2)\ldots(n\text{-}M\text{+}1)$
If $M < n$ then the product we wrote, as well as the number of colorings, is 0. So, number of colorings corresponding to certain configuration is a polynomial (which we wrote explicitly) and since we have finite number of configurations, the total number of colorings is a sum of finite number of polynomials, which is a polynomial.

2. A disc of radius $^1/_N$ is rolling inside the circular box of radius 1, where $N > 2$. (The friction between the edge of the disc and the wall of the box is very high so the disc doesn't slip with respect to the box at the point of tangency). A red point on the boundary of a small circle goes along a star-shaped closed trajectory.
Compute the area, bounded by this trajectory (as a function of N).

First solution. Let us start by building a parametrical equation of the star.
The center of the disc goes in circles of radius $1-\frac{1}{N}$ so it can be described as $v=\left(\left(1-\frac{1}{N}\right)\cos t,\left(1-\frac{1}{N}\right)\sin t\right)$. The vector which goes from the center of the disc to the red point goes around a circle of radius $\frac{1}{N}$ in the opposite direction, so it can be described as $u=\left(\frac{1}{N}\cos s,-\frac{1}{N}\sin s\right)$. Both parameters depend linearly on the length of the arc that we cover, t.
While the center goes around one time, the red point meets the boundary N times. This means the small discs rotates around itself $N-1$ times, hence
$u=\left(\frac{1}{N}\cos\left((N-1)t\right),-\frac{1}{N}\sin\left((N-1)t\right)\right)$.
The point on the star can be described as $w = u + v$, which is also a vector function of t. A simple way to check we wrote it correctly – differentiating vectors u, v shows that their velocities are equal in their absolute value and that they cancel each other when the red point is near at the boundary (and then its velocity should be 0, because of the friction).
Of course, since $\dot{u}$ looks always directly clockwise and $\dot{v}$ is of the same absolute value the vector will always go clockwise so the star won't have self-intersections.
Integrating $-ydx$ around the star should, as usual, give the area inside.
Minus sign is because the trajectory, the way we have parameterized it, goes clockwise, so the upper boundary must be consider with plus and the lower with minus. So we get the following integral:

$$\int_0^{2\pi}-\left(\left(1-\frac{1}{N}\right)\sin t-\frac{1}{N}\sin\left((N-1)t\right)\right)\frac{d}{dt}\left(\left(1-\frac{1}{N}\right)\cos t+\frac{1}{N}\cos\left((N-1)t\right)\right)dt=$$

$$=\frac{1}{N^2}\int_0^{2\pi}\left((N-1)\sin t-\sin\left((N-1)t\right)\right)(N-1)\left(\sin t+\sin\left((N-1)t\right)\right)dt=$$

$$=\frac{N-1}{N^2}\int_0^{2\pi}(N-1)\sin^2 t-\sin^2\left((N-1)t\right)+(N-2)\sin t\sin\left((N-1)t\right)dt$$

To finish this, it is useful to know the following exercises:

Exercise 1. $\int_0^{2\pi}\sin^2 t dt=\int_0^{2\pi}\sin^2\left((N-1)t\right)dt=\pi$

(Hint: $\sin^2+\cos^2=1$)

Exercise 2. $\int_0^{2\pi}\sin t\sin\left((N-1)t\right)dt=0$

(Hint: $2\sin a\cdot\sin b=\cos(a-b)-\cos(a+b)$)

So, the answer is $\frac{(N-1)(N-2)}{N^2}\pi$.

Second solution. Like before, we describe the position of red point as the sum of two vectors $w = u + v$ where u goes clockwise in a circle of radius $1-\frac{1}{N}$ one time and v goes counter-clockwise in a circle of radius $\frac{1}{N}$, $N-1$ times, but we don't write the coordinates explicitly.

For any to vectors $k=(k_x,k_y),m=(m_x,m_y)$ denote the oriented area of the parallelogram they form $k=(k_x,k_y),m=(m_x,m_y)$. So, in time dt the vector w sweeps area $\frac{dw\times w}{2}$ (since its clockwise) which gives total area

$$\int_0^{2\pi}dw\times w=\int_0^{2\pi}\frac{1}{2}(du+dv)\times(u+v)=\frac{1}{2}\int_0^{2\pi}du\times u+dv\times v+du\times v+dv\times u$$

The integrals $\int_0^{2\pi}du\times v,\int_0^{2\pi}dv\times u$ are 0, since the angle between du and v, as well as dv and u rotates uniformly around 0 and makes several full circles.

$\frac{1}{2}\int_0^{2\pi}du\times u=\pi\left(1-\frac{1}{N}\right)^2$ since u sweeps one circle of radius $\left(1-\frac{1}{N}\right)$.

$\frac{1}{2}\int_0^{2\pi}dv\times v=-(N-1)\pi\left(\frac{1}{N}\right)^2$ since v sweeps $N-1$ circles of radius in the opposite direction. So the total integral is:

$$\frac{1}{2}\int_0^{2\pi} du\times u+dv\times v=\pi\left(1-\frac{1}{N}\right)^2-(N-1)\pi\left(\frac{1}{N}\right)^2=\pi\left(\left(\frac{N-1}{N}\right)^2-(N-1)\left(\frac{1}{N}\right)^2\right)=$$

$$=\pi\frac{N-1}{N^2}(N-1-1)=\pi\frac{(N-1)(N-2)}{N^2}$$

3. A natural number k is considered ***good***, if for each N the number $1^k+2^k+\ldots+N^k$ is divisible by $1+2+\ldots+N$.
Describe the set of all good numbers.

Solution. If k is good, then it 1^k+2^k is divisible by 3. So $1+(-1)^k = 0 \pmod 3$ hence k can't be even.
Suppose now k is odd. $1^k+2^k+\ldots+N^k$ is divisible by $1+2+\ldots+N$ if and only if $2(1^k+2^k+\ldots+N^k)$ is divisible by $2(1^k+2^k+\ldots+N^k)=N(N+1)$.
N and $N+1$ are co-prime, so it is sufficient to verify separately that it is divisible by N and by $N+1$. It is enough to prove $2(1^k+2^k+\ldots+N^k)$ is divisible by $N+1$ for all N, then $2(1^k+2^k+\ldots+(N-1)^k)$ is divisible by N and $2(1^k+2^k+\ldots+N^k)$ also. We shall use "Gauss trick":
$2(1^k+2^k+\ldots+N^k) = 2((1^k+N^k) +(2^k+(N-1)^k) +\ldots+(N^k+1^k))$.
But this is definitely divisible by $N+1$ since a^k+b^k is always divisible by $a+b$ for odd k since $a^k+b^k =(a+b)(a^{k-1} - a^{k-2}b + a^{k-3}b^2 - \ldots + a^{k-1})$.

4. Let $\boldsymbol{A_1, A_2,\ldots,A_N}$ be nonzero matrices $M\times M$ (a matrix is called ***nonzero*** if at least one of its elements is nonzero). Prove that there exists a matrix B of the same size such that $\boldsymbol{BA_1BA_2B\ldots BA_NB}$ is a nonzero matrix.

Solution. The key is to consider the kernel and image spaces of matrices. We shall construct projection matrix B of rank 1, which satisfies the conditions. Projection matrix of rank 1 is defined by 2 linear subspaces: kernel of codimension 1 and image of dimension 1, which doesn't contain kernel. Each vector can be uniquely decomposed into sum of two vectors – one from the space of dimension 1 and second from the space of codimension 1. So, the projection can be described as taking the first vector of that decomposition.
For this product to be non-zero, all we need is that the image of B won't be sent into its kernel. So, we have to prove that we can choose a nonzero vector v (or the one-dimensional space) and a space W of codimension that neither A_i will send v into W.

To do this, we must achieve 2 things:
a) Find a vector v which don't belong to kernel of A_i for all i.
b) Find a hyperplane (containing 0) which doesn't contain $A_i v$ for all i.
So, it remains to prove 2 lemmas:
Lemma 1. There exists a vector which is not contained in all given linear subspaces, where number of subspaces is finite.
Lemma 2. There exists a hyperplane (containing 0), which doesn't intersect with a given finite set of points.

Since any subspace can be enlarged to hyperspace, lemma 1 is equivalent to its special case:
Lemma 3. There exists a vector which is not contained in all given hyperplanes, where number of hyperplanes is finite.

Lemma 2 is also follows from lemma 3, since if we replace a hyperplane $a_1x_1+a_2x_2+\ldots+a_nx_n = 0$ by a vector $(a_1, a_2, \ldots, a_n)$ and vice versa, the condition "a hyperplane contains the vector" turns into "a vector belongs to the hyperplane".
So, it is enough to prove lemma 3.

Remark. All this works only for infinite fields.

Proof of lemma 3. Apply the induction over dimension of the space.
The base of induction: space of dimension 1 can't be covered by finite number of points (field is infinite).
The step of induction: assume it is proven for spaces of dimension smaller than n. So, we have finite number of hyperplanes, and we try to prove they don't cover the space. There is infinite number of hyperplanes in the space, so we can choose a huperplane H which is different from all given hyperplanes. Intersection of H with other hyperplanes are sub-hyperplanes in H, so, by induction, they can't cover it.

5. An infinite sequence of real numbers $\{x_i\}$ will be called ***nice*** if $\sum x_i^2$ converges. Let $\{a_i\}$ be a sequence, such that for each nice sequence $\{x_i\}$ the series $\sum a_i x_i$ converges. Prove that the sequence $\{a_i\}$ is nice.

Solution. Assume $\{a_i\}$ isn't nice. So $\sum a_i^2$ diverges. We can cut the sequence $\{a_i\}$ into infinite number of segments, each of which is greater than 1.
(That is done by induction, simply sum up the numbers from the end of segment number k until it exceeds 1, and that will be segment $k+1$.)
Let segment number k start at m_k and have n_k elements.

Then, by construction, $b_k = \sum_{j=m_k-n_k+1}^{m_k} a_j^2 > 1$. We shall use a nice lemma:

Now, construct a sequence $x_j = \dfrac{a_j}{k \cdot b_k}$, for each j which belongs to segment number k. Then

$$\sum_{j=1}^{\infty} a_j x_j = \sum_{k=1}^{\infty} \sum_{j=m_k-n_k+1}^{m_k} a_j x_j = \sum_{k=1}^{\infty} \sum_{j=m_k-n_k+1}^{m_k} \frac{a_j^2}{k \cdot b_k} =$$

$$= \sum_{k=1}^{\infty} \frac{\sum_{j=m_k-n_k+1}^{m_k} a_j^2}{k \cdot b_k} = \sum_{k=1}^{\infty} \frac{b_k}{k \cdot b_k} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

But sequence $\{x_i\}$ is good because

$$\sum_{j=1}^{\infty} x_j^2 = \sum_{k=1}^{\infty} \sum_{j=m_k-n_k+1}^{m_k} \frac{a_j^2}{k^2 \cdot b_k^2} = \sum_{k=1}^{\infty} \frac{b_k}{k^2 \cdot b_k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2 \cdot b_k} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

First stage of Israeli students competition, 2009.

1. Let C be a convex polygon and P a point inside it. Let N be number of vertices, such that an interval connecting P to the vertex divides the angle of C into two acute angles. Denote n number of sides of C, such that the foot of perpendicular from P to that side is strictly inside that side.
Proof that $N = n$.

2. Let A be a 2×2 invertible matrix with real coefficients. One of its coefficients is 200. Can it happen that all the coefficients of matrices A^{-1}, A^2, A^3, A^4, …, A^{100} belong to the interval (-10, 10) ?

3. The sequence $\{x_i\}$ is defined by the initial value $x_0 \subset [0,1]$ and recursive formula $x_{n+1} = \dfrac{1-\sqrt{1-x_n}}{2}$.
Find $\lim\limits_{n\to\infty}\left(x_n \cdot 4^n\right)$.

4. Two players play a game on the infinite chess-board. First player plays with 3 white pieces called sheep, and the second player plays with 3 black pieces, called wolves. They move in turn. In his move each player can move only one piece to an adjacent cell (having a common side with its previous cell). Sheep can be moved only horizontally. If a wolf and a sheep happen to be in the same cell, the wolf eats the sheep. Is it always possible for wolfs to catch at least one sheep?

5. When in three-dimensional space the center of the ball of radius r goes along a circle of radius R (here $R>r>0$), the ball covers a three-dimensional body called torus. Compute the surface area of that torus as a function in r and R.

Good luck!

Second stage of Israeli students competition, 2009.

1. Which is bigger: arctan(e) or $\frac{\pi}{4}+\frac{1}{2}$?
Calculator is not allowed.

First solution. It is the same as to ask what is greater $\arctan(e)-\frac{\pi}{4}$ or $\frac{1}{2}$.

$$\arctan(e)-\frac{\pi}{4}=\arctan(e)-\arctan(1)=\int_1^e \arctan'(x)\,dx=$$

$$=\int_1^e \frac{dx}{1+x^2}<\int_1^e \frac{dx}{2x}=\frac{1}{2}\ln(x)\Big|_1^e=\frac{1-0}{2}=\frac{1}{2}$$

That is because for all $x > 1$ we have $1 + x^2 > 2x$ since it is the same as $(x - 1)^2 > 0$.

Therefore, $\arctan(e) < \frac{\pi}{4}+\frac{1}{2}$.

Second solution. It is the same as to compare e versus $\tan\left(\frac{\pi}{4}+\frac{1}{2}\right)$,

since arctan is monotonously increasing.

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{1}{2}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{1}{2}\right)}=\frac{1+\tan\left(\frac{1}{2}\right)}{1-\tan\left(\frac{1}{2}\right)}$$

The expression $\frac{1+x}{1-x}$ is monotonously increasing for $0 < x < 1$. Indeed, when x is increasing then $1-x$ is decreasing so $\frac{1}{1-x}$ is increasing, and $1+x$ is increasing too, so their product is increasing. But $\tan\left(\frac{1}{2}\right)>\frac{1}{2}$ hence

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{1+\tan\left(\frac{1}{2}\right)}{1-\tan\left(\frac{1}{2}\right)}>\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=3>e$$

QED.

2. Prove that $\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+...+\frac{1}{3n+1}$ is non-integer for any n.

Solution. Consider 2^N, the greatest power of 2 which appears in the sequence of denominators 4, 7, 10, … , $3n + 1$. The question is, whether this sequence of denominators contains other numbers divisible by this power of two.
If not, then multiplying by $2^{N-1}\cdot 1\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot(4n + 1)$ will turn all the summands except one into integer numbers, so number will be non-integer even after multiplying by such a large integer number, so in this case the problem is solved.
If yes, then the sequence of denominators contains another number of the form $k2^N$. Here k cannot be 2 or 3 because multiplying by k turns number of the form $3m+1$ into another number of the form $3m+1$, so k is 4 or greater. But $4\cdot 2^N$ is a greater power of 2, and it turns out to be in the sequence of denominators.
This is a contradiction, since we have chosen the greatest power of 2.

3. A triangle is contained by an 11-dimensional unit cube inside P^{11}. What is the maximal possible perimeter of that triangle?

Answer. $\sqrt{7}+\sqrt{7}+\sqrt{8}=2\left(\sqrt{7}+\sqrt{2}\right)$

Solution.
Lemma 1. The perimeter will be the greatest, if the vertices of triangle are the vertices of the cube.

It follows directly from:

Lemma 2. If two points are fixed, than the third point giving maximal sum of distances from the first two is a vertex of the cube.

Proof. Sum of distances from two given points is a convex function. That happens because each of those is a convex function, and sum of the convex function is a convex function.

Reminder. A convex function is a function, for which above-the-graph domain is convex (above-the-graph domain is $\{ (x , y) \mid f(x) < y \}$, here x may be a vector). Considered on a closed interval, convex function has maximum in one of the ends. So, considered on the bounded polygon, convex function has maximal value at on of its vertices. Distance function is convex since above-the-graph is a cone over a ball, which is a convex body.

Now, back to 11 dimensions. Because of lemma 1, the problem degenerates into a combinatorial problem. Instead of trying to find 3 points, we have 3 sequences of zeroes and ones. The distance between two points in each coordinate is 0 or 1 also, so the distance is square root of number of differences.
Between all 3 points in a given coordinate there are not more than 2 differences. Therefore, in all 11 coordinates, there are no more than 22 differences. So, if numbers of differences between 3 coordinate sequences are K, M, N then the perimeter is $\sqrt{K}+\sqrt{M}+\sqrt{N}$, which should be maximal while $K+M+N \leq 22$.

Lemma 3. If $N > K+1$, then $\sqrt{K}+\sqrt{N} < \sqrt{K+1}+\sqrt{N-1}$. (Actually, this kind of lemma is true for any concave function, not just for square root)

Proof of lemma 3. Reformulate it:

$$\sqrt{N}-\sqrt{N-1} < \sqrt{K+1}-\sqrt{K}$$

Multiply and divide by adjoint:

$$\frac{1}{\sqrt{N}+\sqrt{N-1}} < \frac{1}{\sqrt{K+1}+\sqrt{K}}$$

$$\sqrt{N}+\sqrt{N-1} > \sqrt{K+1}+\sqrt{K}$$

So, $\sqrt{K}+\sqrt{M}+\sqrt{N}$ the number will be maximal none among K, M, N differ by more than 1. Of course, we may also assume $K+M+N = 22$, otherwise adding 1 to one of the numbers will improve $\sqrt{K}+\sqrt{M}+\sqrt{N}$.
So, all K, M, N are equal to either L or $L+1$ and $22 = K+M+N = 3L + R$, where R is 0, 1, 2, or 3. So, $L = 7$, R = 1, and K, M, N are 7, 7, 8 in some order.

Of course, after seeing that 7, 7, 8 is the best under algebraic restriction we got from the cube, we have to check that these lengths are attainable in our cube.
For example:
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
(1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0)
(0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1)
So, the greatest possible perimeter is $\sqrt{7}+\sqrt{7}+\sqrt{8} = 2\left(\sqrt{7}+\sqrt{2}\right)$.

4. Can a polynomial with rational coefficients have $-\sqrt{2}$ as its minimal value?

First solution. Let us try $p'(x) = k\left(x^2-2\right)\left(x-a\right) = x^3 - ax^2 - 2x + 2a$.

$$p(x) = k\left(\frac{x^4}{4} - \frac{ax^3}{3} - x^2 + 2ax\right) + c.$$

The extremal points are $\pm\sqrt{2}$, a, so when we substitute them into $p(x)$ we have good chances to get something with $\pm\sqrt{2}$. Of course, if $k > 0$, then the middle extremum is a maximum, and the other two are minima.

$$p\left(\pm\sqrt{2}\right) = k\left(1 \mp \frac{2a\sqrt{2}}{3} - 2 \pm 2a\sqrt{2}\right) + c = k\left(-1 \pm 2a\sqrt{2}\,\frac{2}{3}\right) + c$$

Choose $a = \frac{3}{4}$. Then that will be the middle extremum.

The local minima are at $\pm\sqrt{2}$, and the global minimal value is the least between $p\left(\pm\sqrt{2}\right) = k\left(-1 \pm \sqrt{2}\right) + c$, which is $p\left(-\sqrt{2}\right) = k\left(-1 - \sqrt{2}\right) + c$. Take $k = c = 1$ and You get a polynomial with rational coefficient satisfying all conditions.

Second solution. Consider $q(x) = \left(x^2 - 2\right)^2 = x^4 - 4x^2 + 4$.

It is zero at $\pm\sqrt{2}$, and positive elsewhere.

Now consider polynomial satisfying $r'(x) = \frac{3}{4}\left(x^2 - 2\right)$, $r(x) = \frac{x^3}{4} - \frac{3x}{2}$.

That polynomial has extrema at $\pm\sqrt{2}$, a local maximum at $-\sqrt{2}$ and a local minimum at $\sqrt{2}$. The coefficient was chosen so that $r\left(\sqrt{2}\right) = \frac{2\sqrt{2}}{4} - \frac{3\sqrt{2}}{2} = -\sqrt{2}$.

Now consider a polynomial $p(x) = r(x) + Aq(x)$, where A is a positive number.

It has local minimum with value $-\sqrt{2}$ at $\sqrt{2}$, and positive value of $\sqrt{2}$ at $-\sqrt{2}$.
The values at far negative numbers are positive, since x^4 is stronger than x^3.
If we enlarge A then values outside small neighbourhoods of $-\sqrt{2}$ and $\sqrt{2}$ become as big as we can wish, say positive. Since values near $-\sqrt{2}$ are also positive, the value at $\sqrt{2}$, which is $-\sqrt{2}$, becomes a global maximum.

5. Consider a shape consisting of a finite number of unit square cells.
We try to cover a board of $m\times n$ cells by equivalent copies of that shape, so that each cell of the board will be covered by the same number of layers.
Prove that it is impossible if and only if we can write a real number in each cell of the board, in such a way that the sum of all those numbers will be strictly negative,

while a sum that can be covered by the given shape is strictly positive (wherever we place it on the board).

Solution. Consider an mn-dimensional linear space of all tables with real values in the cells. For each cell we can take a coordinate unit vector in that space, and a scalar product between that vector and the vector of that table will give the value in that cell.
To each subset of cells we match the sum of the unit vectors. The scalar product with that vector will give sum of numbers in the corresponding set.
Suppose that we have a set of some shapes inside the board, and we construct a vector corresponding to each. To tile the board in $k > 0$ layers is the same as to find integer nonnegative coefficients such the linear combination will be $(k, k, k, \dots, k)$. Which is the same as to express $(1, 1, 1, \dots, 1)$ as a linear combination of some of those vectors with positive rational coefficients.
The rest of it follows from two lemmas:

Lemma 1. Consider vectors with rational coordinates $v_1, v_2, \dots, v_q$ and v.
If v is a linear combination of $v_1, v_2, \dots, v_q$ with positive real coefficients then it is a linear combination of $v_1, v_2, \dots, v_q$ with positive rational coefficients.

Lemma 2. Consider vectors with nonnegative coordinates $v_1, v_2, \dots, v_q$ and v.
If v is not a linear combination of $v_1, v_2, \dots, v_q$ with positive real coefficients then there exists a vector u such that $(u, v) < 0$ and $(u, v_i) > 0$ for $i = 1, 2, \dots, q$.

From the first lemma we see, that if there is no tiling, then the vectors corresponding to our shapes don't generate the vector of the whole board as a linear combination with nonnegative coefficients. From the second lemma we see that in such case there is a table which has negative sum over all the cells and positive sum over each shape. That proves the problem in a non-trivial direction. (The other direction is obvious: if such a table exists, then the tiling doesn't, since the sum in the cells of that tiling should be negative, but it will be positive.)
So, it remains to prove the lemmas.

Proof of lemma 1. Solution of the system of linear equations, which is written by one vectorial equation $x_1v_1 + x_2v_2 + \dots + x_qv_q = v$ is a shifted linear subspace in the q-dimensional space. So, if it exists (and it is given it has a positive real solution), it can be solved by Gauss method and we shall have an answer:
$(x_1, x_2, \dots, x_q) = u + y_1u_1 + y_2u_2 + \dots + y_tu_t$, where $u, u_1, u_2, \dots, u_t$ are some q-dimensional vectors, and $y_1, y_2, \dots, y_t$ are arbitrary real numbers.

Since Gauss method is an algebraic procedure, all the coordinates of u, u_1, … , u_t will be rational.
It is given that for some values of y_1, y_2, … , y_t all the coordinates will be positive real numbers, so they will also be positive if we change y_1, y_2, … , y_t by sufficiently small numbers, since linear functions are continuous. But in any neighborhood of each real number there is a rational number. So we can shift coordinates slightly so that x_1, x_2, … , x_q will remain positive and y_1, y_2, … , y_t will be rational, but then all x's will also be rational since they are algebraic expressions in y's and coordinates of u's.

Proof of lemma 2. Let S be a hyperplane defined by the equation:
Sum of all coordinates = 1.
Positive linear combinations of v_1, v_2, … , v_q cut S along a convex body C.
This convex body is bounded, since it is inside a simplex, whose vertices are coordinate vectors, because the coordinates are positive (a simplex is a multidimensional generalization of triangle).
A ray generated by vector v cuts S at a point P which is not in C.
We shall prove that there is a sub-hyperplane T in S, such that P is on one side of T and P is on one side, and C is on the other side.
From that it will follow that a hyperplane, passing through T and 0, such v will be on one side, and v_1, v_2, … , v_q on the other side, and the equation defining that hyperplane will have one sign on v_1, v_2, … , v_q and another sign on v.

So, it remains to prove the following statement inside hyperplane S, which is Euclidean space by itself:
Lemma 3. Let C be a compact convex body in a Euclidean space, and P a point outside C, then there exists a hyperplane T that defines that C is which separate P from C.

Proof of lemma 3. Let Q be the point of C closest to P (it exists since C is compact).
Let T be a perpendicular bisector to interval QP. (Perpendicular bisector is a hyperplane cutting the interval perpendicularly in the middle, it is also the set of all points which are at the same distance from both ends).
We shall prove that T separates P from C. Suppose not: there is a point R in C either on T itself or on the same side of T as P. The whole interval QR is in C, since P is convex. But the angle PQR is acute. So, if we start going by QR from Q to R we get closer to P, at least at first. But Q is the point of C that is closest to P, that is a contradiction. QED.
Remark. We don't really need the convex set to be compact for lemma 3, enough to require that it is closed. Infinite-dimensional version of lemma 3 is called Hahn-

Banach theorem, and it is considered one of the central theorems of functional analysis.

Olympiad of Israel Mathematical Union

Selection of the team for IMC 2009

Please write your solutions in English

1. Denote A be number of ways to paint the cells of the 8×8 chessboard in 3 colors, so that no two adjacent cells are of the same color (by adjacent cells we mean cells having common side). Denote X the number of ways to write integer numbers in the cells of the chessboard, so that the number in the bottom left corner is 0, and the difference between numbers in any two adjacent cell is 1 (here by difference of x and y we mean $|x - y|$).
Express X via A.

2. Let ABCD be a convex planar cyclic quadrilateral (מרובע קמור חסום) and P a point in space. Show that $PD^2 \cdot S_{ABC} + PB^2 \cdot S_{ACD} = PA^2 \cdot S_{BCD} + PC^2 \cdot S_{ABD}$.

3. It is given that $\sum_{i=1}^{\infty} x_i$ converges, and $\{x_i\}$ is a sequence of real numbers.

Can we claim that $\sum_{i=1}^{\infty} \sin(x_i)$ converges?

4. Suppose A is an $m \times n$ matrix and B is an $n \times m$ matrix. Prove that the set of nonzero eigenvalues of AB coincides with the set of nonzero eigenvalues of BA.

5. (a) Find a function defined on closed interval [-1,1], which has only finite number of discontinuity point, such that its graph is invariant under rotation by the right angle around the origin.
(b) Prove that there is no function on open interval (-1,1) which satisfies the same conditions.

Good luck!

Third stage of Israeli students competition, 2009.

1. Denote A be number of ways to paint the cells of the 8×8 chessboard in 3 colors, so that no two adjacent cells are of the same color (by adjacent cells we mean cells having common side). Denote X the number of ways to write integer numbers in the cells of the chessboard, so that the number in the bottom left corner is 0, and the difference between numbers in any two adjacent cell is 1 (here by difference of x and y we mean $|x - y|$).
Express X via A.

Answer. $X = A / 3$.
Solution. Assume we have a table of numbers, satisfying the condition. If we paint the cells having numbers of type $3k$ red, cells having numbers of type $3k+1$ green, and cells having numbers $3k+2$ blue, we get a coloring of the board in 3 colors, and the left-bottom cell a1 is red.
So, we get a coloring of the board satisfying the condition, with the specified color at a1, and that is only 1/3 of all possible colorings (it is easy to see that all colors for a specific cell have equal probabilities, since we can rotate colors, by replacing red → green → blue → red).
So, each one of permitted X tables of numbers can be turned into one of $A/3$ coloring. It remains to prove that this correspondence is 1-1. To show it, we should explain why given a coloring of the board we can reconstruct – and uniquely – the table of numbers.
Firstly, if we know the coloring and we know the number at a certain cell, we can reconstruct the number at an adjacent cell. That is because we have only 2 options ($x + 1$ or $x - 1$, where x is the number in the first cell), and these two options have different remainders mod 3, so the coloring allows us to distinguish those two options. Notice, that each of two remainders different from x are attainable, and they correspond to both colors different from the color of x.
We start with the cell a1 and write 0 in that cell, since this is given. Then, as we described before, we can reconstruct numbers of b1, c1, … , h1 and also numbers a2, a3, … , a8.
From now on we shall reconstruct b2, b3, …, b7, c2, c3, …, c7 and so on. As we saw before, reconstruction given one neighbor and color exists and unique, so the question is whether reconstruction given 2 neighbors (down and left) and color exists, because if it is, it is also unique. Suppose WLOG that we try to reconstruct b2 when the colors, and the numbers of a2 and b1 are given. The conditions that should be satisfied: the remainder mod 3 is given, and the difference with both numbers, of a2 and b1, are given. If the numbers of a2 and b1 are equal, then the

two conditions are the same, so the whole reconstruction is the same as before, so there's nothing new to prove here.
So it remains to consider the case when the numbers a2 and b1 are different. Then the difference between them is 2, since they differ by 1 from the same number. So they have two different colors, so there is only one choice for the color of b2, to be different from the both colors. The average of a2 and b1 is the only number that differs by 1 from both and it is of color different from both, so that is the only possible reconstruction of that cell.
So, reconstruction is in both cases feasible and unique. QED.

2. Let ABCD be a convex planar cyclic quadrilateral (מרובע קמור חסום) and P a point in space. Show that $PD^2 \cdot S_{ABC} + PB^2 \cdot S_{ACD} = PA^2 \cdot S_{BCD} + PC^2 \cdot S_{ABD}$.

Solution. Choose a Cartesian coordinate system such that P is the origin and plan ABCD corresponds is $z = k$. The coordinates of our points are
$A(x_a, y_a, k), B(x_b, y_b, k), C(x_c, y_c, k), D(x_d, y_d, k)$.
Since the points belong to the same circle, the pairs
$(x_a, y_a), (x_b, y_b), (x_c, y_c), (x_d, y_d)$ all satisfy the equation:

$$x^2 + y^2 + \alpha x + \beta y + \gamma = 0.$$

Consider the matrix

$$\begin{pmatrix} x_a & y_a & 1 & x_a^2 + y_a^2 + k^2 \\ x_b & y_b & 1 & x_b^2 + y_b^2 + k^2 \\ x_c & y_c & 1 & x_c^2 + y_c^2 + k^2 \\ x_d & y_d & 1 & x_d^2 + y_d^2 + k^2 \end{pmatrix} = \begin{pmatrix} x_a & y_a & 1 & PA^2 \\ x_b & y_b & 1 & PB^2 \\ x_c & y_c & 1 & PC^2 \\ x_d & y_d & 1 & PD^2 \end{pmatrix}$$

The vector $\begin{pmatrix} \alpha \\ \beta \\ \gamma - k^2 \\ 1 \end{pmatrix}$ belongs to its kernel, so it is degenerate either way.

So determinant is 0. Decompose the determinant along the last column and You get the required identity (multiplied by 2), since minors are twice the areas of triangles.

Second solution (from the work of Dan Carmon). Perform inversion with center at P (and radius 1). Points A, B, C, D will go to points A', B', C', D'.
It is well-known that inversion turns generic spheres (i. e. spheres or planes) to generic spheres. So, intersection of generic spheres (which are circles or lines) are

turned into intersections of generic spheres. So, A'B'C'D' is still cyclic (or collinear). WLOG, it is cyclic: if we prove the formula in the case when P is not on the circle, the degenerate case follows by continuity of the both sides of the identity.

We shall use the famous formula for distance after inversion: A'B' = AB/(PA·PB). If you don't know it, please prove it (hint: similarity of triangles).

All triangles ABC, ABD, ACD, BCD are inscribed in the same circle of radius R, so their area might be computed as S_{ABC} = AB·BC·CA/(4R).

Substitute all areas with that formula to the identity we need to prove, and multiply by 4R. We get an expression with lengths only, without areas:

$$PD^2 \cdot AB \cdot BC \cdot CA + PB^2 \cdot AC \cdot CD \cdot DA = PA^2 \cdot BC \cdot CD \cdot DB + PC^2 \cdot AB \cdot BD \cdot DA$$

Divide by $PA^2 \cdot PB^2 \cdot PC^2 \cdot PD^2$. You get, by formula of distance after inversion,

$$A'B' \cdot B'C' \cdot C'A' + A'C' \cdot C'D' \cdot D'A' = B'C' \cdot C'D' \cdot D'B' + A'B' \cdot B'D' \cdot D'A'$$

If A'B'C'D' is circumscribed, we divide by 4R' where R' is the radius of A'B'C'D' circumcircle, we get

$$S_{A'B'C'} + S_{A'C'D'} = S_{B'C'D'} + S_{A'B'D'}$$

That is obvious, since both are equal to the area of quadrilateral A'B'C'D'.

3. It is given that $\sum_{i=1}^{\infty} x_i$ converges, and $\{x_i\}$ is a sequence of real numbers.

Can we claim that $\sum_{i=1}^{\infty} \sin(x_i)$ converges?

Answer. No.

Solution. For any *y*, consider triple: $2y, -y, -y$.

Apply sine to all numbers in triple: $\sin(2y), \sin(-y), \sin(-y)$.

Denote $f(y) = \sin(2y) + \sin(-y) + \sin(-y) = 2\sin(y)(\cos(y) - 1)$.

It is nonzero when *y* is sufficiently close to 0.

If for some *y* we repeat $2^n \left\lceil \frac{1}{|f(y)|} \right\rceil$ triples of that kind in the series, then sum of the corresponding interval in series $\sum x_i$ is 0, and the sum of corresponding interval in the $\sum \sin(x_i)$ has absolute value above 2^n.

So, take intervals of $2^n\left\lceil \dfrac{1}{|f(1/n)|} \right\rceil$ triples constructed from $y=\pm\dfrac{1}{n}$.

The series $\sum x_i$ will be converging, since each triple gives 0, and elements tend to zero, so the estimate on the absolute value of every tail of this series is 4ε, if ε is the estimate on absolute value.

At the same time, $\sum \sin(x_i)$ diverges, since it consists of intervals, and contribution of each interval is above 2^n by absolute value.

4. Suppose A is an $m\times n$ matrix and B is an $n\times m$ matrix. Prove that the set of nonzero eigenvalues of AB coincides with the set of nonzero eigenvalues of BA.

First solution. By symmetry, WLOG, $m \le n$. Let A' and B' be $n\times n$ matrixes created from A and B by adding 0 rows below and columns on the right. Then B'A' = BA, and A'B' is a block matrix, first block is AB, second block is 0-matrix.
Anyway, eigenvalues of BA and of B'A' are the same, and nonzero eigenvalues of AB and of A'B' are the same, so from now on WLOG we may assume that the matrixes A and B were square matrixes from the beginning, i. e. $m = n$.
If B is invertible, then AB = B(AB)B^{-1} so the matrixes AB and BA are similar, so their eigenvalues coincide.

Lemma. The set of invertible matrixes is dense in the set of matrixes, in other words for any non-invertible matrix B there is a sequence of matrices $\{B_n\}$ such that $B_n \xrightarrow[n\to\infty]{} B$.

This lemma allows to extend the claim from invertible matrixes to non-invertible. Indeed, for any non-invertible B we have a sequence of invertible matrixes $B_n \xrightarrow[n\to\infty]{} B$. For any element of this sequence, AB_n is similar to B_nA, in particular AB_n and B_nA have the same characteristic polynomial. The coefficients of the characteristic polynomial are polynomials in matrix elements, so characteristic polynomials of AB and of BA are limits of characteristic polynomials of AB_n and B_nA respectively, so they are equal. Since AB and BA have the same characteristic polynomials, their polynomials should have the same sets of nonzero roots, QED.

It remains to prove the lemma.

Proof of lemma. The lemma is a direct result of the combination of 3 facts:

a. The set of invertible matrixes is non-empty.
b. Non-invertible matrixes are defined by a polynomial in coefficients.
c. In $\mathbb{R}^N$, the set of non-zeroes of given polynomial is either dense or empty.

The fact a' is obvious, the b' is also obvious, so it remains to prove c'.

So, assume a polynomial $p(x_1, x_2, ..., x_N)$ has nonzero value at point $(g_1, g_2, ..., g_N)$ and we need to find a sequence of such points converging to a given point $(x_1, x_2, ..., x_N)$. Draw a line in space passing via this 2 points, in parametric form $(x_1 + tk_1, x_2 + tk_2, ..., x_N + tk_n)$, here t is the parameter. Consider our polynomial restricted to that line $q(t) = p(x_1 + tk_1, x_2 + tk_2, ..., x_N + tk_n)$, it is a polynomial in t. The polynomial q is nonzero for at least one value (when the line goes via th point $(g_1, g_2, ..., g_N)$, so it is a nonzero polynomial of one variable and it has only finite number of roots, so the non-zeroes are dense in this line and we can construct the sequence. QED.

Second solution. Assume λ is not an eigenvalue of AB. Then AB – λI is an invertible matrix and its inverse is C, i.e. I = (AB – λI)C = ABC – λC.
Consider matrix BCA.

$$\begin{aligned}(BA - \lambda I)(BCA) &= B(ABC)A - \lambda BCA = B(\lambda C + I)A - \lambda BCA = \\ &= \lambda BCA + BA - \lambda BCA = BA\end{aligned}$$

Therefore (BA – λI)(BCA – I) = BA – BA + λI = λI.
So, if λ is nonzero, then (BCA – I)/λ is the inverse matrix of BA – λI.
To summarize: if a nonzero λ is not an eigenvalue of BA, then it is also not an eigenvalue of AB. Vice versa is also true by symmetry.

Third solution (from the work of Gal Dor). Let $m(x)$, $n(x)$ be the minimal polynomials of AB, BA respectively.
So, by definition $m(AB) = a_k(AB)^k + \ldots + a_2(AB)^2 + a_1AB + a_0I = 0$
Multiply by B from the left and by A from the right. You get:

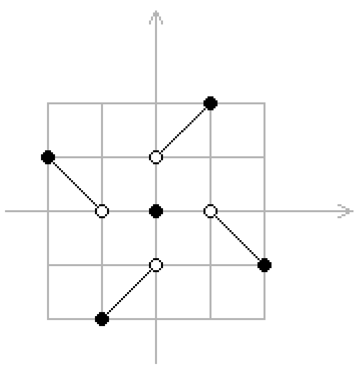
$$a_k(BA)^{k+1} + \ldots + a_2(BA)^3 + a_1(BA)^2 + a_0BA = 0$$

This is what happens when you apply polynomial $xm(x)$ to BA.
Any polynomial that nullifies BA is divisible by $n(x)$.
Therefore $xm(x)$ is divisible by $n(x)$.
For the same reason $xn(x)$ is divisible by $m(x)$.
Hence $m(x)$ and $n(x)$ have the same nonzero roots, QED
(and the same multiplicities, and multiplicity of 0 differs by 1 at most).

Fourth solution (from the work of Ilya Gringlaz). Assume λ is a nonzero eigenvalue of AB, so for a curtain vector v we have AB$v = \lambda v$.
Notice that Bv is nonzero, otherwise ABv would be 0 and not λv.
But BABv = B$\lambda v = \lambda$Bv, so vector Bv is nonzero and it gets multiplied by λ when we multiply it by BA, so BA has λ as an eigenvalues with eigenvector Bv.

5. **(a)** Find a function defined on closed interval [-1,1], which has only finite number of discontinuity point, such that its graph is invariant under rotation by the right angle around the origin.
(b) Prove that there is no function on open interval (-1,1) which satisfies the same conditions.

Solution. **(a)** One of the possible examples:



$$f(x)=\begin{cases} -\tfrac{1}{2}-x & x\in[-1,-\tfrac{1}{2}) \\ x-\tfrac{1}{2} & x\in[-\tfrac{1}{2},0) \\ 0 & x=0 \\ x+\tfrac{1}{2} & x\in(0,\tfrac{1}{2}] \\ \tfrac{1}{2}-x & x\in(\tfrac{1}{2},1] \end{cases}$$

(b) First solution. The graph consists of the finite number of continuous intervals, open, closed and half-open (the isolated points will be considered as very short closed intervals), because there is only finite number of discontinuity points.
On each interval function is strictly monotone, since if some value is accepted twice then after 90° rotation we would see 2 values for the same x.
The continuity interval of the graph containing 0 is an isolated point. Would it be longer, than for x sufficiently close to 0 from one side the sign of $f(x)$ would be the same, and that would contradict invariance with respect to rotation by the right angle.
Except for isolated point at 0, no continuity interval of the graph will go to itself after rotation by 180° around the origin. Indeed, if it would, than it would contain 0, and we proved it is impossible.
Also, except isolated point at 0, no continuity interval of the graph would go to itself after rotation by 90° around the origin, since then it would also go to itself after two rotations of that kind, and that is impossible.

Let **S** be the set of all continuity intervals of the graph except the isolated point at 0. Rotation by 90° around the origin divides **S** into orbits of four. Consider two ends of each element of **S**: each of them can be either open or closed. Consider the total number of open ends in **S** minus total number of closed ends.

Each element of S contributes 2, -2 or 0 to this quantity, so each orbit of four contributes something divisible by 8. On the other hand, each non-integer discontinuity point gives 1 open end and 1 close end which cancel out, and at 0, 1, -1 we have 4 open ends, so the total quantity is 4.
Contradiction: 4 is not divisible by 8.

Second solution. Like in the first solution, we explain that there are orbits of four and one separate point. Union of all continuity intervals is the domain.
Now we count Euler characteristic. Euler characteristic is additive. Euler characteristic of a point is 1, and of an open interval is -1.
So, Euler characteristic of the domain should be 4k+1, and it isn't.

Remark. Another way to formulate the main condition of this problem, about the rotation by right angle, is $f(f(x)) = -x$.

Targil 1 - polynomials.

1. A polynomial $p(x)$ of degree n has only integer values in integer points.
(a) Show that $n!p(x)$ has integer coefficients.
(b) Can we claim that $p(x)$ has integer coefficients?

2. Let $p(x)$ be a polynomial with integer coefficients, and $a_1 < a_2 < \ldots < a_n$ integer numbers.
(a) Prove that there always exists an integer a such that $p(a)$ is divisible by $p(a_1)$, $p(a_2)$, … , $p(a_n)$.
(b) Can we claim that there always exists an integer a such that $p(a)$ is divisible by $p(a_1)p(a_2)\cdot \ldots \cdot p(a_n)$?

3. Let $P(x)$ be polynomial with integer coefficients of degree $n > 1$.
Consider a polynomial $Q(x) = P(P(P(\ldots P(P(x))\ldots)))$, where P occurs n times.
Show that Q has no more than n integer stable points, i. e. no more than n integers such that $Q(z) = z$.

4. Consider a graph of a polynomial $p(x)$ of degree n on a plane, and a point P on the same plane. Show that there are no more than n tangent lines to the graph of $p(x)$ passing through P.

5*. Prove that $5765^{5765}+5766$
(a) is not a prime number
(b) is a product of three numbers which are greater than 1.

Targil 1 - polynomials.

1. A polynomial $p(x)$ of degree n has only integer values in integer points.
(a) Show that $n!p(x)$ has integer coefficients.
(b) Can we claim that $p(x)$ has integer coefficients?

Solution.
(b) No. For example $p(x) = x(x+1)/2$ or, more generally, the binomial coefficient $p(x) = x(x-1)(x-2)\ldots(x-n+1)/n!$ (By the way, why is it integer for integer x?)

(a) For every k from 0 to n consider a polynomial

$$p_k(x) = \frac{x}{k}\cdot\frac{x-1}{k-1}\cdot\frac{x-2}{k-2}\cdot\ldots\cdot\frac{x-k+1}{1}\cdot\frac{x-k-1}{-1}\cdot\ldots\cdot\frac{x-n}{k-n}.$$

This polynomial is equal 1 at k and 0 at all other integer points from 0 to n.
And all these polynomial have degree n.
For each polynomial $p(x)$ of degree n consider polynomial

$$q(x) = p(0)p_0(x) + p(1)p_1(x) + p(2)p_2(x) + \ldots + p(n)p_n(x)$$

Notice, that $p(x), q(x)$ coincide at 0, 1, 2, … , n so $p(x) - q(x)$ have at least n+1 roots, that it impossible for polynomial of degree n unless it is identically zero.
So $q(x)$ is $p(x)$.
Hence it is enough to show that $n!q(x)$ or even $n!p_k(x)$, has integer coefficients.

$$n!p_k(x) = \pm\frac{n!}{k!(n-k)!}x(x-1)(x-2)\ldots(x-k+1)(x-k-1)\cdot\ldots\cdot(x-n)$$

And $\dfrac{n!}{k!(n-k)!}$ is integer (did I ask You how to prove that)?

2. Let $p(x)$ be a polynomial with integer coefficients, and $a_1 < a_2 < \ldots < a_n$ integer numbers.
(a) Prove that there always exists an integer a such that $p(a)$ is divisible by $p(a_1)$, $p(a_2)$, … , $p(a_n)$.
(b) Can we claim that there always exists an integer a such that $p(a)$ is divisible by $p(a_1)p(a_2)\cdot \ldots \cdot p(a_n)$?

Solution. **(b)** No. $p(n) = 4n + 2$ is always divisible by 2 and never divisible by 4.
(a) It is enough to prove for $n = 2$ (that there is a such that $p(a)$ is divisible by $p(a_1)$ and $p(a_2)$), and then the statement is obvious by induction.

The most useful lemma about polynomials with integer coefficients:
$p(x) - p(y)$ is divisible by $x - y$.

Since that fact is so important, we shal see 2 proofs:
First proof. $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \ldots xy^{n-1})$ is divisible by $x - y$.
So sums of those expressions with integer coefficientsare also divisible by $x - y$.
Second proof. $x = y \pmod{x - y}$ hence $x^n = y^n \pmod{x - y}$ for all n,
therefore $p(x) = p(y) \pmod{x - y}$

So, back to the problem. All $p(a_1 + kp(a_1))$ are divisible by $p(a_1)$, while all $p(a_2 + mp(a_2))$ are divisible by $p(a_2)$. So if $a_1 + kp(a_1) = a = a_2 + mp(a_2)$, we won. By the inverse part of Euclidean algorithm, we know we can do it if $p(a_1)$, $p(a_2)$ are coprime (don't have a common divisor > 1).

If they are not coprime, "make them coprime". Let s be a product of all highest powers of primes in the decomposition of $p(a_1)$ which are higher than corresponding powers in the decomposition of $p(a_2)$. Let t be the product of all highest powers of all other primes in the decomposition of $p(a_1)$.
Then $\gcd(s, t) = 1$, while $\operatorname{lcm}(s, t) = \operatorname{lcm}(p(a_1), p(a_2))$
(Here lcm is the least common multiple, gcd is the greatest common divisor.)
Then we can find a such that $a_1 + ks = a = a_2 + mt$, then $p(a)$ is divisible by st which is lcm(s, t) which is the same as $\operatorname{lcm}(p(a_1), p(a_2))$. That's it.

3. Let $P(x)$ be polynomial with integer coefficients of degree $n > 1$.
Consider a polynomial $Q(x) = P(P(P(\ldots P(P(x))\ldots)))$, where P occurs n times.
Show that Q has no more than n integer stable points, i. e. no more than n integers such that $Q(z) = z$.

Solution. It follows from the most useful lemma on polynomials with integer coefficients (at the top of this page) that applying P to two different integers performs one of the following 3 operations:

a. Glues them together
b. Keeps the distance between them
c. Magnifies the distance between them.

Consider two stable points of $P(P(P(\ldots)))$. Each time we apply P to both points, they cannot be glued together, and cannot become more distant (since afterwards after applying P more times they won't get closer unless they'll be glued together). So P keeps the distance between each two stable points of $P(P(P(\ldots)))$.

So, if we have several stable points of $P(P(P(\ldots)))$, then P keeps distance between them all, so action of P on that set of points is the same as action of a linear function L(x) of slope 1 or -1. So all those points satisfy the equation $P(\mathrm{x}) = L(\mathrm{x})$. But this is polynomial equation of degree n, so it has no more than n roots.

4. Consider a graph of a polynomial $p(x)$ of degree n on a plane, and a point P on the same plane. Show that there are no more than n tangent lines to the graph of $p(x)$ passing through P.

Solution. Shifting in both x and y direction doesn't influence the degree of polynomial, so we may assume that P is the origin (0, 0).
The equation of tangent line to $p(x)$ at $(z,p(z))$ is $y - p(z) = (x - z)\, p'(z)$
If it passes via 0 we get $- p(z) = - z\, p'(z)$
That is a polynomial equation of z of degree n.
It cannot have more than n solution!
Unless… it is constantly 0.
But the highest degree term coefficient in the left hand side is $-az^n$ and in the right hand side is $-naz^n$ and they don't cancel out, unless $n = 1$ and then degree is 1 and tangent line is unique (though there are infinite number of tangent points).

5*. Prove that $5765^{5765}+5766$
(a) is not a prime number
(b) is a product of three numbers which are greater than 1.

Solution. (a) $5765 = 5600 + 140 + 21 + 4 = 4 \pmod 7$.

$4^3 = 2^6 = 1 \pmod 7$ because of Fermat little theorem.

$5765 = 2 \pmod 3$ so

$5765^{5765}+5766 = 4^2 + 5 = 0 \pmod 7$

So, it is divisible by 7.

(b) Polynomial $x^{3n+2} + x + 1$ accepts zero values at $\dfrac{1 \pm \sqrt{3}}{2}$ (those are the numbers such that $x^3 = 1$ but $x \neq 1$, i. e. $\dfrac{x^3-1}{x-1} = x^2 + x + 1$).

Since the set of roots of $x^{3n+2} + x + 1$ contains the set of roots of $x^2 + x + 1$ and the last has roots of multiplicity 1, the first is divisible by the last. Since the first coefficient of the last is 1 and other coefficients of both polynomials are integer, we see that the result of division will be a polynomial with integer coefficients.

Therefore $5765^{5765}+5766$ is divisible by $5765^2 + 5766$. QED.

Targil 2 - some linear algebra.

1. Let R be a 3×3 matrix representing rotation of Euclidean space.
How to compute the angle of rotation? And the axis?

2. Assume $\alpha \neq 0$ is a real number and *F*, *G* are two linear maps (operators) on $\mathbb{R}^n$ such that $FG - GF = \alpha F$.
(a) Prove that $F^k G - GF^k = \alpha k F^k$.
(b) Prove that is $F^k = 0$ for certain *k*.

3. (a) Is it true that for each couple of square matrices A, B, matrices AB, BA are similar?
(b) Is it true that A and A^T are always similar?
(Reminder: matrices X and Y are similar iff $X = PYP^{-1}$ for some invertible P, that means, the matrices represent the same linear transformation in a certain basis.)

4*. (a) Let $A_1A_2\ldots A_n$ be a regular polygon, O its center. For any point X, consider perpendiculars from X to the lines of sides of the polygons as vectors starting at X and ending on corresponding sides. Prove that sum of those vectors is *n*XO/2.
(b) In similar problem in a platonic solid of *n* faces, the answer is *n*XO/3.

5**. Consider an anti-symmetric ($A = -A^T$) matrix with integer coefficients. Show that the determinant is a perfect square.

Targil 2 - some linear algebra.

1. Let R be a 3×3 matrix representing rotation of Euclidean space.
How to compute the angle of rotation? And the axis?

Solution. Angle can be computed as arccos((trace – 1)/2)
Indeed, if the axis of rotation is z axis of the space the matrix would take certain form, on the diagonal we would have two times cos(angle) and 1,
so trace = 2cos(angle) + 1.
But trace doesn't depend on the choice of basis, so this formula holds in any basis.
Axis is a solution of linear system $Ax = x$ and can be found by Gauss method.
Or, since that system is degenerate, and has one-dimensional solution, as a vector product of two linearly independent lines of the matrix.

And yes, for unit matrix axis of rotation is undefined.

2. Assume $\alpha \neq 0$ is a real number and F, G are two linear maps (operators) on $\mathbb{R}^n$ such that $FG - GF = \alpha F$.
(a) Prove that $F^k G - GF^k = \alpha k F^k$.
(b) Prove that is $F^k = 0$ for certain k.

Solution. (a) Induction over k. Base of induction is given. Step from k to k+1:

$$F^{k+1}G - GF^{k+1} = F^{k+1}G - FGF^K + FGF^K - GF^{k+1} =$$

$$= F\left(F^k G - GF^k\right) + \left(FG - GF\right)F^k =$$

$$= F\left(\alpha k F^k\right) + \left(\alpha F\right)F^k = \alpha k F^{k+1} + \alpha F^{k+1} = \alpha\left(k+1\right)F^{k+1}$$

(b) Consider a linear operator over the linear space of $n \times n$ matrices
$L(X) = XG - GX$.
If all F^k are nonzero, than all of them are eigenvectors of that operator corresponding to different eigenvalues. But linear operator on finite-dimensional space can have only finite number of eigenvalues. QED.

3. (a) Is it true that for each couple of square matrices A, B, matrices AB, BA are similar?
(b) Is it true that A and A^T are always similar?
(Reminder: matrices X and Y are similar iff $X = PYP^{-1}$ for some invertible P, that means, the matrices represent the same linear transformation in a certain basis.)

Answers (a) no (b) yes.

Solution. (a) For example

$$\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\cdot\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

(b) Every matrix A is similar to its Jordan form $J=PAP^{-1}$.
Then A^T is similar to $J^T = (P^T)^{-1}A^TP^T$ (by the way, why $(A^T)^{-1} = (A^{-1})^T$?).
So, it is enough to prove, that a Jordan cell is similar to its transpose.
The similarity is performed by a matrix R, having 1s on the secondary coordinate and zeroes elsewhere. That is permutation matrix, it reverts the order of all coordinates, and $R^{-1} = R$. Multiplying by R from the left reverts the order of rows, and multiplying by R from the right reverts the order of columns, so conjugation by R rotates the matrix by 180^o. That will bring a Jordan cell C to C^T.

4*. (a) Let $A_1A_2\ldots A_n$ be a regular polygon, O its center. For any point X, consider perpendiculars from X to the lines of sides of the polygons as vectors starting at X and ending on corresponding sides. Prove that sum of those vectors is nXO/2.
(b) In similar problem in a platonic solid of n faces, the answer is nXO/3.

Solution. Everything depends linearly on X, so the formula in Cartesian coordinates should be MX + v, where M is a matrix, and v is a vector.
If the O is the origin, the result is O, since it is preserved by many rotations, and the only vector that is preserved by all those rotations is 0.
So the formula is linear, MX, multiplication by a certain matrix.
If X is on a perpendicular from O to a face, then rotation or symmetry that keeps this perpendicular line sends the polytope/polygon to itself, so X should be sent to aX, where a is a constant. This constant is the same for all perpendiculars to faces, because of the symmetry. So, our matrix acts as multiplication by a on all those vectors, and they span the whole space, so our matrix is a times unit matrix.
But what is a?
a = trace divided by dimension. Our linear transformation is some of projectional linear transformation – projecting vector to a line perpendicular to a certain face. Trace of each summand is 1, since trace doesn't depend on coordinates, and if the axis of projection would be a coordinate axis, matrix would have 2 in one corresponding diagonal call and 0 in all other sells.

So, we have summands, number of summands is equal to number of faces of the polytope, and trace of each is 1, so total trace = number of faces, hence a = number of faces / dimension.

5.** Consider an anti-symmetric ($A = -A^T$) matrix with integer coefficients. Show that the determinant is a perfect square.

Remark. $\det A = \det A^T = (-1)^n \det A$, so it is nonzero (and non-obvious) only for even dimension.

First solution. Determinant is integer, so it is enough to prove the it is a square of rational number, then we shall know it is a square of integer. If we apply a certain permutation on rows and the same permutation on columns, matrix will remain anti-symmetric and will keep the same determinant.

So we may assume that unless the matrix consists of zeroes only, then cells near the left-top corner (1,2) and (2,1) are non-zero: one is a, another is $-a$. Then by adding linear combinations of first and second rows to all other rows, we can eliminate all numbers in the first and second columns after the second row. These Gauss method operations are equivalent to multiplying the matrix from the left by an invertible matrix.

If A is anti-symmetric, then it is easy to see that BAB^T is also anti-symmetric. Let B be the matrix that is doing Gauss method operation to eliminate the first two columns under the top-left 2×2 block. Then B^T does the same operations on the columns. Obviously, both B and B^T are rational, so determinant is multiplied by a square of rational number. That number is nonzero, since B is invertible.

But now we get a block matrix, that consists of 2 anti-symmetric blocks, so the statement follows by induction over dimensions.

Second proof. It is known, that over anti-symmetric multi-linear forms the wedge product is defined, that makes a $k+m$-form out of k-form and m-form.

$$(\kappa \wedge \mu)(v_1, v_2, \dots, v_m) = \frac{1}{k!m!} \sum_{\sigma \in \mathbb{S}_{k+m}} \operatorname{sgn}(\sigma)\, \kappa\left(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}\right) \cdot \mu\left(v_{\sigma(m+1)}, v_{\sigma(m+2)}, \dots, v_{\sigma(m+k)}\right)$$

(here we divide by $k!m!$ to kill ambiguity – no need to sum equivalent summands several time, so this formula is actually integer).

This product is super-commutative and associative.

Any anti-symmetric 2-form can be represented in a general form as $\sum_{i<j} a_{ij} x_i \wedge x_j$, where x_i are basic linear functionals corresponding to “taking i’th coordinate”, or, when suitable basis is chosen, in a canonic form:
$\omega = k_1 x_1 \wedge x_2 + k_2 x_3 \wedge x_4 + k_3 x_5 \wedge x_6 + \ldots + k_n x_{2n-1} \wedge x_{2n}$.

Actually, that was what we have proven in the first solution.

But since the definition of the wedge product doesn’t use coordinates, as well as some definitions of determinant, if we prove certain equality between those in the canonical basis, we shall know it for any basis.

Consider the product $\frac{\omega \wedge \omega \wedge \ldots \wedge \omega}{n!}$, where ω is multiplied by itself n times.

When we open brackets, all products with similar factors cancel out. So we get $n!$ equivalent products, so after dividing by $n!$ we get an expression which is integer and not fractional in the coefficients, and that is $(k_1 k_2 k_3 \ldots k_n) x_1 \wedge x_2 \wedge x_3 \wedge \ldots \wedge x_{2n}$, product of all coefficients time standard volume form.

The determinant of the anti-symmetric matrix is $k_1^2 k_2^2 k_3^2 \cdot \ldots \cdot k_n^2$. It is the square of the coefficient before the volume form of $\frac{\omega \wedge \omega \wedge \ldots \wedge \omega}{n!}$. So it will be not necessarily in the canonical basis.

Example. Consider $n = 4$. Matrix $A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$ is represented by a form $\omega = a_{12} x_1 \wedge x_2 + a_{13} x_1 \wedge x_3 + a_{14} x_1 \wedge x_4 + a_{24} x_2 \wedge x_4 + a_{23} x_2 \wedge x_3 + a_{34} x_3 \wedge x_4$.

Then $\frac{\omega \wedge \omega}{2} = (a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23}) x_1 \wedge x_2 \wedge x_3 \wedge x_4$.

(When computing this things, just multiply each couple of terms once and don't divide by 2).

So $\det A = \left(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\right)^2$.

Outline of third solution (Ofir Gorodetzky)

We know (either by guessing or from previous solution) the formula for the expression whose square is the determinant: it is a sum over all ways to decompose the set of all indices into pairs, of product of cells corresponding to that pairs (one index is of row, another of column), signs are chosen by the sign of a permutation which is formed when we write down all those pairs in a row, pair after pair.

So, we can prove combinatorially, that the square of that expression is the determinant. The determinant is a sum of all products over all permutations (or maximal rook arrangements). Some of those permutations contain odd cycles, others only even cycles. We can show that any permutation containing at least one odd cycle will cancel out with another permutation because the matrix is anti-symmetric (by transposing only that specific cycle).

So, we remain with permutations having even cycles only. Sides of even circle might be colored into black and white. That splits the permutation into two perfect matchings. Each of those perfect matchings can be considered as a summand in the polynomial we described, so the determinant is what we get after multiplying that expression by itself (since each time we unite 2 pair decompositions, we get a permutation with even cycles). Working out the signs is left as an exercise ☺.

Targil 3 - functions.

1. Consider a function $f(x) = x - \frac{1}{x-a_1} - \frac{1}{x-a_2} - \ldots - \frac{1}{x-a_n}$, where $a_1, \ldots a_n$ are some real constants. Compute the total length of $f^{-1}([a,b])$.
(Here $[a,b]$ is an interval between a and b, $f^{-1}(set)$ denotes the inverse image of that *set* under *f*, that is all points that are sent by *f* to that set, and total length of several intervals is the sum of their lengths).

2. Let $f(x) = 2x(1-x)$, for $x \in \mathbb{R}$. Define $f_n(x) = f(\ldots f(f(x))\ldots)$, where *f* is applied *n* times.

(a) Compute $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

(b) Compute $\int_0^1 f_n(x)\,dx$ for all natural *n*.

3. Prove that there is no function $f:\mathbb{R}\to\mathbb{R}$ such that $f(0)>0$ and $f(x+y) \ge f(x) + y\cdot f(f(x))$ for all $x, y \in \mathbb{R}$.

4. Prove that for every continuous function $f:[0,1]\times[0,1]\to\mathbb{R}$,

$$\int_0^1\int_0^1 (f(x,y))^2\,dxdy + \left(\int_0^1\int_0^1 f(x,y)\,dxdy\right)^2 \ge \int_0^1\left(\int_0^1 f(x,y)\,dx\right)^2 dy + \int_0^1\left(\int_0^1 f(x,y)\,dy\right)^2 dx$$

5*. Can a minimal value of a polynomial with rational coefficients be $\sqrt{2}$?

By minimal value here we mean the value at a point of global minimum.

Targil 4 – parity and divisibility.

1. We are given a herd 2009 cows. For each cow, if it is taken aside, others can be divided into two sub-herds of 1004 cows and equal total weight.
Prove that all the cows have the same weight.

2.** (a) A square is divided into N triangles of equal area. Prove that N is even.
(b) Generalize it for higher dimensions (a cube is divided into simplexes).

3. On infinite empty chessboard, a rectangle of $m \times n$ pieces is placed. One type of operation is allowed: a piece can jump above the piece in adjacent cell to the next cell after it, which should be free, and then the piece above which it jumped is removed.
By adjacent cells we mean cells with common side.
The purpose of the game is to leave only one piece on the board.
For which m, n is it possible?

4. Two players play a game on the standard empty chessboard.
They have a chess knight (horse). The first player places it on the chessboard at any cell he wishes, then the second makes a legal move with the knight, then the first makes a legal move and so on. In addition to standard chess rules, the knight is forbidden to step on the same cell twice.
The player that can't make a move in his turn loses. Who of the two players has a chance to win?

5*. Let T be the set of all numbers of the form m^n, where $m > 1$ and $n > 1$ are integer. Compute $\sum_{t \in T} \frac{1}{t-1}$.

(Since T is a set, a number which can be represented both as m^n and a^b is counted only once.)

Targil 4 – parity and divisibility.

1. We are given a herd 2009 cows. For each cow, if it is taken aside, others can be divided into two sub-herds of 1004 cows and equal total weight.
Prove that all the cows have the same weight.

Solution. Let us fix a partition into two halves for each 2008 cows. This choice of partitions will be called ***configuration***. The condition that both halves of the partition will be of the same weight specifies a linear equation for the weights of cows, $x_1, x_2, \ldots, x_{2009}$. So the complete configuration specifies a system of 2009 linear equations. Those equations might be linearly dependent. However, with Gauss method, we can solve it: we can take several unknowns as parameters and obtain get the other unknowns as linear expressions in those parameters. All coefficients in those expressions will be rational, since Gauss method is an algebraic procedure.

So, for each configuration, and for each possible set of weights of cows, we can find ε-close rational weights of cows which satisfy all the condition with the same configuration. So, without loss of generality, we can assume that if not all cows will have the same weight, then they have rational weights. So, by multiplying weight of each cow by the denominator (or, alternatively, by choosing appropriate measurement units, since multiplying weight of a cow by a large number might not have physical meaning), we can assume, WLOG, that all cows have natural weights.

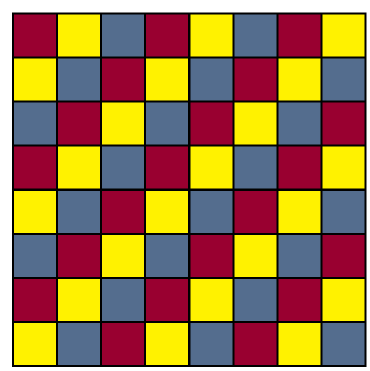
Out of all solutions with natural weights of cows, with not all cows of the same weight, take the minimal one, with minimal total weight. Then all weights cannot be simultaneously even, since otherwise we would be able to divide by 2.
Also, all weight all weights cannot be simultaneously odd, otherwise we could add 1 and divide by 2 and get smaller solution, unless all cows are of weight 1.
But all cows have to be of the same parity, since if we take any cow apart, we remain with the herd of even weight (since it can be divided into two equal parts).
That is a contradiction. QED.

2.** (a) A square is divided into N triangles of equal area. Prove that N is even.
(b) Generalize it for higher dimensions (a cube is divided into simplexes).

Alexey believes he had a very nice proof of this, which we cannot reconstruct.
Several ideas he remembers from that prove:

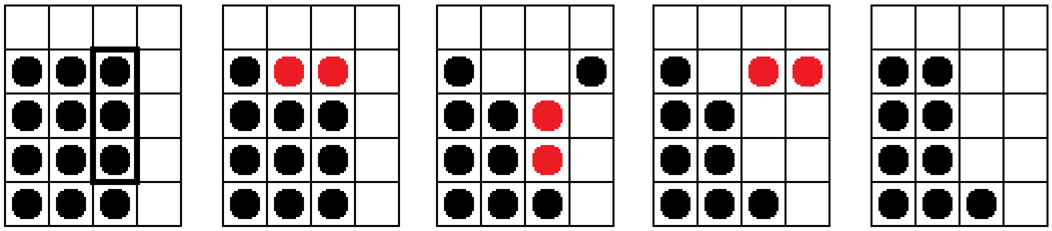
(1) Assume first coordinates are rational and arrive to a contradiction with that.
(2) If all coordinates are rational, multiply everything by the common denominator and make them integer. If common denominator was odd, it turns out that vertices of each triangle are on one line mod 2. Because an expression in coordinates with vector products is equal twice the area, so if number of triangles is odd those expressions turn out to be even. But the vertices of the square are not on one line mod 2.
(3) If the denominator is even, shift all odd coordinates by 1 mod 2, so that all conditions will still hold, and then divide all coordinates by 2.
(4) Though we cannot assume all coordinates are rational, we can assume all coordinates are algebraic. Algebraic numbers are similar to rational in some ways, and the proof for rational numbers can be modified to work with algebraic numbers.

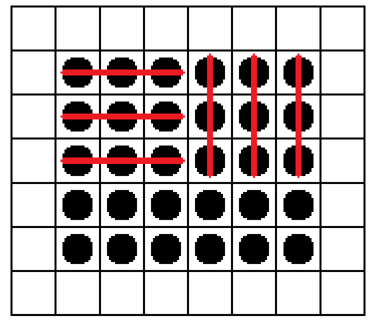
3. On infinite empty chessboard, a rectangle of $m\times n$ pieces is placed. One type of operation is allowed: a piece can jump above the piece in adjacent cell to the next cell after it, which should be free, and then the piece above which it jumped is removed.
By adjacent cells we mean cells with common side.
The purpose of the game is to leave only one piece on the board.
For which m, n is it possible?



Answer. When both m,n are >1 and not divisible by 3, or when they are 1 and 2.
Solution. We can paint the board into 3 colors, like on the picture. Each move is equivalent to inversion of 1×3 rectangle, so number of pieces on each color will change by 1, so parity of pieces on each color will change. If one side of rectangle is divisible by 3, then in the beginning there is the same number of pieces on each color, so on any stage of the game, we shall have either odd number on all colors or even number of all colors, so we won't arrive to 0, 0, 1 and so we shall never remain with just one piece.
If one side of rectangle is 1 we are trapped in just one line. We have just one sequence, so we can make a move on each edge outside, and after each move we shall still have just one sequence and a few separated discs, and we shll not able

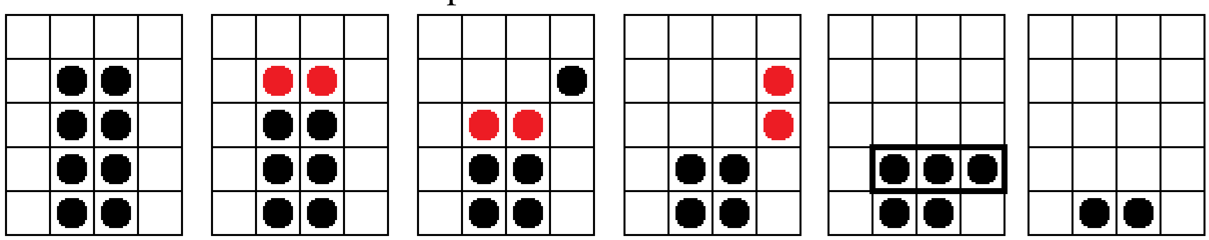
Now for the good case. Here is a way to eliminate a rectangle 1×3 near the corner:





Using this elimination, we can turn a rectangle $m\times n$ into rectangle $m\times(n\text{-}3)$ for $m \geq 3$, $n > 3$: eliminate top 3 discs in each column one by one, from right to left until only 3 columns remain, then eliminate top 3 discs with one move 3 times.

We can also turn a rectangle $m\times n$ into rectangle $m\times(n\text{-}3)$ for $m = 2$, $n > 3$, using 3 moves and elimination of a triple:

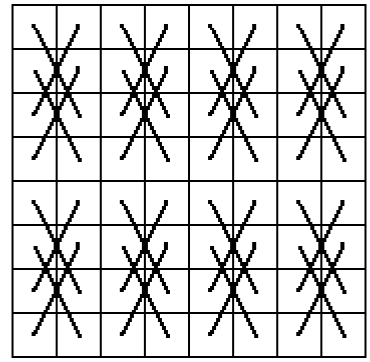


Also, for $m,n > 3$ we can turn rectangle $m\times n$ into rectangle $(m\text{-}3)\times(n\text{-}3)$ in a method, similar to the described above: from right to left, eliminate top 3 discs in each column until only 3 remain, then from top to bottom, eliminate left 3 discs in each row until only 3 remain, then from left to right, eliminate 3 leftmost discs 3 times.

After several procedures of that kind (reducing the biggest side of the rectangle each time if the difference between sides is bigger than one, and if sides are almost equal reducing them simultaneously), if m and n are not divisible by 3, we shall get either rectangle 2×2, or 2×1, or 1×1. The case 2×2 is reduced to 2×1 by 2 horizontal moves in the same direction. The case of 2×1 is reduced to 1×1 by 1 move.

4. Two players play a game on the standard empty chessboard.
They have a chess knight (horse). The first player places it on the chessboard at any cell he wishes, then the second makes a legal move with the knight, then the first makes a legal move and so on. In addition to standard chess rules, the knight is forbidden to step on the same cell twice.

The player that can't make a move in his turn loses. Who of the two players has a chance to win?



Answer. The second.
Solution. Divide the cells into pairs, connected by the move of the knight (for example, like in the picture). Each time the first player takes one of the cells of a certain pair, the second will take another pair.

5*. Let T be the set of all numbers of the form m^n, where $m > 1$ and $n > 1$ are integer. Compute $\sum_{t\in T}\frac{1}{t-1}$.

(Since T is a set, a number which can be represented both as m^n and a^b is counted only once.)

Remark. This problem was proposed to us by Dan, he also wrote down both solutions below.

First Solution Denote for any number $t \in T$, $c_1(t) = \#\{(m,n) \mid m,n \geq 1, m^n = t\}$ the number of ways to represent t as a power. Also denote by $c_2(t) = \#\{(m,n) \mid m,n \geq 2, m^n = t\}$ the number of ways to represent t non-trivially as a power, and $c_3(t) = \#\{(m,n) \mid m \in T, n \geq 1, m^n = t\}$ the number of ways to represent t as a power of a perfect power.

It is easy to verify that for any $t \in T$, $c_2(t) = c_3(t) = c_1(t) - 1$.

We now have, using summations of geometric and telescopic series:

$$\sum_{t\in T}\frac{1}{t-1}=\sum_{t\in T}\sum_{n=1}^{\infty}\frac{1}{t^n}\overset{s=t^n}{=}\sum_{s\in T}\sum_{(t,n):t\in T,n\geq 1,t^n=s}\frac{1}{s}=\sum_{s\in T}\frac{c_3(s)}{s}=$$

$$=\sum_{t\in T}\frac{c_2(t)}{t}=\sum_{t\in T}\sum_{(m,n):m,n\geq 2,m^n=t}\frac{1}{t}=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{m^n}=\sum_{m=2}^{\infty}\frac{1}{m(m-1)}=$$

$$=\sum_{m=2}^{\infty}\left(\frac{1}{m-1}-\frac{1}{m}\right)=1$$

Second Solution (Due to Euler and Goldbach)

Denote by $S=\mathbb{N}\setminus T\setminus\{0,1\}=\{n\geq 2\mid n\notin T\}$ the set of all numbers greater than 1 which are not perfect powers. Let $\boldsymbol{x}=\sum_{n=1}^{\infty}\frac{1}{n}$ denote the harmonic series. Then we my rearrange and write $\boldsymbol{x}=\sum_{t\in T}\frac{1}{t-1}+\sum_{s\in S}\frac{1}{s-1}$, as $T\cup S=\{2,3,4,...\}$. On the other hand, we have the property that any number $n\geq 2$ can be expressed <u>uniquely</u> in the form $n=s^m$ where $s\in S$ is a non power and $m\geq 1$ some natural. Hence by using the geometric series we get $\sum_{s\in S}\frac{1}{s-1}=\sum_{s\in S}\sum_{m=1}^{\infty}\frac{1}{s^m}=\sum_{n=2}^{\infty}\frac{1}{n}=\boldsymbol{x}-1$, from which we can derive $\sum_{t\in T}\frac{1}{t-1}=\boldsymbol{x}-\sum_{s\in S}\frac{1}{s-1}=\boldsymbol{x}-(\boldsymbol{x}-1)=1$.

Note that this proof is far from rigorous: The value of the harmonic series $\boldsymbol{x}$ is well known to be infinite, and so it is not quite necessary that rearrangements as those we did by adding and subtracting $\boldsymbol{x}$, as well as manipulating the terms of the divergent $\sum_{s\in S}\frac{1}{s-1}$, will really yield the correct answer. However, the proof can be modified (with some work) to a rigorous proof, as is done in this article.

(http://www.recercat.net/bitstream/2072/920/1/776.pdf)

Targil 5 – double counting.

1. In one country, there are 5 big and 19 small cities. The country is divided into 9 regions. Each big city is connected by bus to at least 14 cities, while each small city is connected by bus to at most 3 cities (each bus goes in both directions). Show that there exists a region in which no two cities are connected by bus.

2. a. Show that each map on a sphere has a country with less than 6 neighbors, and conclude that each map can be painted in 6 colors, so that countries having common border of positive length will be of different colors.
b. Show that each map on a sphere can be painted in 5 colors.
c. What is the maximal number of necessary colors for a map on a torus?

3. There are l unit vectors in n-dimensional space which are pair-wise orthogonal, and the orthogonal projection of each vector to a given k-dimensional subspace is longer than ε. Show that $l \leq k / \varepsilon^2$.

4. In a table there are N columns and M rows, N > M.
Some cells are marked by stars, and in each column there's at least one star.
Show that there is a star for which there are less stars in its column than in its row.

[Another way to formulate essentially the same problem: the books in the library were rearranged, so that for each book we have more books on the same shelf with it than before; show that now there is an empty shelf].

5*. Show that in a group of 50 people there are two that have an even number of common friends (maybe 0), assuming that friendship is symmetric.

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Show that there exists a region in which no two cities are connected by bus.

Solution. Estimate the connections with 19 small cities in two ways:

(a) Each small city is connected to at most 3 big ones, totally at most 57 connections.

(b) Each big city is connected to 14 cities at least, only 4 of those may be the other big cities, so at least 10 of connected cities are small, which is at least 50 connections.

We see that (b) is just slightly greater then (a), while (a) counts each (b)-type connection of big and small cities once, and each connection of two small cities twice. So twice the number of connections between 2 small cities is at most 7.
Therefore there can be at most 3 connections between pairs of small cities.
Since there are 9 regions, there's a region that contains neither those 3 connections between small cities nor any of the 5 big cities. Therefore, in that regions there are only small cities which are not connected to each other.

2. a. Show that each map on a sphere has a country with less than 6 neighbors, and conclude that each map can be painted in 6 colors, so that countries having common border of positive length will be of different colors.
b. Show that each map on a sphere can be painted in 5 colors.
c. What is the maximal number of necessary colors for a map on a torus?

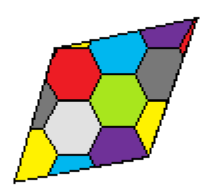
Solution. a. As in the famous Euler formula, denote: F number of faces, E number of edges, V number of vertices.
Count the edge-vertex incidence relations X two ways: it is exactly 2E, and at least 3V, since each vertex is at least on 3 edges. Hence $\boxed{2E/3 = X/3 \geq V}$.
Now count the number Y of face-edge incidence relations: it is again 2E, since each edge is of 2 faces, and at least 6F if we assume that each country has at least 6 neighbors, therefore $\boxed{E/3 = Y/6 \geq F}$.
If we sum up the two inequalities we get $E \geq V + F$, therefore the Euler expression $V - E + F$ is not positive, but it is equal to 2 by Euler formula.
Exercise. Find a map on sphere which is a counter-example to Euler formula and complete the proof for those cases too (hint in the end of the solution).

Remark. This statement (and Euler formula itself) can be proved by double-counting of angles in either Euclidean, spherical or hyperbolic geometries, but then an extra explanation is required that we can make all the borders straight line.
For example in Euclidean geometry: total sum of all angles is 360 degrees times number of vertexes, so the average angle is at most 120 degrees, so the average number of vertexes in the polygon is at most 6, but if you do it carefully you make it less than 6.

Theorem about 6 colors is proved by induction over the number of countries.
Since there's a country of less than 6 neighbors, if one of the neighbors would temporary take that country, we would be able to paint it into 6 colors by induction assumption, and when that country gains independence again, we can choose one of 6 colors which is different from all of its neighbors.

Remark. As for b' - we can't prove that there's a country of less than 5 neighbors: a dodecahedron is just one counter-example, and there's a lot of others, so a more subtle approach is required.

b. If the map has a country of less than 5 neighbors, we can use the induction. If not, consider a country C with just 5 neighbors (it can't be that all countries have at least 6 neighbors, as we have proven before).
It cannot happen that all 5 neighbors of C are connected to each other, since a full graph with 5 vertexes, K_5, is not a planar graph. So A and B, some two neighbors of C, are not connected.
Temporary unite A, B, C into one country. Since number of countries is smaller, now the map can be painted in 5 colors. Then make C a separate unpainted country. It has 5 neighbors, out of those 2 have the same color, so its neighbors have only 4 colors, so we can choose for C one of the 5 colors different from all its neighbors. A and B already have a color, and it doesn't make a problem since they are not neighbors.



c. Every map on torus can be painted into 7 colors, and there are maps that require all 7 colors. An example of a map which requires 7 colors is on the picture. It is cut out of periodic hexagonal pattern of 7 colors, each color touches every other color. When we glue the corresponding opposite sides of the parallelogram, we get a map of 7 countries on the torus, where each country is a neighbor of each.
As in a', double count and Euler formula prove that there is a country with no more than 6 neighbors, and after that we can apply induction.

Euler formula on torus is $V - E + F = 0$.
By double-counting we get: $2E/3 \geq V$ and $E/3 > 2E/7 \geq F$ (if each country has at least 7 neighbors).
So $E > V + F$ and Euler expression is negative when it should be 0.

Hint to the exercise: consider countries with holes.

3. There are l unit vectors in n-dimensional space which are pair-wise orthogonal, and the orthogonal projection of each vector to a given k-dimensional subspace is longer than ε. Show that $l \leq k / \varepsilon^2$.

Solution. For any set of less than n vectors, we can find a unit vector orthogonal to them all. So, we can complete our l unit vectors to an orthonormal system of n vectors. Choose a rotated coordinate system such that the mentioned k-dimensional space would contain the first k coordinate axis.
Now write down each vector in coordinates as a column, one after another, first the l given vectors and then the rest, and you get an orthogonal matrix.
Within this matrix, consider a sub-block of first k rows in the first l columns.
Apply double-counting to the sum of squares of the numbers in that sub-block.
Each of l first vectors in projection to the first k coordinates is of length ε^2 at least, so the sum is at least $l\varepsilon^2$. On the other hand, sum of squares in each row (even the whole row of any orthogonal matrix, not to talk of the sub-block), is at most 1, so the total sum of squares in the sub-block is at most k.
Conclusion: $l\varepsilon^2 \leq k$. QED.

4. In a table there are N columns and M rows, $N > M$.
Some cells are marked by stars, and in each column there's at least one star.
Show that there is a star for which there are less stars in its column than in its row.

[Another way to formulate essentially the same problem: the books in the library were rearranged, so that for each book we have more books on the same shelf with it than before; show that now there is an empty shelf].

Solution. Draw two tables, A and B, of the same size as the original table.
The cells that correspond to the empty cells of A or B will still be empty.
The cells that correspond to the marked cells of the original table will contain numbers according to the following rule:
If in the original table the row of the original cell contains k stars, the number that in table A will replace each star of that row is 1/k.

If in the original table the column of the original cell has l stars, the number that in table B will replace each star of that column is $1/l$.
Sum of all numbers in B is equal N, the number of columns, since each column had a star and sum in each is 1. Sum of numbers in A is at most M, the number of rows, which is less than N. So for some cell, the corresponding number in A is smaller than in B, so for that cell $1/k > 1/l$, and $l > k$, so for that star number of stars in the same row is bigger than the number of stars in the same column.

Remark. A combinatorial solution that doesn't use double counting can be devised for that problem, but neither as short nor as elegant, see for instance
http://taharut.org/Solutions/Ikarit/T11/Autumn/S_11_A_O_I6.doc

Of course, we could have allowed to mark more than one star in each cell, then there would be more than one number in both A and B table, but the proof would remain the same.
The story about books: if the rows of the table are shelves before the reordering, the columns of the table are the shelves after the reordering, and the stars are the books, we get again the same problem.

5*. Show that in a group of 50 people there are two that have an even number of common friends (maybe 0), assuming that friendship is symmetric.

First solution. Assume there's someone (denote him Yossi) who has an odd number of friend. Consider the subgroup of friends of Yossi. If there's anyone with even number of friends inside that subgroup, we got someone who actually has even number of friends with Yossi, QED.

If not, then we have that between odd number of people, each having odd number of friends in that group, so total number of pairs "someone and his friend" is odd, but it cannot be since it is equal to twice the number of friendships in that group.

So, it remains to consider the case where each person among the 50 has even number of friends. Then consider someone called Kobi. The rest of the crowd can be divided into two parts: even number of the friends of Kobi, denote them F, and odd number of the others, that group will be denoted as O.

If someone in has even number of friends in then he has an even number of the common friends with Kobi and we are done. So, it remains to assume that each person in has an odd number of friends in F. In particular, each person in F has even number of friends: odd number in F, Kobi himself, and even number from O.

So the number of friendships between F and O is even number of even numbers, and that is even.

On the other hand, each person in O has odd number of friends in F, and O is of odd order, so the number of friendships between F and O is an odd number of odd numbers, and that is odd.

That is a contradiction: the number of friendship connections between O and F is both even and odd.

Second solution. Consider the incidence matrix A: each number in column i row j is 1 if people i and j are friends and 0 otherwise. It can be considered in two ways: either as a matrix over integers or as a matrix with coefficients in the {0,1} field of two elements; we shall use the field of two elements.

Let v be a vector all coordinates of which are 1.

Like we have shown in the beginning of the previous solution, WLOG we can assume that each person has even number of friends. That means, each row/column is orthogonal to v, since the sum in each row is even, so $Av = 0$.

Consider A^2: On the cell $(i\ ,\ j)$ for $i \neq j$ You will have the number of common friends of i and j which is always 1 unless we get what we want, and on diagonal cell You get the number of friends of someone mod 2, which is 0.

So we know the A^2 matrix precisely, and $A^2v = v \neq 0$, which contradicts $Av = 0$.

Targil 6 – discrete derivative.

1. Which functions satisfy the following condition: for every triple of different real numbers, x, y, z, the inequality $\frac{f(x)}{(x-y)(x-z)}+\frac{f(y)}{(y-x)(y-z)}+\frac{f(z)}{(z-x)(z-y)}\geq 0$ holds?

2. A grasshopper performs an infinite sequence of jumps on the straight line. The length of the jump number n should be n^{5769}, but it is allowed to choose a direction of each jump. Show that it can visit all integer points if it wants.

3. Is it possible to divide [0, 1] into black and white intervals so that for each polynomial of degree < 5769, we shall have $\int\limits_{white} p(x)dx = \int\limits_{black} p(x)dx$?

4. a. Show that each integer number can be written as a sum of 5 cubes of integer numbers.
b. Find some natural number N (as small as You can), such that each integer number is a sum of N numbers of type k^{2009}, for integer k.

5*. Consider a set of points (x, y, z) such that x, y, z are integer nonnegative numbers not bigger than n, which cannot be simultaneously 0. What is the minimal number of planes not passing through (0, 0, 0) that contain all those points?

Targil 6 – discrete derivative.

1. Which functions satisfy the following condition: for every triple of different real numbers, x, y, z, the inequality $\frac{f(x)}{(x-y)(x-z)}+\frac{f(y)}{(y-x)(y-z)}+\frac{f(z)}{(z-x)(z-y)}\geq 0$ holds?

Answer. Convex functions.

First solution. WLOG $x < y < z$. Multiply by $(y - x)(z - x)(z - y)$.

$$(z-y)f(x)+(x-z)f(y)+(y-x)f(z)\geq 0$$

The LHS (left hand side) is twice the formula for oriented area of the triangle $(x,f(x)),(y,f(y)),(z,f(z))$.

Therefore, that triangle is positively oriented. Since the points go from left to right, it means that they are on a convex (smiling) function. Since the condition of the convex function is enough to verify on triples of points, this means that the original inequality means convexity.

Remark. If we wouldn't specify the order of points, the sign of the inequality might turn during multiplication, depending on the order of the points. Then we would get equivalence between the sign of the permutation of the first coordinates and the orientation of the triangle, which is essentially the same thing, but the explanation would be slightly longer.

Second solution. Consider a quadratic function $p(t)=a+bt+ct^2$, which coincides with f at x, y, z.

To find the coefficients, you have to solve the system of linear equations:

$$\begin{pmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}f(x)\\ f(y)\\ f(z)\end{pmatrix}$$

The solution might be found by Leibniz-Cramer. For example, the most significant coefficient, of second degree, the specifies the type of the parabola (smiling/sad) is

$$c=\left\|\begin{pmatrix}1 & x & f(x)\\ 1 & y & f(y)\\ 1 & z & f(z)\end{pmatrix}\right\| \Big/ \left\|\begin{pmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{pmatrix}\right\|=\frac{(z-y)f(x)+(x-z)f(y)+(y-x)f(z)}{(z-y)(y-x)(z-x)}$$

That is precisely the left hand side of the expression from the problem, so each such parabola is smiling (or at least not sad), so each triple is convex and the function itself is convex.

Very generic remark. Let function be given at points $x_0, x_1, \ldots, x_n$ and we want to compute the main coefficient of the polynomial of degree n accepting given values at given nodes. So, we write the system of equation (as we did above, but bigger) and now we want to solve it.
There are two standard ways to solve a square system of linear equations: Gauss method and Leibniz-Cramer rule. Gauss method is more practical, since it has smaller computational complexity, while Leibniz-Cramer formula is more elegant (at least if you like symmetric expressions), and both are applicable in our case.

If You apply Gauss method, You inductively obtain nice symmetric expressions, called Newton's divided differences http://en.wikipedia.org/wiki/Divided_differences . They are easy to compute, but hard to penetrate (which is something You would expect from Gauss method).

If you apply Leibniz-Cramer, it goes very similar to what we did in the second solution. You get a ratio of two determinants. The denominator is the usual Vandermonde, while the numerator is the Vandermonde with last column replaced by values of function. When you expand the numerator determinant along the last column, you get values of function multiplied by Vandermonde minors. If you write these coefficients and the denominator as products, and cancel what can be cancelled, you get Lagrange formula, which is equal to Newton's divided differences but much more symmetric:

$$\frac{f(x_0)}{(x_0-x_1)\cdot\ldots\cdot(x_0-x_n)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdot\ldots\cdot(x_1-x_n)}+\ldots+\frac{f(x_n)}{(x_n-x_0)\cdot\ldots\cdot(x_n-x_{n-1})}$$

In term k all brackets that didn't contain x_k were cancelled out.
For $n = 1$ you get the simple formula for slope, and for $n = 2$ you get the formula which appeared in this problem.

2. A grasshopper performs an infinite sequence of jumps on the straight line. The length of the jump number n should be n^{5769}, but it is allowed to choose a direction of each jump. Show that it can visit all integer points if it wants.

Solution. Consider a combination: jump in one direction and then jump to another direction. This will be called a combination of the first type. The length of the total jump will be $(n + 1)^{5769} - n^{5769}$ which is a polynomial of degree 5768 in n, $p(n)$.

Now consider a combination of the second type, which consists of combination of the first type in one direction and the combination of the first type to another.
It gives a total jump of $p(n + 2) - p(n)$ which is a polynomial of degree 5767.
Generalize that definition by induction: when the combination of type k is defined, define a combination of type k+1 as performing the combination of type k to one direction and then to another. By induction, we see that a combination of type k starting with step n leads to the jump by $P_k(n)$, where P_k is a polynomial of degree $5769 - k$. So, combination of type 5769 results in a jump by a constant number M to a direction of our choice.

So if we make a N combinations of type 5769 to the left and $2N$ combinations of type N to the right, we shall cover all integer points having the same remainder mod M as the original points, which are in the NM neighborhood of the original points. The only thing that remains to do is to learn how to switch to any given remainder mod M. If we learn that, we can switch to every remainder mod M one after another, and for each remainder cover a large interval of representatives, while this intervals will become larger and larger every time.

Assume the current position of the grasshopper is L mod M, and we want it to be K mod M. If M is even and K, L are of different parity, the first thing to do is to change parity. That is a simple thing to do: parity is preserved by every even jump and changed by each odd jump, so after one or two moves the parity will be the same. Now, it is enough to learn how to add 2 to our position mod M, and repeat it several times.
We can make several combinations of type 5796 without changing our remainder mod M. The numbers which are 1 mod M will be called a good number. If we see that the next combination of type 5769 doesn't contain a step with a good number, we shall simply do one more combination of type 5769 in arbitrary direction.
If we shall see that the next combination of the type 5769 will contain a step with a good number g we shall do something slightly different. We shall consider the combination of type 5769, direction chosen so that the jump number g will be to the left, and perform this combination, but with the jump number g reversed.
If we wouldn't reverse that jump we would have a combination of moves which keeps the same remainder mod M, but since we have reversed the g jump, and it is of length 1 mod M, we shall actually perform step 2 mod M, QED.

3. Is it possible to divide [0, 1] into black and white intervals so that for each polynomial of degree < 5769, we shall have $\int\limits_{white} p(x)dx = \int\limits_{black} p(x)dx$?

Answer. Yes.

Solution. For every interval, painted into black and white, and every function f on that interval, the *zebra integral* of the function will be $\int\limits_{white} f(x)dx - \int\limits_{black} f(x)dx$.

We have to construct a coloring of [0,1] such that the zebra integral will be 0 on every polynomial of deg < 5769. It really doesn't matter if we work with the interval [0,1] or any other interval, since linear substitution $ax+b$ can send interval [0,1] to any interval $[a, a+b]$ and if $p(x)$ is a polynomial of degree k iff $p(ax+b)$ is a polynomial of degree k.

We shall prove the statement by induction. If we paint first half of the interval black and the second half of the interval white, the zebra integral of every 0-degree polynomial is 0. Now assume we have constructed the coloring of [0,1] which is zero on all polynomials of degree less than k but non-zero on x^k.

Let $p(x)$ be a polynomial of degree k. Then its zebra integral with the given coloring depends only on the first coefficient. Then $p(x+1)$ has the same zebra integral. So, if we would shift the colored interval by 1 the zebra integral would be the same.

Now consider the following coloring of the interval [0,2]: is it equal to the original coloring on the interval [0,1] and inverse to the shifted version of the original coloring on the interval [1,2]. Then the zebra integral over [0,2] for each polynomial of degree k is a difference between 2 equal numbers, so it is 0, QED.

Remark. In the coloring that we've constructed is a color in each point may be described as the checksum of the first k bits in the binary expansion of the point coordinate.

4. a. Show that each integer number can be written as a sum of 5 cubes of integer numbers.

b. Find some natural number N (as small as You can), such that each integer number is a sum of N numbers of type k^{2009}, for integer k.

Solution. a. Consider the second discrete derivative of n^3,

$$(n + 1)^3 - 2n^3 + (n - 1)^3 = 6n$$

So every number which is divisible by 6 is a sum of 4 cubes

$$(n + 1)^3 + (-n)^3 + (-n)^3 + (n - 1)^3 = 6n$$

Now it is enough to find one example of integer k for each remainder mod 6, such that k^3 represents that remainder, and all other numbers would be obtained by adding multiples of 6 to those numbers. We could have written a list of 6 numbers with their cubes, but instead let us say that $n^3 = n \pmod 6$, since

$$n^3 - n = (n-1)n(n+1)$$

and that is divisible by both 2 and 3.

b. We shall prove it for N which is not really small.
First, consider the discrete derivative of order 5768 applied to n^{5769}.
By the discrete derivative of a function $f(n)$ defined at integer points we mean a function $f(n+1) - f(n)$. It reduces the power of a polynomial by 1.
By the discrete derivative of order k we mean what happens after you apply the discrete derivative k times, which is

$$f(n+k) - \binom{k}{k-1} f(n+k-1) + \ldots \pm \binom{k}{2} f(n+2) \mp \binom{k}{1} f(n+1) \pm f(n)$$

If the function is smooth, the discrete derivative is equal to the actual derivative at some point.
So, the discrete derivative of order 5768 applied to n^{5769} is a polynomial of order 1, of type $ax + b$. Among its values we have all the numbers which have certain remainder mod a (to be precise, all numbers which are divisible by 5769!, but that is not important for the proof), and all of them are sums of no more than 2^{5768} numbers of type n^{5769}.
By adding to those no more than a ones or zeroes, we get all integer numbers.

5*. Consider a set of points (x, y, z) such that x, y, z are integer nonnegative numbers not bigger than n, which cannot be simultaneously 0. What is the minimal number of planes not passing through $(0, 0, 0)$ that contain all those points?

Answer. $3n$.
Solution. It is not hard to construct an example of $3n$ planes that cover all the points:
Example 1. $\{x + y + z = m\}$, where m is any integer between 1 and $3n$.
Example 2. $\{x = m\}$ and $\{y = m\}$ and $\{z = m\}$, for m between 1 and n.

There are many others. The tricky part is to prove the minimality of those examples. Denote G_n the set of points with integer coordinates between 0 and n. Each plane has an equation which can be written in a form $ax+by+cz+d = 0$.

Multiply the LHSs (left hand sides) all those equations, and you get a polynomial in x, y, z that nullifies at all the points of G_n except (0,0,0). The degree of the polynomial is equal to the number of equations.

We shall prove the more general statement: if a polynomial $p(x, y, z)$ nullifies at all points of G_n except (0,0,0), its degree is at least $3n$.

The solution is simple: consider the discrete derivative in all 3 directions.

The discrete derivative in x, is $p(x+1, y, z) - p(x, y, z)$, the discrete derivative in y is $p(x, y+1, z) - p(x, y, z)$, and the discrete derivative in z is $p(x, y, z+1) - p(x, y, z)$.

It is easy to see that discrete derivative in each direction reduces the degree at least by 1 (enough to check it no monomials and that is really easy).

After we apply discrete derivatives 3 times, once in each direction, we get polynomial which nullifies on all elements of G_{n-1} except (0,0,0).

So, if we do that triple discrete differentiation n times, we get a polynomial which is non-zero at (0,0,0), so it is nonzero, so in the beginning we had a polynomial of degree $3n$ at least.

Remark. This was the last problem of IMO 2007 in Vietnam. It was solved only by two contestants: Peter Scholtze from Germany and Konstantin Matveev from Russia. The solution of Peter was approximately what was described above; as for Kostia, he wrote that the bound of the degree on polynomial follows from a theorem by Noga Alon, which was known to him, so-called combinatorial Nullstellenzsatz, (http://www.cs.tau.ac.il/~nogaa/PDFS/null2.pdf) and reproduced the theorem and the proof in his work.

Targil 7 – discrete convolution.

1. Without computer or calculator, find the decimal representation of $\frac{1}{81}$.

2. (a) Is it true, that for each polynomial with real coefficients p(x) there exists a polynomial q(x), such that q(x^3) is divisible by p(x) ?
(b) A point will be called ***even*** if both coordinates are integer even numbers. Function $f:\mathbb{Z}^2 \to \mathbb{R}$ will be called ***discretely harmonic***, if

$$f(x,y) = \frac{f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)}{4}.$$

Suppose we are given the values of a discretely harmonic function f at all even points except (0,0). Can we reconstruct $f(0,0)$?

3*. Prove that $x^{13}+2x^{12}-6x^{11}+2x^{10}-10x^8+4x^6+60x^5-44x^4-4x+4$ is irreducible over $\mathbb{Z}$.

4. You are given an N×M table of real numbers. The sum in every sub-square of 3×3 cells is positive, and the sum in any sub-square of 5×5 cells is negative.
What can we claim about M and N?
In other words, for which M and N a table satisfying the conditions exists?

5*. Consider the polynomial $1+x^2y^4+x^4y^2-3x^2y^2$. Prove that it is non-negative for any real *x*, *y* and that this polynomial cannot be represented as a sum of squares of polynomials with real coefficients.

Targil 7 – discrete convolution.

1. Without computer or calculator, find the decimal representation of $\frac{1}{81}$.

Answer. 1/81 = 0.0123456790123456790123456790… = 0.(012345679)
(when we take a sequence of digits in brackets, it means it is repeated infinite number of times).

Solution. It is easy to see that 1/9 = 0.(1) = 0.111111111111… .
So $1/81 = (1/9)^2 = 0.1\cdot(1/9) + 0.01\cdot(1/9) + 0.001\cdot(1/9) + \ldots =$
  0.0111111111111111111… +
+ 0.0011111111111111111… +
+ 0.0001111111111111111… +
+ 0.0000111111111111111… + …
= 0.123545…

Let us do it a little bit more carefully / rigorously.
First of all, if $10^K - 1$ is divisible by N, then 1/N has period K, since after moving K position in long division the remainder will be 1.
$10^1 = 1 \pmod 9$, that is why period of 1/10 is 1.
So, 10^M goes not over all remainders mod 81 but only over those that are 1 mod 9, and there are only 9 of those. They form an abelian group with multiplication, powers of 10 mod 81 form a subgroup, so the number of elements in the subgroup divides 9, so $10^9 = 1 \bmod 81$, and the period of 1/81 is no longer than 9.
Conclusion: we need to compute only 9 first digits after the decimal point, and the repeat them periodically.
Denote by $o(x)$ positive number not bigger than x.
So $1/9 = 10^{-1}+10^{-2} + \ldots + 10^{-12} + o(10^{-12})$.
$1/81 = (1/9)^2 = 10^{-2} + 2\cdot10^{-3} + 3\cdot10^{-4} + \ldots + 7\cdot10^{-8}+ 8\cdot10^{-9} + 9\cdot10^{-10}+10\cdot10^{-11} +$
$+ 11\cdot10^{-12} +12\cdot10^{-13} + 11\cdot10^{-14} + 10\cdot10^{-15} + \ldots +10^{-24} + o(10^{-12}\cdot2/9) =$
$= 10^{-2} + 2\cdot10^{-3} + 3\cdot10^{-4} + \ldots + 7\cdot10^{-8}+ 8\cdot10^{-9} + 10\cdot10^{-10} + 11\cdot10^{-12}+$
$+ o(\cdot10^{-12}\cdot12/9) + o(10^{-12}\cdot2/9) = 0.012345679011 + o(2\cdot10^{-11})$
From here we see the first 10 digits, and as we have explained before, that is enough.

2. (a) Is it true, that for each polynomial with real coefficients p(x) there exists a polynomial q(x), such that $q(x^3)$ is divisible by p(x) ?
(b) A point will be called ***even*** if both coordinates are integer even numbers.
Function $f:\mathbb{Z}^2 \to \mathbb{R}$ will be called ***discretely harmonic***, if

$$f(x,y) = \frac{f(x+1,y)+f(x-1,y)+f(x,y+1)+f(x,y-1)}{4}.$$

Suppose we are given the values of a discretely harmonic function f at all even points except (0,0). Can we reconstruct $f(0,0)$?

Answers. Yes and yes.

(a) First solution. Consider a linear map from a linear space of polynomials of x^3 to the linear space of polynomials of degree < deg p : remainder of division by p. This map has a non-trivial kernel. QED.

Second solution. Consider $\omega = \dfrac{-1+i\sqrt{3}}{2}$, a number such that $\omega^3 = 1$.

Notice that for any polynomial $p(\omega x) = p(x)$ iff powers of all nonzero monomials are divisible by 3.

So, for each polynomial p consider the polynomial $p(x)\,p(\omega x)\,p(\omega^2 x)$.

It is divisible by $p(x)$, it is real since $\overline{\omega} = \omega^2$ and it is stable under multiplication of x by ω so all its monomials have powers divisible by 3.

(b) We want to reconstruct $f(0,0)$ given values of even points. The values of a discretely harmonic function satisfy infinite number of linear relations, spanned by $f(m+1,n) + f(m-1,n) + f(m,n+1) + f(m,n-1) - 4f(m,n) = 0$.

The trick is to construct a linear combination of those, in which all non-even values cancel out and only even remain.

Consider polynomials in x, y, x^{-1}, y^{-1}; some call polynomial with negative powers Laurent polynomials.

For each linear combinations of the values of our harmonic function we shall correspond a Laurent polynomial in the following way:

$f(m,n)$ corresponds to $x^m y^n$,

linear combination of those corresponds to linear combination of corresponding monomial with the same coefficients.

For example $f(1,0) + f(-1,0) + f(0,1) + f(0,-1) - 4f(0,0)$ corresponds to $p(x,y) = x + y + x^{-1} + y^{-1} - 4$.

Another example: $f(m+1,n) + f(m-1,n) + f(m,n+1) + f(m,n-1) - 4f(m,n)$ corresponds to $x^m y^n p(x,y)$. Therefore, the basic relations and their linear combinations that can be computed from them correspond to multiples of $p(x,y)$.

So, in the new language, the question is: construct an example of Laurent polynomial, which is even in both variables and divisible by $p(x,y)$.

The simples construction is $p(x,y)\cdot p(-x,y)\cdot p(x,-y)\cdot p(-x,-y)$.

3*. Prove that $p(x) = x^{13} + 2x^{12} - 6x^{11} + 2x^{10} - 10x^8 + 4x^6 + 60x^5 - 44x^4 - 4x + 4$ is irreducible over $\mathbb{Z}$.

First solution. Assume that $p(x) = (a_k x^k + ... + a_0)(b_l x^l + ... + b_0)$.
First of all, look on this equation mod 2. You conclude that all the coefficients except a_k and b_l are 0 mod 2, so they are even, while are a_k and b_l odd.
$a_0 b_0$ is divisible by 4, but not by 8, and both factors are even, so neither of them is divisible by 2. So, a_0 and b_0 are even but not divisible by 4.
WLOG (without loss of generality), $k < l$.
Consider k'th coefficient of p :

$$a_k b_0 + \sum_{j=1}^{k} a_{j-k} b_k .$$

The first term is divisible by 2 but not by 4, all other terms are divisible by 4. So coefficient number k, which is in the lower half of polynomial p is divisible by 4 but not by 2. That is not our case: all coefficients from degree 0 to 6 are divisible by 4.

Second solution. For each polynomial $a_k x^k + ... + a_0$, we shall build an unbounded convex polygon in the following way.
First, we consider all nonzero coefficients. For each coefficient a_m let l be the order of 2 in prime decomposition that coefficient, which means that 2^l is the highest power of 2 that divides a_m. On the plane, we mark the point (m, l) and construct the ray going straight upward, starting at this point.
The convex hull of these rays is the unbounded polygon.

A polygon like that can be constructed for any polynomial. It is easy to see that for any pair of polynomials p and q, the polygon of pq is Minkowski sum of the polygon of p and the polygon of q.

Reminder. Minkowski sum of two sets A, B in the plane is the set of all $a + b$, such that a belongs to A and b belongs to B.
It is easy to see that Minkowski sum of two convex polygons is also a polygon, and that each side of it is either translated copy of a side of one of the polygons, or, if two sides of the added polygons are parallel, it can be a side in the same direction, and its length is sum of lengths of two sides.

Indeed, the points on the diagram can come from either product of coefficient in the diagrams, which is sum of both degree and order of 2. Contributions from product of different coefficients can cancel out partially or completely. If things cancel out completely, we have less coefficients to draw; if things cancel out partially, for example sum of two things that are divisible by 2 can sum up to something divisible by 4 or by 8, but it will still be a point inside the diagram, because we added those rays in the construction.

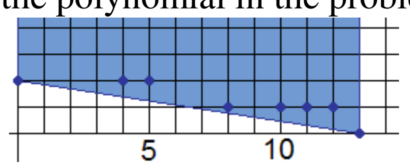
The important thing is, that points at the vertexes of the Minkowski sum don't cancel out. Indeed, each vertex of Minkowski sum come from two vertexes of the two original polygons, both are the extreme points in the same direction, so we have only one contribution at this vertex. Of course, we can have contributions with the same power of x and higher power of 2, but it won't influence the order of divisibility by powers of 2.

These diagrams lead us to the nice

Dumas criterion. If the lower boundary (without the vertical parts) of the diagram we have constructed doesn't contain integer points except the ends, then the polynomial is irreducible.

Indeed, a product of two polynomials would create some vertexes on the diagram, or at least a side with integer points in the middle.

Remark 1. We have constructed the diagrams with order of divisibility by 2, but it is possible to do the same thing with any prime number p instead 2.
Remark 2. The famous Eisenstein criterion is only a special case of Dumas criterion.
Now look at the diagram of the polynomial in the problem:



The lower boundary does not contain integer point between the ends, so it is irreducible.

4. You are given an N×M table of real numbers. The sum in every sub-square of 3×3 cells is positive, and the sum in any sub-square of 5×5 cells is negative. What can we claim about M and N?
In other words, for which M and N a table satisfying the conditions exists?

Answer. Either M or N should be less then 7.

First solution. In a table 7×7 sum of all sub-squares 3×3 is equal to sum of all sub-squares 5×5. This can be checked directly. By condition, that number should be both positive and negative. So, the table cannot contain 7×7 sub-square, so one of the dimensions should be at most six.
Consider the following line of six numbers:
3 -5 3 3 -5 3
Notice, that the sum of any 3 consequent numbers is 1, and the sum of any 5 consequent numbers is -1. By writing this line many times, we get N×6 table, such that the sum in each 3×3 sub-square is positive, and the sum in any 5×5 sub-square is negative.
We can reduce number of columns, and so we get an example of a table of no more than 6 rows. Rotation of those examples by the right angle produces examples for no more than 6 columns.

Second solution. To prove that the table doesn't exists for certain M and N, we shall try to arrive to a contradiction in the following way. We shall try to build a linear combination with positive coefficients of 3×3 sub-squares which is at the same time a linear combination of 5×5 sub-squares with positive coefficients. The expression which satisfies these requirements should be positive and negative at the same time, and that is a contradiction.
Any linear combination of cells can be coded as a polynomial in two variables, where cell in row a, column b is represented by $x^a y^b$ (both rows and columns have nonnegative numbers starting with 0) and linear combinations of cells will be written as linear combinations of corresponding monomials with the same coefficients.
Sum of sub-square 3×3 corresponds to some monomial times
$$p(x, y) = (1 + x + x^2)(1 + y + y^2)$$
Sum of sub-square 5×5 corresponds to some monomial times
$$q(x, y) = (1 + x + x^2 + x^3 + x^4)(1 + y + y^2 + y^3 + y^4)$$
Therefore, the polynomial corresponding to contradiction polynomial is, on one hand, $a(x, y)\, p(x, y)$ and on the other hand $b(x, y)\, q(x, y)$, where a and b are nonzero polynomials with nonnegative coefficients.
The product $p(x, y)\, q(x, y)$ certainly is of that kind, it has degrees 6 in x and in y, so we have built a contradiction for M, N $\geq$ 7.
The question is, whether there exists a contradiction polynomial of lower degree in one variable at least. The answer is no: indeed, there is unique factorization for polynomials, so any contradiction polynomial must be divisible by $(1 + x + x^2)$, by $(1 + y + y^2)$, by $(1 + x + x^2 + x^3 + x^4)$ and by $(1 + y + y^2 + y^3 + y^4)$, the polynomials

are coprime, so it must be divisible by their product. So any contradiction polynomial must be divisible by $p(x, y)\, q(x, y)$, so it requires M, N $\geq$ 7.

This doesn't complete the proof yet. Indeed, even though we can't find the contradiction with linear combinations of conditions, it still doesn't obviously follow that the table of numbers exists. One way out of this is to present an example (like we did in the first solution, another way is to prove a generic linear algebra statement (which reminds the last problem of second ***shlav***).

Theorem. Let F and G two families of cellular shapes on a board of K cells. Assume we try to put a number in each cell so that sum of numbers in any shape from F will be positive and in any shape from G will be negative. That is possible if and only if it is not possible to find an equality between linear combination of F shapes with positive coefficients and linear combination of G cells with positive coefficients.

One direction is obvious (if the table exists then linear combinations of F shapes give positive values on the table and linear combinations of G shapes give negative values). The other direction is what we actually need.
We shall consider cells as basic unit vectors in K-dimensional space, and shapes as their linear combinations. Linear combinations of F-shapes form a convex cone, as well as linear combinations of G-shapes. The statement we come to is geometric: if we have two non-intersecting convex cones, and we want to separate them by a hyperplane via the origin (so linear combination of F will be on the positive side, and linear combinations of G will be on the negative side. The values in the cells of the table are the values of the linear functional, which defines the hyperplane, on the basic unit vectors.
In our case, all the rays of both cones intersect the hyperplane

$$H = \{\text{sum of all coordinates} = 1\}$$

in the compact simplex, whose vertexes are basic unit vectors. So, in hyperplane H intersection with both cones are compact convex polytops, P and Q. If we separate the two polytops in H by a K – 2 plane S, then the hyperplane passing through S and the origin separates the cones. Hence somewhat unexpectedly, the theorem is reduced to a

Lemma. Two disjoint compact convex bodies P and Q in K – 1 dimensional space can be separated by a hyperplane.

Proof of lemma. Let X, Y be points in P, Q respectively, such that XY is the minimal. A pair like that exists, since P and Q are compact, hence P×Q is compact too, hence each function on P×Q has a minimal value.
The hyperplane that will be taken is perpendicular bisector of interval XY.

5*. Consider the polynomial $1+x^2y^4+x^4y^2-3x^2y^2$. Prove that it is non-negative for any real x, y and that this polynomial cannot be represented as a sum of squares of polynomials with real coefficients.

Solution. The non-negativity is Cauchy inequality for $n = 3$:
$$\frac{1+x^2y^4+x^4y^2}{3} \geq \sqrt[3]{x^6y^6} = x^2y^2$$
So, it remains to prove that it is not a sum of squares.

Definition. For each polynomial $p(x_1, x_2, \dots, x_k)$, each monomial specifies an integer point in $\mathbb{R}^k$ such coordinate j of the point is the power of x_j in that monomial. Convex hull of those points is called ***Newton polytope***.

Basic fact. If p, q are two polynomials, then the Newton polytope of pq is the Minkowski sum of Newton polytopes of p and of q.

Explanation. Consider generic vector v in $\mathbb{R}^k$. The point having highest scalar product with v on given polytope is unique. In that way, for each direction we have one vertex of Newton polytope of p and of q. Sum of this two points is a vertex on Minkowski sum, and all vertexes of Minkowski sum can be described that way. Product of corresponding monomials cannot cancel out with products of other monomials in pq, since it is the extreme point is some direction.
Exercise. Make sure this explanation can be made into a proof.

Squaring any polynomial multiplies its Newton polytope by 2.
After squaring, the coefficients corresponding to the vertexes of Newton polytope are positive. Indeed, they are squares of coefficients corresponding to the vertexes of Newton's polytope of the original polynomial.

Consider several polynomials, such that coefficients corresponding to the vertexes of their Newton polytope are positive. Then the Newton polytope of their sum of is the convex hull of their Newton polytopes. (In general, without the positivity condition on vertexes, it is wrong). Indeed, if any vertex of Newton polytope of one of the polygons cancels out with some coefficient of another polynomial, then the vertex is internal for another polytope and isn't on the convex hull of all polytopes anyway.

From now on we shall talk about Newton polygons, to indicate that we came down from general facts about polynomials to our specific polynomials of two variables.

Newton polygon of the polynomial which was mentioned in the problem is triangle

$$(0,0)\ ,\ (2,4)\ ,\ (4,2)$$

Assume that it was sum of squares. So, each square that participated in the game had Newton polygon covered by that triangle, therefore Newton polygons of each original polynomials before squaring was contained in the triangle

$$(0,0)\ ,\ (1,2)\ ,\ (2,1)$$

There are only 4 integer points in it: the 3 vertexes and (1,1). So each polynomial before squaring was of form $a + bxy + cxy^2 + dx^2y$.
When we take the square, the only contribution to coefficient (2,2) is $b^2x^2y^2$.
So, if our polynomial would be sum of squares then the coefficient of x^2y^2 woud be sum of squares, and hence nonnegative. And it is negative, contradiction.

Historical remark. This kind of examples was in fact known to Hilbert, and led him to formulate 17th problem in his famous list
http://en.wikipedia.org/wiki/Hilbert%27s_seventeenth_problem

Group action.

For definition see http://en.wikipedia.org/wiki/Group_action

This targil contains a crucial error, but I won't say which problem.

1. **(a)** Assume that prime number p divides $x^{16} + 1$. Then p is of form $32k + 1$.
(b) Without using the general Dirichlet theorem or L-functions, prove that for any k there is infinite set of primes of form $kn + 1$.

2. **(a)** Pizza consists of p sectors (p is a prime number). Each triangle can be one of a types (onions, olives, mushrooms and so on). Compute the number of possible nonequivalent pizzas.
(b)* Let p be a prime number. Compute the quantity of subsets of p elements of a set $\{1, 2, 3, \ldots, 2p\}$ such that sum of subset elements is divisible by p.

3*. The order of group G is $2^k(2l+1)$, and the group has an element of order 2^k. Prove that elements of odd order form a subgroup.

4**. Prove that the group of rotations of dodecahedron is A_5 (the group of even permutations of 5 elements), and groups of all dodecahedron symmetries is S_5 (the group of all permutation of 5 elements).

5. Let p be a prime number.
(a) Show that group of order p^k has nontrivial center (in other words, it has elements other than unit that commute with all the other elements).
(b) Show that any group of p^2 elements is commutative.
(c) Can a group of p^3 elements be non-commutative?

Group action - solutions.

The wrong statement was the second part of problem 4.

1. (a) Assume that prime number p divides $x^{16} + 1$. Then p is of form $32k + 1$.
(b) Without using the general Dirichlet theorem or L-functions, prove that for any k there is infinite set of primes of form $kn + 1$.

Solution. (a) Modulu p, the equation $x^{16} = -1$ has a solution. So, there is a remainder $x \bmod p$, which is of order 32. By Fermat little theorem, order of each element mod p divides $p - 1$, therefore $p - 1 = 32k$.

(b) We shall need the following notion:
Definition. Cyclotomic polynomial is.
In other words, it is a monic polynomial that has simple roots which are "roots of 1 of degree n precisely".
A simplest way to construct it is to take $x^n - 1$ and divide it by all common divisors with $x^k - 1$ for $k < n$. From this it can be seen that cyclotomic polynomial has integer coefficients.

Generic remarks on cyclotomic polynomials:
a. If You write down first one-hundred-something cyclotomic polynomials, you will get an impression that all their nonzero coefficients ±1. That impression is wrong.
b. It can be shown that all cyclotomic polynomials are irreducible. But that's an exercise with 2 stars, and we won't discuss it right now.
c. Cyclotomic polynomials have many applications, for example to geometry of regular polygons and to Galois theory.

Back to the original problem. Assume that there is only a finite number of primes of type $kn + 1$. Let P be the product of all those primes.
Consider the number $\Phi_n(P)$. It is not divisible by the divisors of P, since the free coefficient of cyclotomic polynomial is 1 for $n > 1$ (why?). But it has a prime divisor, p. We claim that p is of form $kn + 1$, which contradicts the assumption.
Indeed, $\Phi_n(P) = 0(\text{mod}\, p)$ hence $P^n = 1(\text{mod}\, p)$ and for no $m < n$ it we shall get $P^m = 1(\text{mod}\, p)$ because $\Phi_n(x), x^m - 1$ are coprime polynomials.
Therefore, the order of P mod p is n and by Fermat little theorem, n divides $p - 1$.

2. (a) Pizza consists of p sectors (p is a prime number). Each triangle can be one of a types (onions, olives, mushrooms and so on). Compute the number of possible nonequivalent pizzas.
(b)* Let p be a prime number. Compute the quantity of subsets of p elements of a set $\{1, 2, 3, \ldots, 2p\}$ such that sum of subset elements is divisible by p.

Answers. (a) $\dfrac{a^p - a}{p} + a$ (b) $\dfrac{\binom{2p}{p} - 2}{p} + 2$

Remark. From these one can get nice conclusions about divisibility.
From (a) we get Fermat little theorem: $a^p = a \pmod{p}$

Obvious generalization of (b) is $\binom{np}{kp} = \binom{n}{k} \pmod{p}$

Solution. (a) If pizza wouldn't rotate, we would obviously get a^p pizzas. However, group of rotations divides pizzas into orbits. As it usually happens with group action, order of orbit × order of stabilizer = order of group, which is p in our case. Since p has only two divisors, we conclude that each pizza belongs to orbit of 1 or of p. Pizzas which belong to the orbit of 1 are monochrome pizzas (all rotations keep preserve it, so all sectors are the same).
There are precisely a monochrome pizzas and $a^p - a$ non-monochrome states of pizza, the second amount must be divided by p since it consists of p-orbits.

(b) Consider the following rotation of numbers: each number smaller than p is increased by 1, and p is replaced by 1.
Like pizza rotations, this operation splits all subsets of p elements into orbits of p or 1. There are only two stable sets of order p, which are invariant with respect to that rotation: $\{1, 2, \ldots, p\}$ and $\{p+1, p+2, \ldots, 2p\}$.
Obviously, sums of elements in those two sets are divisible by p.
The other $\binom{2p}{p} - 2$ sets are divided into orbits of p. Consider one orbit. Sets in it have k elements not bigger than p, where $0 < k < p$. Each rotation adds k to the sum of elements mod p. So, in each orbit of p there will be only one set with the sum divisible by p.

3*. The order of group G is $2^k(2l+1)$, and the group has an element of order 2^k. Prove that elements of odd order form a subgroup.

Solution. We shall use induction on k. for k = 0 the statement is obvious.

Consider left action of group on itself. Each element of a group, when you multiply group elements by it, defines a permutation of group elements. Any element of odd order defines only odd cycles, so it defines an even permutation. An element of order 2^k defines 2m+1 cycles of even order (2^k), and that is odd permutation.

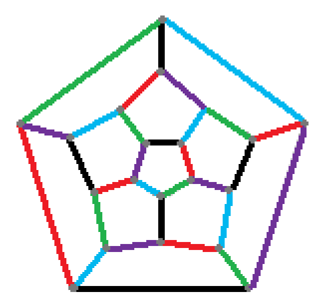
Consider a subgroup of those group elements that correspond to even permutations.

Those elements contain all elements of odd order, but not all elements in the group.

It is a subgroup of order 2, so we reduced our problem to a problem on a smaller group, which follows from induction assumption.

4.** Prove that the group of rotations of dodecahedron is A_5 (the group of even permutations of 5 elements), and groups of all dodecahedron symmetries is S_5 (the group of all permutation of 5 elements).

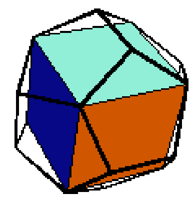
Solution. The statement about S_5 is false. Dodecahedron happens to have a transformation of central symmetry. Choose dodecahedrons center as a origin of Cartesian system, central symmetry is multiplication by -1 and all other transformation are linear, and multiplication by -1 commutes with any linear transformation.
So, the group of symmetries of dodecahedron has a nontrivial center, while S_5 doesn't.
Now we shall prove that is the group of rotations of dodecahedron is A_5.



Remark. A convenient way to draw a convex polytope on a plane is to perform stereographic projection from the center of some face.

We paint the edges of dodecahedron in 5 colors (see the picture). When you look at any face, each edge has the same color as the edge containing the opposite vertex of

this face and lying outside this face. In that way, any edge defines all the edges of the same color.
Every symmetry of the dodecahedron defines some permutation of colors.
That gives us a mapping from group of rotations of dodecahedron to the S_5.
There are 60 rotations: a given face can be sent to every face in 5 ways.
There are also 60 even permutations in S_5.
There are 24 possible cyclic orders of 5 elements; on the faces of our dodecahedron we see 12 different cyclic orders. Therefore, all rotations correspond to different permutations. Therefore the mapping that we constructed is injective.
It remains to prove that all the permutations we get are even.
Rotating of any face around its axis gives cycle of order 5, which is an even permutation. Any other rotation can be decomposed into product of those simple rotations, so it also corresponds to an even permutation. QED.



Remark. Another ways to formulate this group action:

a. Middle points of edges form 5 regular octahedrons. Rotations permute those octahedrons.
b. It is possible to inscribe 5 cubes into dodecahedron, so that the vertexes of the cubes are also vertexes of dodecahedron. Rotations permute those cubes.

I've decided that it is easiest to explain when picture can be made planar.

5. Let p be a prime number.
(a) Show that group of order p^k has nontrivial center (in other words, it has elements other than unit that commute with all the other elements).
(b) Show that any group of p^2 elements is commutative.
(c) Can a group of p^3 elements be non-commutative?

Solution. (a) Consider conjugation action of group by itself. That is, each element g in the group G defines a permutation: h goes to ghg^{-1}.
So, each element h has certain orbit: orbit is the set of all elements, to which h is sent by different g. A very naïve, but also a very powerful fact: when group acts on the set, the order of orbit × order of stabilizer (a subgroup of elements which don't move the element) = order of group.
In our specific case, order of group is p^k, so any orbit is of size power of p. That means, each orbit is either divisible by p or 1. Orbit of 1 under conjugation is of size 1, so there are several more orbits of size one, since one orbit of size 1 + several orbits of size divisible by p will never give you p^k.

Elements having orbit of size 1 are central elements, so center is nontrivial.

(b) Consider center C of group G. By (a), it is nontrivial. So C has p or p^2 elements. If it is p^2, we won, so from now on we shall assume it is p.
Let a be an element outside C. Then a and C span a commutative group. Its order is more than p, but still a divisor of p^2, so it is p^2. So it is the whole G and like we said, it is commutative.

(c) For example, consider matrixes over $\mathbb{Z}_p$ of the form $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$.

It is easy to check they form a non-commutative group of p^3 elements.

Targil 9 - unique factorization.

Reminders:
(a) A ***ring*** is a set of numbers with 3 arithmetic operation: +, –, ·, and some obvious axioms (for lists of axioms, use google/wikipedia).

(b) $\mathbb{Z}[\alpha]$ = the ring generated by integers and α.

(c) ***Unit***s of the ring are elements invertible under multiplication (such a that there exists b such that $ab = 1$).
(d) Element a is ***irreducible***, if for any decomposition $a = bc$ either b or c is a unit (but not both).
(e) ***Irreducible factorization*** of some element of a ring is a representation of it as a product of irreducible elements.
(f) A ring has ***unique factorization*** property, if any two irreducible factorizations of any element differ only by permutation of factors and multiplication of factors by units.

1. Consider ring $\mathbb{Z}\left[i\sqrt{d}\right]$, where d is a positive integer.

(a) Prove that if $d < 3$ there is unique factorization property.
(b) Prove that if $d \geq 3$ there is no unique factorization property.

2.* Find all integer solutions of equations:
(a) $x^2 + 4 = y^3$
(b) $x^2 + 2 = y^3$

3. Represent 1234321 and 123454321 as $a^2 + b^2$, where a, b are positive integers (using computer here is the moral equivalent of driving the morning run, and has the same effect but on a different group of muscles☺).

4. **(a)** How many ways are there to represent $2^k p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n} \cdot q_1^{j_1} q_2^{j_2} \cdot \ldots \cdot q_m^{j_m}$ as a sum of 2 integer squares, if p_i are prime numbers of type $4a + 1$ and q_j are prime numbers of type $4b + 3$?
(b)* For R > 0, show that number if integer points in the disc { $x^2 + y^2 \leq$ R } is

$$1+4\left(\left[R\right]-\left[\frac{R}{3}\right]+\left[\frac{R}{5}\right]-\left[\frac{R}{7}\right]+\left[\frac{R}{9}\right]-\left[\frac{R}{11}\right]+\left[\frac{R}{13}\right]-\ldots\right).$$

(c)* Consider "triangular" lattice, formed by points of $\mathbb{Z}[\omega]$, where $\omega \neq 1,\ \omega^3 = 1$.
Show that number of points of this lattice in the disc { $x^2 + y^2 \leq$ R } is

$$1+6\left(\left[R\right]-\left[\frac{R}{2}\right]+\left[\frac{R}{4}\right]-\left[\frac{R}{5}\right]+\left[\frac{R}{7}\right]-\left[\frac{R}{8}\right]+\left[\frac{R}{10}\right]-\left[\frac{R}{11}\right]+\ldots\right).$$

5. Suppose $n > m > 0$ are integers, $\phi = \arctan(m/n)$, prove $\left\{\frac{k\phi}{\pi}\right\} > \frac{1}{\pi}\left(\frac{1}{\sqrt{m^2+n^2}}\right)^k$.

($\{x\}$ denotes fractional part of x, which is a number in [0,1) equivalent to x mod 1)

Targil 9 - unique factorization.

1. Consider ring $\mathbb{Z}\left[i\sqrt{d}\right]$, where d is a positive integer.

(a) Prove that if $d < 3$ there is unique factorization property.

…

Solution. For any z, denote $\|z\| = |z|^2 = z \cdot \bar{z}$. If $z = a + ib\sqrt{d} \in \mathbb{Z}\left[i\sqrt{d}\right]$ then $\|z\| = a^2 + b^2 d$ is a nonnegative integer.

The proof is based on a sequence of lemmas, known as division with remainder and Euclidean algorithm, most of which must look familiar.

The ring that we investigate, $\mathbb{Z}[i]$ or $\mathbb{Z}\left[i\sqrt{2}\right]$, will be denoted R.

Lemma 1. If $z, w \in R, w \neq 0$, there exist $q, r \in R$, such that $z = qw + r$ and $\|w\| > \|r\|$.

Lemma 2. For every z, w there exists a common divisor $c \in R$ such that any common divisor of z and w in R is also divisor of c, and $c = mz + nw$, where $m, n \in R$. This c is unique up to multiplication by a unit and it is called greatest common divisor.

Lemma 3. If p is irreducible, and st is divisible by p (while $s, t, p \in R$), then either s or t is divisible by n.

After we prove the, life becomes easy. Indeed, take two different irreducible factorizations of the same number: $p_1 p_2 \cdot \ldots \cdot p_n = q_1 q_2 \cdot \ldots \cdot q_m$.

Both sides of the inequality are divisible by p_1, so by lemma 3 one of the factors q_i in the right hand side is divisible by 1. If one irreducible number is divisible by another, the quotient is a unit. So we may cancel out p_1 with q_i and a unit will remain in RHS, but this unit can be hidden into another element of the product. So we shall have a shorter identity, for which equivalence is already known by induction (base of induction: if one side of identity had only one element of the product, we get irreducible = something, so something is also irreducible, so both factorization have 1 element and it is the same).

So, it remains to prove the lemmas:

Proof of lemma 1. Consider $s = z / w$, a complex number but not necessary in R. Let q be the number in R closest to s. Since R geometrically is a rectangular lattice, $|s - q|$ is at most half the diagonal of a small rectangle, which is less than 1.

$$|s - q| < 1$$
$$|z - qw| = |sw - qw| = |s - q|\cdot|w| < |w|$$
If we denote $r = z - qw$ we got $|r| < 1$ and
$$z = qw + r$$
QED.

Proof of lemma 2. Using lemma 1, we build a sequence of identities:
$$z = qw + r_1$$
$$w = q_1 r_1 + r_2$$
$$r_1 = q_2 r_2 + r_3$$
$$r_2 = q_3 r_3 + r_4$$
$$\ldots$$
$$r_{n-2} = q_{n-1} r_{n-1} + r_n$$
The sequence $\|r_k\|$ is decreasing. A decreasing sequence of positive integers must stop, so at some moment the remainder will be 0:
$$r_{n-1} = q_n r_n$$
If certain k in R divides both z and w, by first identity of the sequence it divides r_1, so by second identity it divides r_2 and so on, hence by induction it divides r_n. Also, by the last identity r_n divides r_{n-1} so by the identity before the last it divides r_{n-2} and hence by the identity before that r_{n-3} and so on, and in the end we see that it divides both z and w.
Since each common divisor k of z and w is a divisor of r_n, so $\|r_n\| = \|kl\| = \|k\| \cdot \|l\|$, therefore $\|r_n\| \geq \|k\|$. So r_n is the longest among all common divisors of z and w.
Any other common divisor of the same length is divisible by r_n, and the ration is of length 1, so it is a unit, so any other divisor of maximal length is equivalent to r_n.
From each identity we can express the r_{k+2} as a linear combination r_{k+1} and r_k with coefficient in r. We start with $r_n = r_{n-2} - q_{n-1} r_{n-1}$.
Substitute $r_{n-1} = r_{n-3} - q_{n-2} r_{n-2}$, and we get $r_n = u_1 r_{n-3} + v_1 r_{n-2}$.
Substitute $r_{n-2} = r_{n-4} - q_{n-3} r_{n-3}$, and we get $r_n = u_1 r_{n-4} + v_1 r_{n-3}$.
This process continues until we get $r_n = uz + vw$.

Remark. $\{ uz + vw \}$ is an ideal, spanned by z and w. All its elements are divisible by all common divisors of z and w. What is more, with Euclidean algorithm we found d, such that $\{ kd \} = \{ uz + vw \}$. Ideals for which such d exists are called principal ideals.
The word "ideal" in ring theory historically appeared from thinking about rings where these statements are wrong, so attempt to find one number generating the ideal spanned by w and z fail completely, so they talked about "ideal numbers", which are not actual numbers we want them to be but subsets of the ring.

Proof of lemma 3. Assume that st is divisible by p but s and t separately are not. The common divisor of p and anything is either a unit or equivalent to p, but s is not divisible by p, so the greatest common divisor of s and p os 1, therefore

$$1 = as + bp$$

For similar reasons

$$1 = mt + np$$

Therefore

$$1 = (as + bp)(mt + np) = amst + (asn + bmt + bnp)p = Mst + Np = Kp$$

Hence p is a unit, which contradicts the assumption.

1. Consider ring $\mathbb{Z}\left[i\sqrt{d}\right]$, where d is a positive integer.

…

(b) Prove that if $d \geq 3$ there is no unique factorization property.

Solution

$$\left(1+i\sqrt{d}\right)\left(1-i\sqrt{d}\right)=d+1$$

$$\left(i\sqrt{d}\right)\left(-i\sqrt{d}\right)=d$$

One of these two numbers is even and hence divisible by 2.

2 is irreducible: indeed, if $2 = kl$, then $\|k\|\cdot\|l\|=4$, so either k or l is a unit or $\|k\|=\|l\|=2$ but we don't have numbers of length $\sqrt{2}$ in our ring.

So, if we would have unique decomposition, either $1+i\sqrt{d}$ or $1-i\sqrt{d}$ or $i\sqrt{d}$ or $-i\sqrt{d}$ would be divisible by 2, but they are not (you can divide them by 2 as complex numbers, but you will get outside R).

2.* Find all integer solutions of equations:

(a) $x^2 + 4 = y^3$

…

Solution. $(x + 2i)(x - 2i) = y^3$

The difference of between the two factors is $4i = -i(1+i)^4$.

$1 + i$ is irreducible, so the greatest common divisor of $x + 2i$ and $x - 2i$ is $(1+i)^N$, where N is nonnegative integer not greater than 4.

Irreducible factorization of $x + 2i$ is complex conjugate to irreducible factorization of $x - 2i$, and $1 + i$ is equivalent to its complex conjugate, so $1 + i$ and its equivalents appears in the same power in $x + 2i$ and $x - 2i$.

The total power should be divisible by 3, therefore it is 0 or 3.

Apart from power of $1 + i$ the two factors $x + 2i$ and $x - 2i$ have no common divisors, so both are cubes (because of unique factorization in the ring of Gaussian numbers $\mathbb{Z}[i]$).

So, $x+2i=(a+bi)^3=(a^3-3ab^2)+(3a^2b-b^3)i$.

$$\begin{cases} x=a^3-3ab^2=(a^2-3b^2)a \\ 2=3a^2b-b^3=(3a^2-b^2)b \end{cases}$$

The last equation implies $b=\pm1$ or $b=\pm2$.

First option: $b=\pm1$.

$$3a^2-1=3a^2-b^2=\pm2$$

$$3a^2=-1 \text{ or } 3\text{, so in previous equation it was + and } b = 1$$

$$a^2=1$$

We get $x=\pm2$, $y = 0$.

Second option: $b=\pm2$.

$$3a^2-4=3a^2-b^2=\pm1$$

$$3a^2=5 \text{ or } 3\text{, so in previous equation it was – and } b = -2$$

$$a^2=1$$

We get $x=(a^2-3b^2)a=\pm(1-12)=\pm11$, $y = 5$.

Verification shows that $(\pm2,0),(\pm11,5)$ are indeed solutions.

2.* Find all integer solutions of equations:

…

(b) $x^2 + 2 = y^3$

Solution. $(x+i\sqrt{2})(x-i\sqrt{2})=y^3$

In $\mathbb{Z}[i\sqrt{2}]$ we also have unique factorization.

The greatest common divisor of both brackets also divides their difference, which is $i2\sqrt{2}=(-i\sqrt{2})^3$. So, it is $(i\sqrt{2})^k$, k not bigger then 3.

Irreducible factorizations of $x+i\sqrt{2},x-i\sqrt{2}$ are complex conjugate and $i\sqrt{2}$ is equivalent to its complex conjugate, so $i\sqrt{2}$ and its equivalents appears in the same power in $x+i\sqrt{2},x-i\sqrt{2}$.

The total power of $i\sqrt{2}$ should be divisible by 3, and apart from that $x+i\sqrt{2}, x-i\sqrt{2}$ have no common divisors, so both are cubes.

$$x+i\sqrt{2}=\left(a+ib\sqrt{2}\right)^3=\left(a^3-6ab^2\right)+\left(3a^2b-2b^3\right)i\sqrt{2}$$

$$\begin{cases} x=a^3-6ab^2=\left(a^2-6b^2\right)a \\ 1=3a^2b-2b^3=\left(3a^2-2b^2\right)b \end{cases}$$

From the last equation:

$$\begin{cases} b=\pm 1 \\ 3a^2-2b^2=\pm 1 \end{cases}$$

The last equation gives two options for $3a^2$: either 3 or –1.
It should be divisible by 3, so in both last equations sign is +, a =

$$a=\pm 1, b=1$$

$$x=a^3-6ab^2=\mp 5$$

Both options give a solution with $y = 7$.

3. Represent 1234321 and 123454321 as $a^2 + b^2$, where a, b are positive integers.

Solution:

$$1234321 = 1111^2 = (101\cdot 11)^2$$
$$101 = 10^2 + 1$$
$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$
$$101^2 = (10^2 + 1)(10^2 + 1) = 99^2 + 20^2$$
$$(101\cdot 11)^2 = (99^2 + 20^2)\cdot 11^2 = 1089^2 + 220^2$$

$$123454321 = 11111^2 = (41\cdot 271)^2$$
$$41 = 25 + 16 = 5^2 + 4^2$$
$$41^2 = (5^2 + 4^2)(5^2 + 4^2) = 9^2 + 40^2$$
$$(41\cdot 271)^2 = (9^2 + 40^2)\cdot 271^2 = 2529^2 + 10840^2$$

Remarks.

I. In this solution, we used the lemma: sum of two squares times sum of two squares is sum of two squares. This can come out of an ingenious algebraic trick,

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

or from a simple-minded observation: $\|a+bi\|\cdot\|c+di\|=\|x+iy\|$.

II. In each case, we had two factors: one was $4k + 1$, another $4m + 3$.
There is no sense even to try decomposing q^2 into sum of two positive squares, if q is a prime number of type $4m + 3$. Because the only solution of the equation

$$x^2 + y^2 = 0 \pmod{q}$$

is (0, 0). If we would have a different solution we would get:

$$(x / y)^2 = x^2 / y^2 = -1 \pmod q$$

Then x/y is an element is of order 4 mod q, so $q - 1$ is divisible by 4 by Fermat little theorem. Therefore, if $a^2 + b^2$ is divisible by 11 (or 271) then both a and b are divisible by 11, so divide each of them by 11 and sum of squares by 11^2 and continue from there. So, when we took 11^2 out of the brackets, it was the only choice.

III. We could factorize each number in Gaussian integers. Decomposing natural prime number only has a chance when it is 2 or of form $4k + 1$.
So, for example

$$(a + bi)(a - bi) = 1234321 = 1111^2 = (101 \cdot 11)^2 = (10 + i)^2(10 - i)^2 11^2$$

Factorization of $a + bi$ is a unit times some factors from the right.
Norms of $a + bi$ and $a - bi$ are the same, so they should get the same number of factors. Each of them should get 11 (since they are conjugate) and two from the following $10 + i$, $10 + i$, $10 - i$, $10 - i$. That leaves us with only two options either each factor gets $10 + i$ and $10 - i$, or one takes $10 + i$ twice and another gets $10 - i$ twice. So, there are only two ways to represent it as a sum of squares of integers. The first way is $1111^2 + 0^2$, and the second is the only answer to our question. Similar with $1089^2 + 220^2$.

4. (a) How many ways are there to represent $2^k p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n} \cdot q_1^{j_1} q_2^{j_2} \cdot \ldots \cdot q_m^{j_m}$ as a sum of 2 integer squares, if p_i are prime numbers of type $4a + 1$ and q_j are prime numbers of type $4b + 3$?

Answer. $4(i_1 + 1)(i_2 + 1)\ldots(i_n + 1)$ if all j_k are even and 0 otherwise.

Solution. As we noticed in the remark II above, if q_k divides $a^2 + b^2$ then it divides both a and b, and we so we can say there are no decomposition into some of squares if $j_k = 1$ and if $j_k > 1$ it can be reduced by 2 and we get an equivalent problem. Repeating this, we get that if at least one jk is odd, then decomposition into 2 squares doesn't exist, and if all are even, the number of decompositions is the same as for $2^k p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n}$.

The crucial fact is
Lemma. A prime number p of type $4k + 1$ is representable as sum of integer squares.

First proof of lemma. (with Gaussian integers and unique decomposition)
Consider the equation $a^2 + b^2 = 0$ (mod p).
It is equivalent to $(a/b)^2 = -1$ (mod p) or to the statement a/b is of order 4 precisely mod p and so it has solution (because primitive root exists mod p *and* $p - 1$ is $4k$).
In other words, we have $a^2 + b^2 = mp$ but neither a, nor b nor m is divisible by p.
Let z be the greatest common divisor of $a + bi$ and p in the ring of Gaussian integers, then $\|z\|$ is a divisor of both $\|a+bi\| = a^2 + b^2$ and p^2, so it is 1, p or p^2.
But $\bar{z}$ is a common divisor of $a - bi$ and p, so $\bar{z}z$ is divisible by p and so z is not a unit. Also $\|z\|$ is not p^2 otherwise $a^2 + b^2$ would be divisible by p^2. So $\|z\| = p$.

Second proof of lemma. (Elementary)
As before, we argue that there is "$\sqrt{-1}$" mod p, i. e. a number c s. t.
$c^2 = -1$ (mod p).
Let S be a set of nonzero integer numbers in $\left[0, \sqrt{p}\right)$. Then $|S| = \left\lceil \sqrt{p} \right\rceil$.

Denote $cS = \{cx \mid x \in S\}$. Mark points on a circle of length p corresponding to the points of cS. You get a circle of length p divided into $\left\lceil \sqrt{p} \right\rceil$ arcs. So one of the arcs is shorter than $\sqrt{p}$. So, we have $0 \le k, l < \sqrt{p}$ such that $ck - cl = b$ (mod p) and $0 < b < \sqrt{p}$. Take $a = k - l$ and we get $|a| < \sqrt{p}, a \ne 0$, and $ca = b$ (mod p).
Therefore $0 < a^2 + b^2 < p + p = 2p$ and $a^2 + b^2 = a^2(1 + c^2) = 0$ (mod p).
So, $a^2 + b^2 = p$. QED

From the first approach we see that this decomposition is in fact unique: indeed if $(a + bi)(a - bi) = p$ then $\|a+bi\| = \|a-bi\| = p$ is prime so $a + bi$ and $a - bi$ are irreducible. So if $c^2 + d^2 = p$ then $c + di$ is equivalent to either $c + di$ or $c - di$, under multiplication by units.
Notice that $a + bi$ and $a - bi$ are not equivalent unless the angle between them is a multiple of 90°, that happens only when $p = 2$. Therefore for 2 we have 4 representations as $a^2 + b^2$, and for prime $p = 4k + 1$ we have 8 representations.

Now consider a composite number $2^k p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n}$.
Assume that $\|z\| = 2^k p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n}$, where z is Gaussian integer.
Up to multiplication by a unit, z is defined by its factorization, so we must count number of possible factorizations and multiply by 4. The only irreducible number of $\|\ \|= 2$ up to equivalence is $1 + i$, so we should have it in power k. Othe

irreducible factors should have $\| \ \| = p_t$. For each t, we should have precisely i_t of those, and there are two types (up to equivalence), any we can have 0, 1, 2, … , i_t of the first type and the rest of the second type. So, for each t we have $i_t + 1$ choices and these choices are independent, so there are $(i_1 + 1)(i_2 + 1)\ldots(i_n + 1)$ factorization. Now multiply by the number of units to get the answer.

4. (b)* For R > 0, show that number if integer points in the disc $\{ x^2 + y^2 \le R \}$ is

$$1+4\left([R]-\left[\frac{R}{3}\right]+\left[\frac{R}{5}\right]-\left[\frac{R}{7}\right]+\left[\frac{R}{9}\right]-\left[\frac{R}{11}\right]+\left[\frac{R}{13}\right]-\ldots\right).$$

Solution. Let us consider again the question that we have solved in 4(a): how many ways are there to represent integer $N > 0$ as $a^2 + b^2$, where a and b are integers (or, in other words, how many integer points does a circle with center at the origin and radius $\sqrt{N}$ have). We had an answer in terms of factorization, but there is a nicer way to formulate the answer.

Notations. Let $Z(N)$ be number of pairs of integers (a, b) such that N as $a^2 + b^2$,
$A(N)$ number of all divisors of N of type $4k + 1$ (not necessary prime),
$B(N)$ number of all divisors of N of type $4k + 3$ (not necessary prime).

Claim. For $N > 0$, $Z(N) = 4(A(N) - B(N))$.

Proof. We shall conclude it from the answer of 4(a) which uses factorization
$N = 2^k p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n} \cdot q_1^{j_1} q_2^{j_2} \cdot \ldots \cdot q_m^{j_m}$.
First of all, both answers disregard even divisors, so it is enough to prove it for odd N, i. e. $k = 0$.
Now there is a matching of divisors of N related to q_1, each couple has a ratio q_1 precisely: for any d which is a divisor of N, if of q_1 in the decomposition of d has the same parity as j_1 we divide by q_1, otherwise we multiply by d.
In each pair of this matching one divisor is of type $4k + 1$, another of type $4k + 3$.
If j_1 is odd then the matching is perfect and $A(N) - B(N) = 0$, and so is $Z(N)$.
If j_1 is even then the matching is not perfect: the things that aren't cancelled out by this matching are precisely divisors of $N' = N / q_1^{j_1}$. But we saw that $Z(N) = Z(N')$ so it is enough to prove for N'.
Doing the same for every q_s we either prove the statement or reduce it to the same statement for number $M = p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n}$. All its divisors are of type $4k + 1$, hence $A(M) = (i_1 + 1)(i_2 + 1)\ldots(i_n + 1)$, $B(M) = 0$, and the rest of it follows form 4(a).

Now that we have proved the claim, we shall prove 4(b).
Number of points in the disc is

$$\sum_{N=0}^{[R]} Z(N) = Z(0) + \sum_{N=1}^{[R]} Z(N) = 1 + 4\sum_{N=1}^{[R]} \left(A(N) - B(N)\right)$$

Notation. $D(d, N) = 1$ if d divides n
0 otherwise.

$$\sum_{N=1}^{[R]} \left(A(N) - B(N)\right) = \sum_{N=1}^{[R]} \left(\sum_{k=0}^{\infty} D(4k+1, N) - \sum_{l=0}^{\infty} D(4l+3, N) \right) =$$

$$= \sum_{k=0}^{\infty} \sum_{N=1}^{[R]} D(4k+1, N) - \sum_{l=0}^{\infty} \sum_{N=1}^{[R]} D(4l+3, N) = \sum_{k=0}^{\infty} \left[\frac{R}{4k+1} \right] - \sum_{l=0}^{\infty} \left[\frac{R}{4l+3} \right]$$

QED.

Remarks.
1. No convergence issues here when interchanging brackets because all summands except finite number of them are zeroes.
2. I know 4(b) from a wonderful book of Hilbert and Con-Vossen "Anschauliche Geometrie" (it has English and Russian translations).
3. Of course, in the limit when we neglect the [], we get the famous Gregory-Leibnitz formula $\pi = 4\arctan(1) = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right)$. It makes sense, since for large circles the discrete area (number of integer points) is close enough to the normal area (the difference of this two is bounded by the number of tiles chopped by the boundary, which is bounded by constant times length of circle, which is O(radius) while area is O(radius^2).
4. Area can be roughly measured also by triangular lattice, so formula of 4(c) also produces a formula for π when applied to large discs, but this time there's an ugly irrational factor related to area of regular hexagonal tile. This is no great wonder since $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \ldots = \int_0^1 \sum_{k=0}^{\infty} \left(x^{3k+1} - x^{3k+2}\right) dx = \int_0^1 \frac{x - x^2}{1 - x^3} dx$, and that is an elementary integral of type $\int \frac{1}{1+t^2} dt$ so it comes to an expression with π.
5. The question "when a number is sum of two squares" is known as Fermat theorem, it has many proofs (thanks Shahar for the link http://en.wikipedia.org/wiki/Proofs_of_Fermat's_theorem_on_sums_of_two_squares#Zagier.27s_.22one-sentence_proof.22)

4. (c)* Consider "triangular" lattice, formed by points of $\mathbb{Z}[\omega]$, where $\omega \neq 1$, $\omega^3 = 1$. Show that number of points of this lattice in the disc { $x^2 + y^2 \leq$ R } is $1+6\left([R]-\left[\frac{R}{2}\right]+\left[\frac{R}{4}\right]-\left[\frac{R}{5}\right]+\left[\frac{R}{7}\right]-\left[\frac{R}{8}\right]+\left[\frac{R}{10}\right]-\left[\frac{R}{11}\right]+\ldots\right)$.

Solution. Open discs of radius 1 centered at points of $\mathbb{Z}[\omega]$ cover the plane, from here we get in $\mathbb{Z}[\omega]$ the logical chain we got in problem 1(a):
Division with remainder => Euclidean algorithm => unique factorization.
Notice, that there are 6 units.

Next we come to a question, which circles of radius $\sqrt{N}$ have points on integer lattice and how many. It is the same as representing N in the form $a^2 - ab + b^2$ (or, which is an equivalent problem, in the form $a^2 + ab + b^2$).

The first answer, similar to 4(a), is: if $N = 3^k p_1^{i_1} p_2^{i_2} \cdot \ldots \cdot p_n^{i_n} \cdot q_1^{j_1} q_2^{j_2} \cdot \ldots \cdot q_m^{j_m}$, where p_i are of type $3s+1$, and q_j are of form $3s+2$, then it depends on the parity of j_l.
If at least one of them is odd, then it is impossible; if all of them are even, then the number of representations is $6(i_1 + 1)(i_2 + 1)\ldots(i_n + 1)$.
Of course, 6 here is the number of units in our ring.

The proof is very similar to 4(b), so we shall say only about the differences.
Firstly, the ring of Gaussian numbers is replaced by $\mathbb{Z}[\omega]$.
Secondly, the modular arithmetic is slightly different: in the previous example, we were reducing $a^2 + b^2 = 0 \pmod{p}$ to $(a/b)^2 = -1$ (mod p); in a similar way it is possible to reduce $a^2 + ab + b^2 = 0 \pmod{p}$ to $(a/b) = -3$ (mod p).
So this time we have to prove that -3 is quadratic mod p if and only if p isn't $3k-1$.
This is known from quadratic reciprocity for example.

After that, we reformulate the answer:

Notations. Let $T(N)$ be number of pairs of integers (a, b) such that N as $a^2 + ab + b^2$,
$U(N)$ number of all divisors of N of type $3k+1$ (not necessary prime),
$V(N)$ number of all divisors of N of type $3k+2$ (not necessary prime).
Claim. For $N > 0$, $T(N) = 6(U(N) - V(N))$.

The proof and the rest of it is similar to 4(b), I won't rewrite it (I could've cut and paste it, but it wouldn't be fair to the readers).

5. Suppose $n > m > 0$ are integers, $\phi = \arctan(m/n)$, prove $\left\{\frac{k\phi}{\pi}\right\} > \frac{1}{\pi}\left(\frac{1}{\sqrt{m^2+n^2}}\right)^k$.

($\{x\}$ denotes fractional part of x, which is a number in [0,1) equivalent to x mod 1)

Solution.

Notation. Given a real number a and a positive number b we shall denote a % b a number x in $[0, b)$ such that $x - a = nb$, where n is integer. It is quite obvious that x is uniquely defined.

The statement of the problem can be rewritten as $(k\phi)\%\pi > \left(\frac{1}{\sqrt{m^2+n^2}}\right)^k$.

We shall need the following lemma.

Lemma. Let $z = x + iy$ be a Gaussian integer. Let B be the union of four lines, coordinate and diagonal B = $\{x = 0\} \cup \{y = 0\} \cup \{x + y = 0\} \cup \{x = y\}$.
If z^k is not in B, then z is not in B.

We shall prove the lemma in the end. Denote $w = n + im$. Then $(k\phi)\%\pi$ is the angle between horizontal axis and w^k, measured counterclockwise. By lemma, it is nonzero. So, $\sin\left((k\phi)\%\pi\right) = \frac{\left|\operatorname{Im}\left(w^k\right)\right|}{\left|w^k\right|} = \frac{\left|\operatorname{Im}\left(w^k\right)\right|}{|w|^k}$.

$$(k\phi)\%\pi > \sin\left((k\phi)\%\pi\right) = \frac{\left|\operatorname{Im}\left(w^k\right)\right|}{|w|^k} \geq \frac{1}{|w|^k} = \left(\frac{1}{\sqrt{m^2+n^2}}\right)^k$$

QED, if we prove the lemma.

First proof of lemma. In the previous questions we have actually classified all irreducible Gaussian numbers:

(a) there are 4 things equivalent to $1+i$.
(b) There are real or imaginary things, equivalent to natural prime numbers of type $4k+3$.
(c) For each natural prime p of type $4k+1$ there are 4 Gaussian irreducible numbers equivalent to $a + bi$ and 4 other prime equivalent to $a - bi$; here $a^2 + b^2 = p$.

If z^k is in B, then z^{4k} is real. Therefore, for each irreducible factor $a + bi$ in z^k of type (c) we have the complex conjugate factor. So, the same is true for z. So, the factorization of z has complex conjugate pairs of type (c) which give a real number in the product times real factors of type (a) times units and factors of type (b) which preserve the British cross B.

Second proof of lemma. Consider the formula cos(2x) = 1 – 2cos^{2}x
It means that if cos x is a rational number of denominator more than 2, then the sequence cos(2x), cos(4x), cos(8x), …, cos(2^nx), … has growing denominators and can't have finite number of states, so x is not rational number degrees.
Cosine of a Gaussian number is a square root of a rational number.
So if $z = re^{i\phi}$ and the argument of z^n is rational in degrees, then the argument of z is rational in degrees, and $\cos(2\phi)$ is also a rational number, so its denominator is 2 at most. This leaves finite number of possibilities: $\cos(2\phi) = 0, \pm 1, \pm\frac{1}{2}$.
Some of these $\left(\phi = \pm 30^\circ, \pm 60^\circ, \pm 120^\circ, \pm 150^\circ\right)$ have irrational tan which cannot appear in Gaussian numbers; others are the British cross.

Targil 10 –algorithms

1. A road in desert area is a real line with camps at integer points. It is a day's walk between two camps. A human can carry 3 packed lunches, while he or she consumes one lunch each day. There is a base at 0. Things can be left only in the camps. It is required to organize an expedition, which will leave a packed lunch at camp 5, and all members of the expedition should return alive to the base. How many packed lunches are required?

2. (a) A soldier is a finite automata: his head has a finite number of states, and he can respond to a finite number of commands. A row consists of N soldiers, which are in the same initial state. Prove that they can be programmed so that several seconds after the leftmost soldier will receive a specific command, they will shoot simultaneously. Each soldier can pass a command to every neighbor during any second.
(b)* Show that for a row of length N no more and no less than 2N – 2 seconds are needed between the first command and the shooting, if the most efficient algorithm is used.

3. Show that in Conway's game "Life" there is a configuration without pre-image.

(The Game of Life http://en.wikipedia.org/wiki/Conway's_Game_of_Life is played on an infinite two-dimensional grid of square *cells*, each of which is in one of two possible states, *live* or *dead*. Every cell interacts with its eight *neighbors*. At each step in time, the following transitions occur:
1. Any live cell with fewer than two live neighbors dies, as if caused by underpopulation.
2. Any live cell with more than three live neighbors dies, as if by overcrowding.
3. Any live cell with two or three live neighbors lives on to the next generation.
4. Any dead cell with exactly three live neighbors becomes a live cell.)

4.* Denote C(n) minimal number of operations required to multiply a segment by n using compass. Denote CR(n) minimal number of operations required to multiply a segment by n using compass and ruler. Prove that C(n)/CR(n) is unbounded.

5.** An infinitely wise but a shortsighted cockroach is trying to find the truth (on the Euclidean plane). If he is in a distance of less than one step from the truth, he will reach it with the next step. After each step he is told whether he got closer to the truth. In the beginning he knows, that the truth is N steps from him.
Prove that the minimal number of steps required to find the truth is
(a) Less than $N + 10 \log_2(N)$
(b) More than $N + \log_2(N)/10$

Targil 10 –algorithms

1. A road in desert area is a real line with camps at integer points. It is a day's walk between two camps. A human can carry 3 packed lunches, while he or she consumes one lunch each day. There is a base at 0. Things can be left only in the camps. It is required to organize an expedition, which will leave a packed lunch at camp 5, and all members of the expedition should return alive to the base. How many packed lunches are required?

Answer. 243 (= 3^5)

Solution. Let us construct an example of expedition which consumes 243 lunches. We shall divide the expedition into 3 equal companies. First two companies will arrive to camp 1, place leave there 1 packed lunch each and come back.
The third company will take the lunches left by the second company and continue with the expedition. After they shall return to camp 1, they shall take packed lunches left by the first company and return home. So we see that if K lunches are enough to leave a lunch at camp N, then 3K lunches are enough to leave a lunch in camp N+1, so by induction 3^L lunches are enough to leave a lunch at camp L.
A proof that in no way we can manage with less than 243 packed lunches, can be constructed by introducing an appropriate energy function (a. k. a. semi-invariant). Define the energy of packed lunch at camp *K* as 3*K*, and the energy of a soldier at camp *K* as $-\sum_{l=1}^{K} 3^l = -\left(3+9+\ldots+3^K\right)$. That is precisely minus the energy required to bring him home, so any soldier returning home without extra supplies is preserving the energy. The total energy is sum of energies of all lunches and all soldiers.
If a soldier that takes 3 lunches from camp *K* to camp *K*+1 the energy of lunches changes from $3\cdot3^K$ to $2\cdot3^{K+1}$ so it goes up by 3^{K+1}, but the energy of the soldier is reduced by the same amount. If a soldier takes less than 3 lunches from K to K+1, or takes less than one lunch from K+1 to K, or stays in the camp and eats a lunch, then the energy is obviously reduced.
In the beginning of the expedition, all soldiers have energy 0, and lunches have energy N, where N is number of lunches acquired at afsanaut. In the end, all soldiers have energy 0 and lunch at camp 5 has energy 243. Since energy is not growing, N is at least 243.

Remarks (1). We used 81 soldiers. In fact, we could do it with one soldier, that would be coming back and forth many times. However, the expedition would take months, instead of ten days.
(2) The problem can be solved "from the end": assume that now we have a lunch at camp 5. Before that we had a man with 2 lunches there and 1 lunch waiting for him at camps 1, 2, 3, 4. Before that, we had a man with 4 lunches at camp 4 and one lunch waiting for him at camps 1, 2, 3. And so on. Of course, explaining minimality would be a mess.

2. (a) A soldier is a finite automata: his head has a finite number of states, and he can respond to a finite number of commands. A row consists of N soldiers, which are in the same initial state. Prove that they can be programmed so that several seconds after the leftmost soldier will receive a specific command, they will shoot simultaneously. Each soldier can pass a command to every neighbor during any second.
(b)* Show that for a row of length N no more and no less than 2N – 2 seconds are needed between the first command and the shooting, if the most efficient algorithm is used.

Solution. (a) First, we shall explain how to find the middle. We send two signals: one with a speed 1 soldier/sec and another with speed 1/3 soldier/sec. It means, the soldiers get an order to pass order A on next second after receiving it, and to pass order B on 3 seconds after receiving it. The rightmost soldier will pass the orders back.
The two signals meet in the middle of the row. Actually, they can almost meet: they can come with a delay of two seconds, but a soldier of finite brain can deal with that. Also, there can be one middle soldier or two middle soldiers.
Anyway, when the signals meet in the middle, the middle soldier/soldiers decides that he is an end-most soldier of a sub-line (or two sub-lines) who have just received an order to organize shooting, and row splits in halves (autonomous but synchronized).
In such case, he starts the process described above recursively.
If a signal comes back too quickly, means that halves of halves of etc. are short enough already (say of two soldiers), the soldier orders his neighbor to shoot and shoots the next second.
The process takes O(n) (about 3n I think) seconds to run.

(b) There is an easy way to prove that it cannot be better than 2n-2, even if soldiers would have infinite memory and brainpower. To synchronize the shooting, the chain should first compute its length (because the shooting time obviously depends on the length of chain). Before soldier 0 received an order, nobody knows nothing about the length of chain. Suppose soldier d was the first to get some indication of chain length. It can happen only after 2n-2-d seconds, because it can only be done after a signal from soldier 0 reached soldier n-1 and got back. Only then the decision can be made about the moment of shooting, and the earliest possible moment of shooting is 2n-2 because otherwise the command to shoot at that moment won't reach soldier 0 in time.

I don't really know the algorithm for 2n – 2, but there are some references and hints in wiki: http://en.wikipedia.org/wiki/Firing_squad_synchronization_problem They claim that the optimal solution was found by Abraham Waxman in 1966 after being an open problem for almost 10 years. I shall lay my hands on that article in a couple of days and send it to the community.

3. Show that in Conway's game "Life" there is a configuration without pre-image.

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3. Any live cell with two or three live neighbors lives on to the next generation.
4. Any dead cell with exactly three live neighbors becomes a live cell.)

First solution. Consider configurations in $N\times N$ square. They depend on the situation in the previous moment of $(N+2)\times(N+2)$ square.
Divide the of $(N+2)\times(N+2)$ square into 3×3 tiles (one or two untiled rows and columns may remain. A tile will be called ***special*** if it has one of the following two configurations:

0	0	0
0	1	0
0	0	0

or

0	0	0
0	0	0
0	0	0

We shall roughly estimate number o configurations with less than K special tiles. There are less than T^K ways to choose K tiles out of T, and the probability that all the tiles that weren't chosen are not special is α^{T-K} where $\alpha = \dfrac{255}{256}$.

So the probability that there are less than K special tiles is less than $T^K\alpha^{T-K}$.
Take $K = N^{3/2}$, and see what happens for large N:

$$\Pr < T^K\alpha^{T-K} = N^{2N^{3/2}}\alpha^{N^2-N^{3/2}} = \left(N^2\alpha^{N^{1/2}-1}\right)^{N^{3/2}} < \left(N^2\alpha^{N^{1/2}}\right)^{N^{3/2}} = $$

$$= \left(M^4\alpha^M\right)^{M^3} = \left(M\beta^M\right)^{4M^3}.$$

Where $M = \sqrt{N}, \beta = \sqrt[4]{\alpha} < 1$. Geometric sequence is stronger than arithmetic, so for sufficiently large *M* (same as sufficiently large *N*), the number $M\gamma^M << 1$ where $\gamma = \sqrt{\beta}$. Therefore the probability we've estimated

$$\Pr < \left(M\beta^M\right)^{4M^3} < \left(\gamma^{M/2}\right)^{4M^3} = \gamma^{2M^4} = \beta^{N^2}$$

Let us divide all configurations on (*N*+2)×(*N*+2) square into two types:

I. Those that have less than $N^{3/2}$ special tiles. According to above estimation, there are less than of $2^{(N+2)^2}\beta^{N^2}$ those. Pre-images of those configurations, for large *N*, can give not more than 1% of all possible configurations in the smaller square. Indeed, for large *N* we have $2^{(N+2)^2}\beta^{N^2} < \frac{1}{100}2^{N^2}$ because $2^{4N+4}\beta^{N^2} < \frac{1}{100}$

$$\left(\beta^N 16\right)^N = 16^N\beta^{N^2} < \frac{1}{1600}$$

Since for large n even $\beta^N 16 < \frac{1}{1600}$.

II. Second type: those that have at least $N^{3/2}$ special tiles. In each special tile, the central cell can be taken 0 or 1, and the outcome at the next stage will be the same. Therefore, this configurations come in families, each family has $2^{N^{3/2}}$ configurations that become the same one on the next step. So, those configurations are pre-images of at most $2^{(N+2)^2}/2^{N^{3/2}}$ configurations. That is also no more than 1% of all possible outcomes for large N. Indeed, $2^{(N+2)^2}/2^{N^{3/2}} < 2^{N^2}/100$ because $2^{4N+4-N^{3/2}} < 2^{\frac{-N^{3/2}}{2}} < 1/100$ for large N.

So, both kinds together give only 2% of configurations, and hence 98% of configurations don't have a pre-image.

Second solution (Alexey Gladkich). Consider a big square of *n*×*n* distinct squares of size 3×3. Chance that center cell of a 3×3 square being alive after one step is less than 50%. Denote it $2^{-1-\varepsilon}$. Therefore, number of big squares such that after a step, center of each of the 3×3 squares is alive is $2^{9n^2-(1+\varepsilon)n^2} = 2^{8n^2-\varepsilon n^2}$.

However, number of squares $(3n-2)\times(3n-2)$ in which these cells are alive is $2^{(3n-2)^2-n^2} = 2^{8n^2-12n+4}$, which is, for large n, much greater (since $\varepsilon n^2 >> 12n$).

4.* Denote C(n) minimal number of operations required to multiply a segment by n using compass. Denote CR(n) minimal number of operations required to multiply a segment by n using compass and ruler. Prove that C(n)/CR(n) is unbounded.

Proof. Recall that diameter of a set is the biggest distance between its points.
With compass only, the diameter can be only doubled by each action. Therefore, logarithm of diameter grows linearly.
With compass and ruler, if you have interval 1 and d you can build also interval d2 using Thales theorem (see the picture). So, we can take squares in constant number of operations. Therefore, we can double the logarithm of diameter, say, each then moves.
In other words, $\mathrm{CR}\left(2^{2^n}\right)$ is only linear is O(n) while $\mathrm{C}\left(2^{2^n}\right)$ is at least 2^n,

5.** An infinitely wise but a shortsighted cockroach is trying to find the truth (on the Euclidean plane). If he is in a distance of less than one step from the truth, he will reach it with the next step. After each step he is told whether he got closer to the truth. In the beginning he knows, that the truth is N steps from him.
Prove that the minimal number of steps required to find the truth is

(a) Less than $N + 10\log_2(N)$
(b) More than $N + \log_2(N)/10$

Solution. **(a)** The set of potential location of the truth is a circle.
First move (in any direction) turns it into an arc of length ~ $\pi N \pm 2$.
During next k moves, the arc can be divided into equal halves by each move.
Indeed, consider a line going through the middle point of the arc on distance ½ from the location of the roach, and step to the symmetric point w. r. t. that line.
Then we shall be told, which half plane (among the two half planes generated by that line) contains the truth. After $\lceil \log_2(\pi N) \rceil$ steps the location of the truth will be limited to an arc shorter than 1. It will take no more than $N+1+\lceil \log_2(\pi N) \rceil$ additional moves to arrive to the middle of that arc, and then the roach will see the truth.
What we got here is $N + 2\cdot\log_2(N) + \text{const}$, where const is a small number.
Actually, it can be made $N + \log_2(N) + \text{const}$, if at first stage we would arrange steps to approximately cancel each other.

(b) **Definition.** During the solution, the location of truth will be denoted T. A step from a point A to a point B will be called ***verification step***, if the angle BAT is at least $\pi/3$.

Obviously, a verification step reduces the distance between the roach and the truth only by ½ at most. Indeed, if BAT is obtuse then the distance even got bigger. Otherwise, if P is a projection of B to AT, then BT > PT = AT – AP > AT – ½.

So, it we have an algorithm of $N + \log_2(N)/10$ steps, no more than $\log_2(N)/5$ are verifications. Otherwise, we would approach by less than $\log_2(N)/10$ during those steps, and by no more than $N - \log_2(N)/10$ during the other steps, so we would not arrive to the truth.

Definitions. (1) Let O be the initial position of the roach. The ***circle of truth*** is a circle with center O and radius N.
(2) The ***arc of truth*** is an arc of the circle of truth having central angle $\pi/3$ and the truth as middle point. The endpoints of that arc will be denoted X and Y.

Lemma. While the roach is still in the circle of radius N/2 and center O, only verification steps can give any information concerning the location of truth within the arc of truth.

Proof. Consider regular (equilateral) triangle TXG, such that G is inside the truth circle. A simple computation of angles shows that OGX = 15^o = GXO, so the triangle OGX is isosceles, GO = GX =GT = TX. Since circle of radius OG passing through G intersects the circle of truth, its radius greater than N/2.
Therefore circle with center O and radius N/2 doesn't contain G. Since the closest point of circle TXG to O is G, we see that circle with center O and radius N/2 don't intersect.
Therefore, for each point C within N/2 steps from O, TCX is less then $60^o = \pi/3$.
Similarly, for each point C within N/2 steps from O, TCY is less then $\pi/3$.
Each step among first (N – 1)/2 generates two half-planes. If that step is not a verification, because of what we proved, X and Y are in the same half-plane as T, and so are all the points between X and Y. QED of lemma.

Let C be the point at which cockroach first time reaches the circle of radius N/2 with center O. Since there are no more than $\log_2(N)/5$, by the moment he arrives to C, the arc of truth will be divided by the generated pairs of half-planes no more than $\log_2(N)/5$ times. Therefore, the arc of *assumed location* of truth by the move N/2 might be at least of length $\frac{N\pi}{3}\Big/2^{\log_2(N)/5}=\frac{\pi N}{3N^{1/5}}$.

Therefore, the roach's idea of location couldn't be better than ½ of that number.
The way that has to be done is at least OC + CT. Let Q be a projection of T to the line OC. We may assume that x = QT is at least $\frac{N}{2N^{1/5}}$ (Although arc of circle is longer than perpendicular, but the ratio between them is less than π/3).

By Pythagoras: $\mathrm{QC}=\mathrm{QO}-\mathrm{N}/2=\sqrt{\mathrm{N}^2-x^2}-\mathrm{N}/2$

$$\mathrm{CT}^2=x^2+\left(\sqrt{\mathrm{N}^2-x^2}-\mathrm{N}/2\right)^2=\mathrm{N}^2+\frac{\mathrm{N}^2}{4}-\frac{\mathrm{N}}{2}\sqrt{\mathrm{N}^2-x^2}=$$

$$=\mathrm{N}^2\left(1+\frac{1}{4}-\sqrt{1-\frac{x^2}{\mathrm{N}^2}}\right)>\mathrm{N}^2\left(1+\frac{1}{4}-1+\frac{x^2}{\mathrm{N}^2}\right)=\mathrm{N}^2\left(\frac{1}{4}+\frac{x^2}{2\mathrm{N}^2}\right)=\frac{\mathrm{N}^2}{4}+\frac{x^2}{2}$$

Lemma 2. If b > a and $c^2 = a^2 + b^2$ then $c > b + a^2/3b$.
The proof is an easy exercise.

So, $\mathrm{CT}>\sqrt{\frac{\mathrm{N}^2}{4}+\frac{x^2}{2}}>\frac{\mathrm{N}}{2}+\frac{x^2}{2}\Big/\frac{3\mathrm{N}}{2}=\frac{\mathrm{N}}{2}+\frac{x^2}{3\mathrm{N}}\geq\frac{\mathrm{N}}{2}+\frac{\mathrm{N}^{3/5}}{12}$.

So the total way is at least $\mathrm{OC+CT}\geq\frac{\mathrm{N}}{2}+\frac{\mathrm{N}}{2}+\frac{\mathrm{N}^{3/5}}{12}=\mathrm{N}+\frac{\mathrm{N}^{3/5}}{12}$.

And that is much longer than we wanted.

Targil 11 – once again, linear algebra

1. Let $A_1A_2\ldots A_n$ be a polygon inscribed in circle. Consider a skew symmetric $n\times n$ matrix (a_{ij}), such that for $i < j$, $a_{ij} = A_iA_j$. Prove that the rank of this matrix is not greater than 2.

2. Let A, B, C be $n\times n$ square matrices. Prove that

$$\mathrm{rk}(AB) + \mathrm{rk}(BC) \leq \mathrm{rk}(ABC) + \mathrm{rk}(B).$$

3.* **(a)** A linear operator A over $\mathbb{C}^n$ can be considered as a linear operator A_r over $\mathbb{R}^{2n}$, because $\mathbb{C}^n$ is a $2n$-dimensional space over $\mathbb{R}$. Prove that $|\det(A)|^2 = \det(A_r)$.

(b) Formulate and prove a more general claim, about finite field extension (field $\mathbb{C}$ is an extension of field $\mathbb{R}$ of degree 2).

4. Consider matrix equation $AX - XB = C$, where A, B, C are given $n\times n$ matrixes, and X is an unknown $n\times n$ matrix. Show that the solution of the equation exists and unique if and only if A and B don't have a common eigenvalue.

5.* Let A be an invertible $n\times n$ real matrix, U, V be linear subsets of $\mathbb{R}^n$. Assume that U and V are ***almost disjoint***, which means they have no more common elements except 0.
Show that there exists an integer k such that A^kU and V are almost disjoint.

Targil 11 – once again, linear algebra

1. Let $A_1A_2…A_n$ be a polygon inscribed in circle. Consider a skew symmetric $n\times n$ matrix (a_{ij}), such that for $i < j$, $a_{ij} = A_iA_j$. Prove that the rank of this matrix is not greater than 2.

Solution. Assume that the circle is unit circle with center at 0 (it just divides the matrix by radius and doesn't affect the rank). Then $A_i = (\cos(2\varphi_i), \sin(2\varphi_i))$.
WLOG, the points go clockwise.
Then $a_{ij} = \sin(\varphi_i - \varphi_j) = \sin(\varphi_i)\cos(\varphi_j) - \sin(\varphi_j)\cos(\varphi_i)$.
Both matrixes $\{\sin(\varphi_i)\cos(\varphi_j)\}$ and $\{\sin(\varphi_j)\cos(\varphi_i)\}$ are of rank 1, hence their difference is of rank 2.
Remark. For the case of 4 points, the determinant is (see targil 2 problem 5): $(A_1A_2\cdot A_3A_4 - A_1A_3\cdot A_2A_4 + A_1A_4\cdot A_2A_3)^2$, and that is equal to 0, so for inscribed quadrilateral $A_1A_2\cdot A_3A_4 + A_1A_4\cdot A_2A_3 = A_1A_3\cdot A_2A_4$. This fact is called Ptolemy's theorem, so problem 1 is sometimes called generalized Ptolemy's theorem.

2. Let A, B, C be $n\times n$ square matrices. Prove that

$$\mathrm{rk}(AB) + \mathrm{rk}(BC) \le \mathrm{rk}(ABC) + \mathrm{rk}(B).$$

First solution. In other words,

$$\mathrm{rk}(AB) - \mathrm{rk}(B) \le \mathrm{rk}(ABC) - \mathrm{rk}(BC).$$

Denote $V = \ker B$, $W = \ker BC$. Obviously $V \subset W$.
For every linear transformation rk = n – dim(ker), therefore the statement may be rewritten as follows:

$$\dim V - \dim\ker(AB) \le \dim W - \dim\ker(ABC)$$

Let $v_1, v_2, \dots, v_k$ be the basis of ker A.
Complete it to the basis $v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_{k+l}$ of ker(AB).
By adding some more vector we can make the basis for ker(ABC):

$$v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_{k+l}, v_{k+l+1}, \dots, v_{k+l+m}.$$

Here k, l, m are nonnegative integers.
We claim that a non-zero linear combination of $Av_{k+l+1}, Av_{k+l+2}, \dots, Av_{k+l+m}$ is not in V. Indeed, if $a_1Av_{k+l+1} + \dots + a_mAv_{k+l+m} = A(a_1v_{k+l+1} + \dots + a_mv_{k+l+m})$ is in $V = \ker B$,

thus $a_1v_{k+l+1} + \ldots + a_mv_{k+l+m}$ is in ker(AB), so it is a linear combination of $v_1,\ldots, v_{k+l}$, but that is impossible since $v_1, \ldots , v_{k+l+m}$ are linearly independent.
Therefore, if $u_1, u_2, \ldots, u_r$ is a basis of V, then $u_1, u_2, \ldots, u_r, Av_{k+l+1}, \ldots, Av_{k+l+m}$ form a linearly independent system in W. Hence $r + m \leq \dim W$.
We wanted to prove that:

$$\dim V - \dim \ker(AB) \leq \dim W - \dim \ker(ABC)$$

In our new notation, that is

$$r - (k + l) \leq \dim W - (k + l + m)$$
$$r + m \leq \dim W$$

QED.

Second solution. This proof will be much shorter, but it uses some higher mathematics, namely quotient spaces. Recall, that if X is a linear subspace of linear space Y, then Y can be divided into equivalence classes: two vectors are equivalent, if their difference is in X. The set of those equivalence classes forms a linear space, which is called quotient space and denoted Y/X.

Like in the first solution, we shall transform the claim into form:

$$\dim V - \dim \ker(AB) \leq \dim W - \dim \ker(ABC)$$

Where $V = \ker(B)$, $W = \ker(BC)$. Also denote $V' = \ker(AB)$, $W' = \ker(ABC)$. Then the claim may be rewritten as follows:

$$\dim W' - \dim V' \leq \dim W - \dim V$$

A maps space W' into space W. If $w \in W'$ and $Aw \in V$ then $w \in V'$.

So, if $w_1, w_2 \in W'$ and $Aw_1 - Aw_2 \in V$, then $w_1 - w_2 \in V'$.

Therefore A induces an injective linear map from W'/V' to W/V. Hence

$$\dim(W'/V') \leq \dim(W/V).$$

The LHS is $\dim W' - \dim V'$, and the RHS is $\dim W - \dim V$.

3.* **(a)** A linear operator A over $\mathbb{C}^n$ can be considered as a linear operator A_r over $\mathbb{R}^{2n}$, because $\mathbb{C}^n$ is a $2n$-dimensional space over $\mathbb{R}$. Prove that $|\det(A)|^2 = \det(A_r)$.

(b) Formulate and prove a more general claim, about finite field extension (field $\mathbb{C}$ is an extension of field $\mathbb{R}$ of degree 2).

Solution. (a) We shall apply Gauss method to simplify the determinant computation of the complex matrix A, and see how will the determinant of the real matrix be transformed in the process.
Permutation of two rows in the complex matrix, that will multiply the complex determinant by -1, will result in permutation of two pairs of rows in the real matrix which won't change its determinant.
Subtracting the multiple of one row from another row in real matrix will result in subtracting linear combinations of two rows from two different rows in the real matrix, so both determinants will be preserved.
Same operations will behave in the same way on columns.
Complex matrix can be diagonalized by those operations. The elements on the diagonal will be x_1+iy_1, x_2+iy_2,…, x_n+iy_n. The real matrix, at the same time, will become a block matrix $\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}, \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n & -y_n \\ y_n & x_n \end{pmatrix}$.
The determinant of the real matrix is product of determinants of the blocks, hence the statement becomes obvious.

(b) Let K[α] be a separable finite field extension over field K. That means α is an algebraic number over K, its minimal polynomial over K is $p(x)$ which is of degree n, and it has n distinct roots in algebraic closure of K (but not in K).

Questions*: Is it true that irreducible polynomial of degree n over a field always has n distinct roots in its algebraic closure? Is it true that the finite field extension is always generated by one number?

So, in simple words K[α] is a set of polynomials of degree less than n in α. These polynomials have natural sums, differences, products and divisions (except by 0) which follow from the relation $p(\alpha)$.
A polynomial $p(x)$ has n distinct roots in the algebraic closure: $\alpha_1=\alpha$, α_2, α_3, …, α_n.
So each number $q(\alpha)$ in K[α] has n distinct conjugate numbers, itself included:

$$q(\alpha_1), q(\alpha_2), \ldots , q(\alpha_n)$$

Product of these n numbers will be called the norm of $q(\alpha)$.

Example. $\mathbb{C} = \mathbb{R}[i]$, it is field extension of degree 2 over $\mathbb{R}$. Any number in $\mathbb{C}$ can be represented as $a + bi$, polynomial of degree 1 in i.
The minimal polynomial $x^2 + 1$ has two roots, i and $-i$. Each element $a + bi$ has a norm $a^2 + b^2$, which is a product of two conjugate numbers, $a + bi$ and $a - bi$.

Now we can formulate the generalization.

Theorem. Let A be a matrix / linear operator over $K[\alpha]^m$. It can be considered as A_K a linear operator over K^{mn}, because $K[\alpha]$ is an n-dimensional linear space. Then the norm of det(A) equals det(A_K).

Proof. Like in (a), the theorem is easily reduced to 1-dimensional case by Gauss method, so we won't repeat it. But here, the one dimensional case is nonobvious. Multiplication by α is a linear operator over $K[\alpha]$.
In the basis $1, \alpha, \alpha^2, \dots \alpha^{n-1}$ it looks as follows:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_o \\ 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & a_{n-1} \end{pmatrix}$$

Here the last column contains minus the coefficients of the minimal polynomial of α, which is $p(x) = x^n - a_{n-1}x^{n-1} - \dots - a_2x^2 - a_1x - a_0$.
Since it is hard to guess eigenvalues of that matrix, we shall take the transposed matrix which has the same eigenvalues (see targil 2, problem 3b), and use it for the guessing. For any root α_k of the minimal polynomial,

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_o & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_k \\ \alpha_k^2 \\ \dots \\ \alpha_k^{n-1} \end{pmatrix} = \begin{pmatrix} \alpha_k \\ \alpha_k^2 \\ \alpha_k^3 \\ \dots \\ \alpha_k^n \end{pmatrix}$$

So, that is an eigenvector and α_k is an eigenvalue and all eigenvalues are different. So the matrix of α is diagonalizable, and has eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$. Therefore the matrix of $q(\alpha) = q$(matrix of α) and its eigenvalues are $q(\alpha_1), q(\alpha_2), \dots, q(\alpha_n)$. Hence the determinant is the product of those.

4. Consider matrix equation $AX - XB = C$, where A, B, C are given $n\times n$ matrixes, and X is an unknown $n\times n$ matrix. Show that the solution of the equation exists and unique if and only if A and B don't have a common eigenvalue.

Solution. I could have written a shorter proof, but I prefer to introduce ideas step by step. First, assume that A and B have a common eigenvalue λ. Then it is also eigenvalue of B^T, since B and B^T have the same characteristic polynomial.

We can find a vector v and a row u such that $Av = \lambda v$ and $uB = \lambda u$ (the latter is equivalent to $B^T u^T = \lambda u^T$).
Take $Y = vu$. Then $AY - YB = Avu - vuB = \lambda vu - \lambda vu = 0$.
So, for $C = 0$ we have an infinite family of solutions kY, and if for a certain C we have at least one solution X, then we also have an infinite family of solutions $X+kY$.

Now consider the case when A and B^T are diagonalizable. So, A has and eigenbasis of vectors $v_1, v_2, \ldots, v_n$ and B^T has eigenbasis of vectors $u_1^T, u_2^T, \ldots, u_n^T$, where u_k are rows. Then $\{v_i u_j\}$ is an eigenbasis of the operator $X \mapsto AX - XB$.
First let us check that it is a basis in the vector space of $n \times n$ matrixes. Since number of elements equals to a dimension, it is sufficient to show that $\{v_i u_j\}$ span the space; linear independence will follow. Denote by $\{e_k\}$ the standard basis of $\mathbb{R}^n$. Since both v_i span the space, for any k, we can write $e_k = \sum a_{ki} v_i$. For the same reason, for every m we can write $e_m^T = \sum b_{mj} u_j$. Therefore, for each k and m we have $e_k e_m^T = \sum \sum a_{ki} b_{mj} v_i u_j$. So, matrixes $\{v_k u_k\}$ span the standard basis for $n \times n$ matrixes (that is, matrixes having 1 in one cell and zero at all other cells) so they really span everything and thus they are basis.
By definition of eigenvector, $Av_i = \lambda_i v_i$, and $u_j B = \mu_j u_j$.
If $X = v_i u_j$, then $AX - BX = Av_i u_j - v_i u_j B = (\lambda_i - \mu_j)X$.
So we see that it is an eigenbasis for the operator $X \mapsto AX - XB$, and its eigenvalues are $\lambda_i - \mu_j$. The operator is invertible iff all eigenvalues are nonzero and that is when all the eigenvalues of A are different from all the eigenvalues of B.

Now, assume that A and B^T are not necessarily invertible. But anyway, for every matrix we can choose a basis (in algebraically closed field) such that the matrix will be upper triangular. In non-coordinate language bringing matrix A to upper triangular form means the following: we can choose a basis $\{v_k\}$, such that Av_k is a linear combination of v_j for $j \le k$, and the coefficient of v_k in that decomposition is the corresponding eigenvalue.
So, we choose such a basis for matrix A and a basis $\{u_j^T\}$ with the same property for matrix B. Then, for the reasons we've already explained, $\{v_i u_j\}$ is a basis for matrixes. We shall index this basis by $i + nj$ (indexing is needed to define upper triangular property).
Then $Av_i u_j - v_i u_j B = (\lambda_i - \mu_j)\, v_i u_j$ + linear combination of previous basis elements, so this basis brings the operator $X \mapsto AX - XB$ to a triangular form, and $\lambda_i - \mu_j$ appear on the diagonal, QED.

5.* Let A be an invertible $n\times n$ real matrix, U, V be linear subsets of $\mathbb{R}^n$. Assume that U and V are ***almost disjoint***, which means they have no more common elements except 0.
Show that there exists an integer k such that A^kU and V are almost disjoint.

First solution. This solution is very short, but it uses some higher math. I shall try to explain it here, but if it is not clear enough, we shall discuss it in greater detail during one of the meetings. The required higher math in this case is exterior power. If we have a linear space W, we can construct k exterior power of that space, $\wedge^k W$ as follows:

First consider expressions $w_1\wedge w_2\wedge\ldots\wedge w_k$, where $w_i \in$ W.

Consider linear combinations of those expressions. Introduce 3 types of relations:

$$a\,(w_1\wedge w_2\wedge\ldots\wedge w_i\wedge\ldots\wedge w_k)= w_1\wedge w_2\wedge\ldots\wedge(aw_i)\wedge\ldots\wedge w_k$$
$$w_1\wedge w_2\wedge\ldots\wedge(u + w_j)\wedge\ldots\wedge w_k = w_1\wedge w_2\wedge\ldots\wedge u\wedge\ldots\wedge w_k + w_1\wedge w_2\wedge\ldots\wedge w_j\wedge\ldots\wedge w_k$$
$$w_1\wedge w_2\wedge\ldots\wedge w_i\wedge\ldots\wedge w_j\wedge\ldots\wedge w_k = -\, w_1\wedge w_2\wedge\ldots\wedge w_j\wedge\ldots\wedge w_i\wedge\ldots\wedge w_k$$

The linear space formed by these linear combinations with these relations, is $\wedge^k W$.

The following properties of exterior powers are easy exercises:

1. If $\dim(W) = n$, then $\dim\left(\Lambda^l W\right)=\binom{n}{l}$.
2. There is a natural distributive product: $\Lambda^l W\times\Lambda^m W\to\Lambda^{l+m}W$, (it is called wedge product and denoted by $\wedge$)
3. A linear operator $A:W\to W$ naturally induces a linear operator $A_*:\Lambda^l W\to\Lambda^l W$. If A is invertible, then A_* is also invertible.

Remark. The last fact gives the most generic way to define determinant.
If you solved these exercises, you can read on.

In our problem, we have sub-spaces V and U in $\mathbb{R}^n$.

Let $v_1, v_2, \ldots, v_l$ be a basis of V and $u_1, u_2, \ldots, u_m$ be a basis of U.
Denote $v = v_1\wedge v_2\wedge \ldots \wedge v_l$ and $u = u_1\wedge u_2\wedge \ldots \wedge u_m$.
What we actually need is to find k such that $(A_*^{\,k} u) \wedge v$ is not zero.
A_* is invertible, so by Cayley-Hamilton theorem: $A_*^{\,N} + k_{N-1}A_*^{\,N-1} + \ldots + k_1A_* + k_0\mathbf{I} = 0$.

Here $\mathbf{I}$ is the identity matrix, $k_0 = \pm\det(A_*) \neq 0$ and $N=\binom{n}{l}$.

Apply this identity to u:

$$A_*^N u + k_{N-1}A_*^{N-1} u + \ldots + k_1 A_* u + k_0 u = 0.$$

Multiply it externally by v:

$$(A_*^N u)\wedge v + (k_{N-1}A_*^{N-1} u)\wedge v + \ldots + (k_1 A_* u)\wedge v + k_0 u\wedge v = 0.$$

The last term in this sum is nonzero, so there must be yet another term in the same sum which is non-zero. QED.

Second solution (Alexey Gladkich). It is sufficient to solve the problem when

$$\dim V + \dim U = n.$$

Otherwise we have a vector v which is not in span of U and V, add it to V and repeat it several times until $\dim V + \dim U = n$.

Let $v_1, v_2, \ldots, v_m$ be a basis of V and $u_1, u_2, \ldots, u_{n-m}$ be a basis of U.
For every k we construct M_k a matrix, first m columns of which are $v_1, v_2, \ldots, v_m$ and last $m - k$ columns are $Au_1, Au_2, \ldots, Au_{n-m}$. The determinant M_k is nonzero iff A^kU and V are almost disjoint.
Also, we may assume that A is of Jordan form in the standard basis.
Recall that the numbers appearing in the k'th power of Jordan cell of eigenvalue λ and size j are $\lambda^k p(k)$ where $p(k)$ is a polynomial of degree less than k (and all eigenvalues are nonzero since the matrix is invertible).
So, the numbers in M_k are sums of expressions of the kind $\lambda^k p(k)$ and so $\det(M_k)$ is sum of products of those so it is also linear combination of expressions of that kind. Then the statement follows to the following theorem, applied to the function $f(k) = \det(M_k)$

Theorem. Consider a function $f(k)$ which is a sum of functions $\lambda_i^k p_i(k)$, where p_i are polynomials, and all $\lambda_i \neq 0$. If $f(0) \neq 1$ then $f(k) \neq 0$ for a positive integer k.

We shall show two proofs for this theorem.

First proof of the theorem. Take terms with highest number $|\lambda_i|$. There can be more than one of that kind.
Out of these, take terms with highest power of x (for example, if you have $10^k k^2$ and $10^k k^3$ take only the last one). These terms after several steps become greater by much than all other terms.
So, the sum of these terms is $k^n\left(a_1\lambda_1^k + \ldots + a_m\lambda_m^k\right) = k^n r^k\left(a_1 e^{ikc_1} + \ldots + a_m e^{ikc_m}\right)$,
where $r = |\lambda_1| = |\lambda_2| = \ldots = |\lambda_m|$.

The bracket $b_k = \left(a_1 e^{ikc_1} + ... + a_m e^{ikc_m}\right)$ can be 0 for all nonnegative integers. In that case, we can easily delete these terms from the sum, and consider a shorter and smaller sum of all the other terms, and do the same thing to it. So, we shall assume that for a certain integer nonnegative q, we have $a_1 e^{iqc_1} + ... + a_m e^{iqc_m} = d \neq 0$. Choose $0 < \varepsilon << |d|$. We shall prove that we can find infinite number of k as large as we want, such that $|b_k - d| < \varepsilon$. From this it will follow that $|b_k| > |d| - \varepsilon$, hence for those k, the expression $k^n r^k \left(a_1 e^{ikc_1} + ... + a_m e^{ikc_m}\right)$ will be growing at least as fast as $k^n r^k \left(|d| - \varepsilon\right)$ and faster than all the other terms in the sum. So absolute values of the whole determinant for those k will be very large and far from zero.

The proof of that statement will be based on the following lemma.

Lemma. Given positive real numbers s_1, s_2, … ,s_m, for any $\delta > 0$, we can find infinitely many positive integers k, that will be as great as we wish, such that for all j the distance from s_j to a positive integer will be less than δ.

First, let us see how this lemma implies the solution of our problem.
Take $s_j = c_j / 2\pi$. If ks_j are close to positive integers, then e^{ikc_1} are close to 1, and b_{k+q} are close to b_k. So to make $| b_{k+q} - b_k | < \varepsilon$, we should simply choose a sufficiently small δ and apply the lemma.
Now it remains only to prove the lemma.

Proof of lemma. It is done by induction over m. For $m = 0$ we have an empty set of s_j and element of empty set satisfies every condition.
Suppose we already have a technology to produce sufficiently large k's that satisfy the condition for all numbers except s_m. Let us build a sequence of such numbers, which is very long and each number is bigger by much than the previous, and all numbers satisfy the condition for s_1, …, s_{m-1} with $\delta/2$ instead of δ:

$$k_1, k_2, \ldots, k_N$$

We may assume that $N > 2 + 1/\delta$.
Then $\{k_j s_m\}$ gives us N points on [0,1) interval, and at least two of them, $i < j$ are closer than δ. Then $k = k_j - k_i$ satisfy the condition. QED.

Second proof of the theorem. We shall apply discrete differentiating operators (like in targil 6). We have a function which is $f(k)=\sum_{j=1}^{m}\lambda_j^k p_j(k)$, which is nonzero at 0 and zero at all positive integers.
Consider the operator $\partial_\lambda : f(k)\mapsto g(k)=f(k+1)-\lambda f(k)$.
Applying such an operator to $\lambda_j^k p_j(k)$ (which is an easy exercise) produces $\lambda_j^k q_j(k)$, where q_j is polynomial of the same power as p_j if $\lambda_j \neq \lambda$, and a polynomial of lower power if $\lambda_j = \lambda$.
Also, when we apply such an operator to a function which is nonzero at 0 and 0 at all positive integers, we get again a function of the same kind.
But application of all ∂_{λ_j} sufficiently many times will turn our function into a constant, which is a contradiction.

Targil 12 – Analytic Geometry.

1. Consider segments AB, such that A is on x axis, B is on y axis, and length of AB is 1. The union of these intervals is a planar shape. Find an equation of the boundary of that shape.

2. For each t, take a line going through two points: $(t, 0)$ and $(0, 1-t)$. When we draw all these line, part of the plane will be painted. Find a curve that separates the painted part of the plain from the unpainted.

3.* We are given an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. A circle with center O is tangent to the ellipse externally (meaning they don't have internal common point); at the same time, there are two parallel lines tangent to both the circle and the ellipse. Find the locus of O satisfying these conditions.

4.* Let P be a point upon the rectangular hyperbola $\{xy = 1\}$.
Let D be a symmetric point to P with respect to 0. Suppose a circle with center at P intersects the hyperbola $\{xy = 1\}$ at 4 points: A, B, C, D. Prove that ABC is equilateral (regular) triangle.

Reminder. Each hyperbola has two asymptotes – straight lines that approximate it very well at all distant points. Hyperbola is called rectangular, if the asymptotes are orthogonal.

5.** For a triangle ABC in plane, consider rectangular hyperbolas, going through A, B and C simultaneously. Each of those hyperbolas has a center of symmetry. Prove that all these centers lie on one circle.

6. ABCD is a tetrahedron in the space. For each edge, consider plane passing via its midpoint and orthogonal to the opposite edge (for instance, a plane via the middle of AB orthogonal to CD). Prove that these 6 planes intersect in one point.

Targil 12 – Analytic Geometry.

1. Consider segments AB, such that A is on x axis, B is on y axis, and length of AB is 1. The union of these intervals is a planar shape. Find an equation of the boundary of that shape.

Solution. Playing a bit with a pencil sliding along the edges of the desk shows you that it sweeps star-like area consisting of 4 symmetric concave triangles. The whole point is to find the envelope of that family of lines.
AOB is a right triangle and AB = 1, we can take

$$A(\cos(t), 0)\ ,\ B(0, \sin(t))$$

So, let us take the intersection between two near intervals:

$$(\cos(t), 0)\ ,\ (0, \sin(t)) \text{ and } (\cos(t+\delta), 0)\ ,\ (0, \sin(t+\delta))$$

In general, the line passing via points $(a, 0)$ and $(0, b)$ has the equation $x/a + y/b = 1$ (it might remind the canonical form of equation of the ellipse equation). So, the two lines are

$$x / \cos(t) + y / \sin(t) = 1$$
$$x / \cos(t+\delta) + y / \sin(t+\delta) = 1$$

This is the same as

$$\begin{cases} \left(\dfrac{\sin(t+\delta)}{\cos(t+\delta)} - \dfrac{\sin(t)}{\cos(t)}\right) x = \sin(t+\delta) - \sin(t) \\ \left(\dfrac{\cos(t+\delta)}{\sin(t+\delta)} - \dfrac{\cos(t)}{\sin(t)}\right) y = \cos(t+\delta) - \cos(t) \end{cases}$$

$$\begin{cases} \dfrac{\sin(t+\delta-t)}{\cos(t+\delta)\cos(t)} \cdot x = \sin(t+\delta) - \sin(t) \\ \dfrac{-\sin(t+\delta-t)}{\sin(t+\delta)\sin(t)} \cdot y = \cos(t+\delta) - \cos(t) \end{cases}$$

When δ tends to 0, we get:

$$\begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \end{cases}$$

And that is parametric description of this curve.
From here we can get also the equation: $\sqrt[3]{x^2} + \sqrt[3]{y^2} = 1$.
When we know the answer already, the solution can be made much shorter: simply compute the tangent and see that it cuts axes where it should.

Remark. This curve is called astroid. Notice that

$$\tfrac{1}{4}\left(\cos(3x)+3\cos(x)\right)=\cos^3(t)$$
$$\tfrac{1}{4}\left(\sin(3x)-3\sin(x)\right)=\sin^3(t)$$

So, another way to describe the astroid is as follows: a trajectory of a point on the boundary of a coin of radius 1/3 which is rolling inside the circular box of radius 1.
There is yet another unexpected way to describe the astroid: the locus of curvature centers of an ellipse.

2. For each t, take a line going through two points: $(t, 0)$ and $(0, 1 - t)$. When we draw all these line, part of the plane will be painted. Find a curve that separates the painted part of the plain from the unpainted.

First solution. Like before, we shall take two close lines and find their intersection point.
The line equations are:

$$\begin{cases} \dfrac{x}{t}+\dfrac{y}{1-t}=1 \\ \dfrac{x}{t+\delta}+\dfrac{y}{1-t-\delta}=1 \end{cases}$$

$$y\left(\frac{t+dt}{1-t-dt}-\frac{t}{1-t}\right)=dt$$

$$y\frac{(t+\delta)(1-t)-t(1-t-\delta)}{(1-t-\delta)(1-t)}=\delta$$

$$y\frac{\delta\cdot(1-t)+t\cdot\delta}{(1-t-\delta)(1-t)}=\delta$$

$$y\frac{\delta}{(1-t-\delta)(1-t)}=\delta$$

$$y=(1-t-\delta)(1-t)$$

When δ tends to 0, we get $y=(1-t)^2$.
There is a symmetry: we can replace x by y, and y by x, and t by $1 - t$, and get $x = t^2$. Alternatively, we can substitute the known value for y into the first equation, and get:

$$\frac{x}{t}+\frac{(1-t)^2}{1-t}=1$$

$$\frac{x}{t}+1-t=1$$

$$x=t^2$$

So, we have the parametric description: $\left(t^2,(1-t)^2\right)$.

It is tempting to write $\sqrt{x}+\sqrt{y}=t+(1-t)=1$. However, it is wrong. Indeed, the square root is the inverse of square only for positive numbers, so that equation only describes the arc of the curve when t and $1 - t$ are both nonnegative.

Also, that would be a sure way to get a contradiction in mathematics. The curve $\sqrt{x}+\sqrt{y}=1$ is contained in the square $[0,1]^2$ so it cannot touch, for instance, the line that goes via (3,0) and (0,–2).

Consider rotated coordinates:

$$\begin{cases} u = x - y = t^2-(1-t)^2 = 2t-1 \\ v = x + y = t^2+(1-t)^2 = 1-2t+2t^2 \end{cases}$$

Clearly, since $t = (u + 1)/2$, that line is a parabola.

So, the answer is: a parabola which is rotated by 45 degrees, and tangent to the axes at (0,1) and (1,0). It remains to check that the outer side of parabola is completely covered by our family of lines, and another isn't. It easy to see from the above computation, that the given family of lines is precisely the family of tangents to the parabola. The rest of it is an exercise (it follows from the convexity of parabola, and the fact that it doesn't have asymptotes).

Second solution. This solution is very simple, but I wouldn't find it if I wouldn't guess the answer first, which was noticed by Markelov.
It is based on the deep similarity between the circle and the parabola.

For example, compare the next two lemmas:
Lemma 1. Let A, B be two different points on a circle such that the lines PA, PB are tangent to the circle. Then PA = PB.
Lemma 2. Let A, B be two different points on a parabola $y = ax^2 + bx + c$ circle such that the lines PA, PB are tangent to the parabola. Then the projections of intervals PA, PB to the x axis are of the same length.

The lemmas are simple exercises (if you didn't know them yet). That similarity is deep: many geometric theorems about circles might be translated into theorems about aligned parabolas. During the translation, the distances must be replaced by the lengths of x–projections.

Consider parabola $y = ax^2 + bx + c$ and consider two points, A and B, such that the slope of tangent lines at those points is 45°. Let P be another point on the parabola. The tangent line at P intersects tangent lines at A

an B at points K and L, respectively, and tangent lines at A and B intersect at point T.
For each vector v, by v_x we shall denote its x projection. So, by lemma 2:

$$KL_x = KP_x + PL_x = AK_x + LB_x$$

But

$$KL_x + AK_x + LB_x = AB_x$$

So

$$KL_x = AK_x + LB_x = AB_x / 2$$

And from this the claim follows directly.

3.* We are given an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. A circle with center O is tangent to the ellipse externally (meaning they don't have internal common point); at the same time, there are two parallel lines tangent to both the circle and the ellipse. Find the locus of O satisfying these conditions.

Answer. A circle with center (0,0) and radius $a + b$.
Solution. The answer is easy to guess. When the pair of tangent lines is rotating, O goes around the 0 by a symmetric curve, which is definitely algebraic (since the conditions look algebraic) and probably of low degree. The vertical and horizontal pairs of tangent lines give yet another clue. So, now that we've guessed what to compute, let's do it.
Let T be the point of tangency between a circle and the ellipse.
Since it is on the ellipse, it can be written as $\left(a\cdot\cos(t),b\cdot\sin(t)\right)$.
The gradient of the function $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ is $\left(\frac{2x}{a^2},\frac{2y}{b^2}\right)$. It is orthogonal to the level sets, one of which is an ellipse. So, the vector $\left(\frac{2\cos(t)}{a},\frac{2\sin(t)}{b}\right)$ is orthogonal to the ellipse at $\left(a\cdot\cos(t),b\cdot\sin(t)\right)$. It is easy to see that vector looks outside the ellipse, so it is proportional to TO with positive coefficient. The same thing can be said about any parallel vector, for instance $\left(b\cdot\cos(t),a\cdot\sin(t)\right)$. Therefore,

$$O = T + TO = \left(a\cdot\cos(t),b\cdot\sin(t)\right)+k\left(b\cdot\cos(t),a\cdot\sin(t)\right)$$

Notice that if $k = 1$, we shall get $\left((a+b)\cdot\cos(t),(a+b)\cdot\sin(t)\right)$, and that is the circle of radius $a + b$, which we have guessed already.
So, we should just check that there is a pair of common parallel tangents to the circle with center at that point and the ellipse.
Theoretically, on one ray TO orthogonal to the ellipse and directed outside, we could get more than one location of O. However, for each pair of parallel tangents at both directions there is only one circle. So, if

$\left((a+b)\cdot\cos(t),(a+b)\cdot\sin(t)\right)$ are solutions, then the pair of tangent lines rotate continuously all the way around the ellipse, and we cover all the possibilities. Therefore it is enough to check that those points satisfy the condition.

The pair of lines, symmetric with respect to 0 and tangent to the circle with center at O=$\left((a+b)\cdot\cos(t),(a+b)\cdot\sin(t)\right)$ is

$$x\cdot\sin(t)-y\cdot\cos(t)=\pm c,$$

since these are lines parallel to the line that goes via 0 and O.

Here c is the distance between 0 and those lines, because sum of the squares of coefficients of x and y is 1.

TO is also a radius of the circle, and it is equal $\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}$.

So, it remains to verify that the lines

$$x\cdot\sin(t)-y\cdot\cos(t)=\pm\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}$$

are tangent to the ellipse, or at least one of them (the other will follow from the symmetry).

Substitute $\left(a\cdot\cos(s),b\cdot\sin(s)\right)$ as (x,y).

$$a\cos(s)\cdot\sin(t)-b\sin(s)\cdot\cos(t)=\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}$$

But by Cauchy-Schwartz inequality,

$a\cos(s)\cdot\sin(t)-b\sin(s)\cdot\cos(t)\le$

$\le\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}\cdot\sqrt{\sin^2(s)+\cos^2(s)}=\sqrt{b^2\cos^2(t)+a^2\sin^2(t)}$

So, the ellipse is one side of that line, and it touches it precisely once, when directions of vectors $\left(b\cos(t),a\sin(t)\right)$ and $\left(-\sin(s),\cos(s)\right)$ coincide. QED.

4.* Let P be a point upon the rectangular hyperbola $\{xy = 1\}$.
Let D be a symmetric point to P with respect to 0. Suppose a circle with center at P intersects the hyperbola $\{xy = 1\}$ at 4 points: A, B, C, D. Prove that ABC is equilateral (regular) triangle.

Reminder. Each hyperbola has two asymptotes – straight lines that approximate it very well at all distant points. Hyperbola is called rectangular, if the asymptotes are orthogonal.

Solution. The following solution belongs to my high-school teacher, Dr. Anatoly Schulman.

Assume P = (u, v). The circle with center P is $(x-u)^2 + (y-v)^2 = R^2$.

$A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ and $D(-u, -v)$ belong to the circle and the hyperbola $y = 1/x$, so their x coordinate satisfies $(x - u)^2 + (1/x - v)^2 = R^2$. If we multiply by x^2 and expand it we shall get an equation of degree 4:

$$x^4 - 2u\,x^3 + k\,x^2 + m\,x + n = 0$$

Notice, that to each value of x only one point on hyperbola may correspond. So the four roots of this equation are precisely x_A, x_B, x_C, $-u$. Then by Vieta theorem, $x_A + x_B + x_C - u = 2u$.

$$x_A + x_B + x_C = 3u$$

Symmetric argument proves $y_A + y_B + y_C = 3v$.
So the mass center of triangle ABC is P, which is also its circumcenter. In other words, the meeting point of medians coincides with the meeting point of perpendicular bisectors of the sides. Thus the medians are the perpendicular bisectors, and hence the triangle ABC is equilateral.

5.** For a triangle ABC in plane, consider rectangular hyperbolas, going through A, B and C simultaneously. Each of those hyperbolas has a center of symmetry. Prove that all these centers lie on one circle.

Solution. First of all, let us understand how does an equation of a rectangular hyperbola look like. Equation of a conic is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Asymptotes are defined by intersection points with the infinite line. So, only the quadratic part, $ax^2 + bxy + cy^2$, influence the asymptotes. It can be decomposed as a product of linear equations, and those lines will be parallel to the asymptotes. If they are orthogonal then

$$ax^2 + bxy + cy^2 = k(mx + ny)(nx - my)$$

$$ax^2 + bxy + cy^2 = k(mnx^2 + (n^2 - m^2)y - mny^2)$$

It is easy to see that for given mn, the expression $(n^2 - m^2)$ may accept all values. So rectangular hyperbolas and couples of orthogonal lines (which are degenerate case of rectangular hyperbolas) are all the quadrics satisfying $a = -c$ and only them.
Consider now rectangular hyperbolas quadrics passing through $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. They satisfy 4 linear equation. First 3 are

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

where i = 1, 2, 3. The last one is

$$a + c = 0$$

All those are linear equations in a, b, c, d, e, f. In 6-dimensional space 4 equation probably define 2-dimensional space, unless one of the equations is linear combination of the previous.
The second is not a multiple of the first, since it is easy to build a conic which contains A and doesn't contain B. It is also easy to find a conic containing A and B but not C. It is also easy to find a conic that contains A, B, C but isn't a rectangular hyperbola (for instance, circumcircle). So,

neither equation is a linear combination of the previous, and indeed we get a 2-dimensional linear space.
That space is spanned by each two non-proportional elements.
Actually, the space of our conics is better described a projective line, since multiplication of an equation by a constant doesn't alter the locus, described by the equation.

From here we can deduce a few conclusions, which are so nice that I cannot pass them by, even though they are not needed for the solutions.

(1) Three altitudes of triangle ABC have a common point. Indeed, consider an equation of quadric q_A which is a product of equations of two lines: BC and the altitude from A. Consider an equation of quadric q_B which is a product of equations of two lines: AB and the altitude from B. Define q_c in the similar way.
Let H be intersection point of altitudes from A and B. Then q_A and q_B have 4 common points: A, B, C, and H. But all rectangular hyperbolas in our family, and q_C among them, can be represented as $\lambda q_A + \mu q_B$, so they pass via A.
(2) Actually, we have generalized that elementary theorem about altitudes: all rectangular hyperbolas containing A, B, C also contain H, which is the orthocenter of triangle ABC.

Anyway, the equations of our rectangular hyperbolas are:

$$0 = ax^2 + bxy + cy^2 + dx + ey + f =$$
$$(a_0+\lambda a_1)x^2 + (b_0+\lambda b_1)xy + (c_0+\lambda c_1)y^2 + (d_0+\lambda d_1)x + (e_0+\lambda e_1)y + (f_0+\lambda f_1)$$

Now we need a way to compute the center of a hyperbola.
Apply parallel shift by (s, t) to the hyperbola. We get the equation

$$0 = a(x-t)^2 + b(x-t)(y-s) + c(y-s)^2 + d(x-t) + e(y-s) + f =$$
$$= ax^2 + bxy + cy^2 + (d - 2at - bs)x + (e - 2cs - bt)y + \mathrm{F}$$

(s, t) is the center of symmetry iff the equation became even. A condition for that is a pair of linear equation: linear coefficients are 0.

$$d - 2at - bs = 0$$
$$e - 2cs - bt = 0$$

In other words

$$2at + bs = d$$
$$bt + 2cs = e$$

The solution is

$$(4ac - b^2)t = 2cd - be$$
$$(4ac - b^2)s = 2ae - bd$$

So the center is

$$\left(\frac{2cd-be}{4ac-b^2},\frac{2ae-bd}{4ac-b^2}\right)=\left(\frac{-2ad-be}{-4a^2-b^2},\frac{2ae-bd}{-4a^2-b^2}\right)=\left(\frac{2ad+be}{4a^2+b^2},\frac{bd-2ae}{4a^2+b^2}\right)$$

The question is whether all those centers belong to one circle, i. e. whether they are described by one equation of the type

$$k(x^2+y^2) + lx + my + n = 0$$

$$\left(\frac{2ad+be}{4a^2+b^2}\right)^2+\left(\frac{bd-2ae}{4a^2+b^2}\right)^2=\frac{(2ad+be)^2+(bd-2ae)^2}{(4a^2+b^2)^2}=$$

$$=\frac{4a^2d^2+b^2e^2+4abde+b^2d^2+4a^2e^2-4abde}{(4a^2+b^2)^2}=$$

$$=\frac{4a^2d^2+b^2e^2+b^2d^2+4a^2e^2}{(4a^2+b^2)^2}=\frac{(4a^2+b^2)(d^2+e^2)}{(4a^2+b^2)^2}=\frac{d^2+e^2}{4a^2+b^2}$$

So, we need to find k, l, m, n such that:

$$k\frac{d^2+e^2}{4a^2+b^2}+l\frac{2ad+be}{4a^2+b^2}+m\frac{bd-2ae}{4a^2+b^2}+n=0$$

Which is the same as:

$$k\cdot(d^2+e^2)+l\cdot(2ad+be)+m\cdot(bd-2ae)+n\cdot(4a^2+b^2)=0$$

Since the possible values of a, b, d, e, f are linear expressions in λ, hence those brackets are quadratic expressions in λ:

$$k\cdot(p_0+p_1\lambda+p_2\lambda^2)+l\cdot(q_0+q_1\lambda+q_2\lambda^2)+$$
$$+m\cdot(r_0+r_1\lambda+r_2\lambda^2)+n\cdot(s_0+s_1\lambda+s_2\lambda^2)=0$$

So, it is enough to find non-zero solution to the system of 3 homogenous equations: $p_ik+q_il+r_im+s_in=0$, for i = 0, 1, 2. These equations have nontrivial solution, hence the centers are on one line or circle.
But the foots of altitudes are not one line, so they are on one circle.

Remark. This circle is famous: it is called Euler's circle, Feuerbach's circle, and nine-point circle. The nine point are: midpoints of the 3 sides, midpoints of the intervals AH, BH, CH where H is the orthocenter, and the foots of the three altitudes. The fact that those nine points are on one circle is considered one of the gems of the elementary geometry.
Problem 5 gives a generic description for all points of the nine-point circle, and not just for 9 of them.

6. ABCD is a tetrahedron in the space. For each edge, consider plane passing via its midpoint and orthogonal to the opposite edge (for instance, a plane via the middle of AB orthogonal to CD). Prove that these 6 planes intersect in one point.

Remark. This point is called Monge point.

Solution. The uniqueness of that point is trivial (otherwise all planes would be parallel to one line, but then all edges of ABCD would be parallel to one plane, then ABCD would be a planar shape and not tetrahedron). We are looking for the point M, such that $M - (A+B)/2$ is orthogonal to $C - D$, along with all the symmetric conditions.
Try $M = (A + B + C + D) / 2$, then $M - (A+B)/2 = (C + D)/2$.
Then we want to require $0 = (C - D , (C + D)/2)$. That is the same thing as $0 = (C - D , C + D) = (C , C) - (D , D)$.
So, if we have chosen the origin to be the center of circumsphere of ABCD, then $|A| = |B| = |C| = |D|$ and it will just work.

First stage of Israeli students competition, 2009-2010.

Duration: 4 hours

1. Compute: a. $\det\begin{pmatrix}1 & 4 & 9\\ 4 & 9 & 16\\ 9 & 16 & 25\end{pmatrix}$ b. $\det\begin{pmatrix}1 & 4 & 9 & 16\\ 4 & 9 & 16 & 25\\ 9 & 16 & 25 & 36\\ 16 & 25 & 36 & 49\end{pmatrix}$

2. a. How many planes are required to cut all the edges of a cube?
b. How many planes are required to cut each edge of a cube twice?
Remark. Edges of a polytope (מקצועות של פאון) are the intervals which are the sides of its faces. We say that a plane cuts an interval if the plane contains precisely one internal point of that interval.

3. Find all continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ satisfying:

$$f(x+y)-f(x-y)=2y\cdot f'(x) \qquad \forall x,y\in\mathbb{R}.$$

4. The Department of Social Equality has 15 workers. In the beginning, each has a salary which is a positive integer number of NIS no greater than 10. Each year, the boss can raise the salaries of exactly 13 workers by 1 NIS simultaneously.
Workers are immortal, they never quit or retire; new workers are never accepted.
The boss wants to make the salaries of all workers in the Department equal.
a. Prove that it is possible.
b. In the worst case, how many years will it take?

5. A function c over the set of natural numbers is defined as follows:
$c(n)=$ 0 if there are even number of ones in binary representation of n,
1 otherwise.
A positive integer number k is given.
Let $l(N)$ be the number of integers n from 0 to N, such that $c(k+n)\neq c(n)$.
Prove that $\lim\limits_{N\to\infty}\frac{l(N)}{N}$ exists and belongs to $\left[\frac{1}{3},\frac{2}{3}\right]$.

Good luck!

First stage of Israeli students' competition, solutions.

1. Compute: a. $\det\begin{pmatrix}1 & 4 & 9\\ 4 & 9 & 16\\ 9 & 16 & 25\end{pmatrix}$ b. $\det\begin{pmatrix}1 & 4 & 9 & 16\\ 4 & 9 & 16 & 25\\ 9 & 16 & 25 & 36\\ 16 & 25 & 36 & 49\end{pmatrix}$

Answers. a. -8 **b.** 0

Solution. a. Subtract second line from the third. After that, subtract first line from the second. We get:

$$\det\begin{pmatrix}1 & 4 & 9\\ 4 & 9 & 16\\ 9 & 16 & 25\end{pmatrix}=\det\begin{pmatrix}1 & 4 & 9\\ 4 & 9 & 16\\ 5 & 7 & 9\end{pmatrix}=\det\begin{pmatrix}1 & 4 & 9\\ 3 & 5 & 7\\ 5 & 7 & 9\end{pmatrix}$$

Now do the same with columns (subtract second from third, first from second):

$$\det\begin{pmatrix}1 & 4 & 9\\ 3 & 5 & 7\\ 5 & 7 & 9\end{pmatrix}=\det\begin{pmatrix}1 & 4 & 5\\ 3 & 5 & 2\\ 5 & 7 & 2\end{pmatrix}=\det\begin{pmatrix}1 & 3 & 5\\ 3 & 2 & 2\\ 5 & 2 & 2\end{pmatrix}$$

Now subtract second row from the third, and then second column from the third:

$$\det\begin{pmatrix}1 & 3 & 5\\ 3 & 2 & 2\\ 5 & 2 & 2\end{pmatrix}=\det\begin{pmatrix}1 & 3 & 5\\ 3 & 2 & 2\\ 2 & 0 & 0\end{pmatrix}=\det\begin{pmatrix}1 & 3 & 2\\ 3 & 2 & 0\\ 2 & 0 & 0\end{pmatrix}$$

Here we have only one permutation with nonzero product, which is the secondary diagonal. The product is 8, but the permutation is negative. So it is -8.

b. The determinant can be written as follows

$$\det\begin{pmatrix}f(1) & f(2) & f(3) & f(4)\\ g(1) & g(2) & g(3) & g(4)\\ h(1) & h(2) & h(3) & h(4)\\ k(1) & k(2) & k(3) & k(4)\end{pmatrix}$$

where $f(x)=x^2, g(x)=(x+1)^2, h(x)=(x+2)^2, k(x)=(x+3)^2$.

Polynomials of degree 2 form 3-dimensional linear space, hence f, g, h, k are linearly dependent. Therefore the rows of matrix are linearly dependent. So the determinant is 0 (of course, we could compute it by subtracting the rows/columns, but it is always better to get the result without computation).

2. a. How many planes are required to cut all the edges of a cube?
b. How many planes are required to cut each edge of a cube twice?
Remark. Edges of a polytope (מקצועות של פאון) are the intervals which are the sides of its faces. We say that a plane cuts an interval if the plane contains precisely one internal point of that interval.

Answer. a. 3. b. 4
Solution. a. It is easy to build an example of 3 planes cutting all the edges of a cube: for instance, for each pair of parallel faces take a plane which is parallel to both and is between them. The tricky part is to show why 2 planes are not enough.

The nicest explanation I saw belongs to Dan Carmon and was invented during the competition. Suppose two planes A, B cut all edges. Consider the third plane C which is orthogonal to both planes. We can rotate the cube slightly, so that planes A and B will still cut the same edges but C won't be orthogonal to any faces of the cube. Project the picture orthogonally to the plane C. In the projection, the cube becomes a convex hexagon, and planes A, B become straight lines. These two straight lines should cut all 6 of the convex hexagon. But each line can cut only two of them. This yields a contradiction.

b. An intersection of a plane and a cube is a polygon with at most 6 sides. The reason is the following: each side is intersection of a plane and a face, and the cube has only 6 faces (maybe less, since the plane doesn't have to intersect all faces).
To cut each of 12 edges 6 times we need at least 24 intersection points, so at least 24 angles in our intersection polygons; each plane contributes at most 6, so at least 4 planes are needed.

A perpendicular bisector plane to any diagonal of the cube (by diagonal of a cube we mean an interval connecting two opposite vertices) cuts precisely 6 edges at their midpoints. The cube has 4 diagonals; so we can have 4 such planes, the picture is symmetric, so each edge is cut by the same number of planes, so it is 2 (since the total number of intersection is $4 \times 6 = 24$). One could complain that the different planes intersect any edge in the same point, precisely in the middle.
To fix this issue, it is enough to shift all 4 planes by very small, but different distances.

3. Find all continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ satisfying:

$$f(x+y)-f(x-y)=2y\cdot f'(x) \qquad \forall x,y\in\mathbb{R}.$$

Solution. Derive by y:

$$f'(x+y)+f'(x-y)=2f'(x)$$

$$\frac{f'(x+y)+f'(x-y)}{2}=f'(x)$$

Therefore, for each two points on the graph, the middle point is also on the graph. Therefore, the interval connecting this two points and the graph of f' coincide on a dense set of points (the middle of the interval, the middle of subintervals formed by the midpoints, the middles of smaller subintervals formed by those points etc.). That fact, along with continuity of the function, implies that f' is linear on any interval, therefore it is linear. That means that f is quadratic function.
It is easy to see that any quadratic function satisfies the original equation (exercise to the reader).

4. The Department of Social Equality has 15 workers. In the beginning, each has a salary which is a positive integer number of NIS no greater than 10. Each year, the boss can raise the salaries of exactly 13 workers by 1 NIS simultaneously. Workers are immortal, they never quit or retire; new workers are never accepted. The boss wants to make the salaries of all workers in the Department equal.
a. Prove that it is possible.
b. In the worst case, how many years will it take?

Answer. 70.
Solution. First of all, raising the salary of 13 by 1 is the same as decreasing salaries of two by 1, at least to the people who think in abstract mathematical terms and are interested only in social equality and not in actual money. That way, we might eventually arrive to negative salaries, but so be it.
a. We can split all people except A into pairs and reduce salary of each pair. This is the same as raising salary to A by 1. Using operations of that kind, we can clearly arrive to the equality.
b. Assume that 13 workers have 10 NIS salary, one worker has 9 NIS salary, and the last one has 1 NIS salary. We reduce salaries of two workers a year. To have an equality, we have to come to a situation, in which all get no more than 1 NIS a year. However, it cannot be that all would get 1 NIS a year, even after a long time,

since the total of all salaries is even and it shall never be odd. So, to achieve social equality, we have to make all their salaries at most 0. Their total salay at the beginning is 140, in the end 0 at most, so at least 70 years are required in this case.

It remains to prove that it cannot be more than 70. Without loss of generality, we may assume that the last worker has salary 1 NIS in the beginning. If the total salary is odd, we might try to arrive to a situation in which the salary of all workers is 1; if the salary is even, we might try to arrive to a situation where the salary of all workers is 0. In the beginning, the largest possible total salary is 141 (all except the last worker get 10), the largest possible even total salary is 141, and the largest possible odd total salary is 140. If it is possible, it would take no more than 70 years to make all salaries to be the same number, which is 1 or 0.

If we would also be allowed to reduce salary to the same person twice during the same year (instead of reducing each time the salaries of two different workers) then it would obviously be possible. We can write a plan, how to arrive to social equality in at most 70 years, with two people in each year, but in some years the same person can be mentioned twice. We can assume this plan is at least for 50 years: if not, we can add 15 more years, while in those additional years the salary is reduced to each worker. Now we shall reorganize this plan so that nobody will be reduced twice in the same year.

Notice, that each worker starts with a salary at most 10, and arrives to the salary at least -9 (because bringing all people to salary -10 means total salary would be -150 and for that 75 years at least would be required). So salary of each worker will be reduced less than 20 times. Therefore, if a salary of some worker according to the plan is reduced twice in one year, it is possible to find a year in which his salary is not reduced (since that plan has 50 years at least). We shall swap one name between these two years, and the number of such bad years will be reduced. We can do it as long as bad years exist, so after a finite number of operations we shall have a plan with the same number of years in which bad years don't exist.

5. A function c over the set of natural numbers is defined as follows:

$c(n) =$ 0 if there are even number of ones in binary representation of n,

1 otherwise.

A positive integer number k is given.
Let $l(N)$ be the number of integers n from 0 to N, such that $c(k+n) \neq c(n)$.
Prove that $\lim\limits_{N\to\infty} \frac{l(N)}{N}$ exists and belongs to $\left[\frac{1}{3}, \frac{2}{3}\right]$.

Solution. Denote $p(k) = \lim\limits_{N\to\infty} \frac{l(N)}{N}$, where k is the number that was used to define function l.
First compute the limit for $k = 1$. The parity is switched if the binary number has in its end 0, or 011, or 01111, or 0111111, … and limit of density of these numbers exists and equals $p(1) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots = \frac{1}{2}\Big/\left(1 - \frac{1}{4}\right) = \frac{1}{2}\Big/\frac{3}{4} = \frac{2}{3}$.
We shall prove the claim by induction. Suppose we have proved the claim for all numbers smaller then k, and now we prove it for given k.
If $k = 2m$ then adding k to the number is the same as adding m the number for which the last binary digit is erased. Since $p(m)$ exists and belongs to $\left[\frac{1}{3}, \frac{2}{3}\right]$, so does $p(k)$.
Now suppose $k = 2m + 1$. Comparing $c(k+n)$ to $c(n)$ splits into two cases: when $n = 2s$, then $c(k+n)$ differs from $c(n)$ if and only if $c(m+s)$ equals $c(s)$, because last binary digit is changed, and to the rest of the number m is added. Therefore the limit of probability that $c(k+n)$ differs from $c(n)$ exists and equals to $1 - p(m)$.
If $n = 2s + 1$, then the last digit is changed anyway, from 1 to 0, and we have a carry, and then we should actually add $m + 1$ to the number which is n after erasing the last digit. So for odd numbers limit density also exists and equals $1 - p(m + 1)$.
If we take both even and odd numbers, the limit also exists and equals the average (since from 1 to N there's almost equal number of evens and odds) which is $p(k) = \frac{(1 - p(m)) + (1 - p(m+1))}{2}$. Since both $1 - p(m)$ and $1 - p(m+1)$ belong to $\left[\frac{1}{3}, \frac{2}{3}\right]$ by induction, their average also belongs to the same interval.

Second stage of Israeli students competition, 2010.

Duration: 4 hours

1. Let $K = \sum_{n \in \mathbb{N}} 2^{-n^2}$ (where $\mathbb{N}$ is a set of positive integers).
Is K rational or irrational?

2. ABCD is a tetrahedron (not necessarily regular). Denote a = distance between the lines AB and CD, b = distance between the lines AC and BD, c = distance between the lines AD and BC. Prove that the volume can't be less than $\frac{abc}{3}$.

3. We have a system of L lamps and B buttons. Each button has one of two states: "on" or "off". Each button is connected to several lamps. A lamp may be connected to more than one button. Pressing a button toggles all connected lamps to the opposite state.
(a) Prove that number of potential states of lamps is a power of 2.
(b) Suppose that for every subset S of lamps there is a button switching state of odd subset of S. Prove that all lamps can be switched off.

4. Compute $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

5. A is a 2×2 matrix with integer coefficients. Absolute values of all entries of A are less than 10. Absolute values of all entries of A^{1000} are less than 10^9. Prove that they are actually less than 10^6.

Good luck!

Second stage of Israeli students competition, 2010.

1. Let $K=\sum_{n\in\mathbb{N}} 2^{-n^2}$ (where $\mathbb{N}$ is a set of positive integers).

Is K rational or irrational?

Answer. K is irrational.
Solution. Write a binary fraction of K. It is not periodic, since it has longer and longer sequences of zeroes. Hence the number is irrational.

2. ABCD is a tetrahedron (not necessarily regular). Denote a = distance between the lines AB and CD, b = distance between the lines AC and BD, c = distance between the lines AD and BC. Prove that the volume can't be less than $\frac{abc}{3}$.

Solution. Move the tetrahedron so that the center of mass of A, B, C, D will be 0. In other words, A + B + C + D = 0.
Construct 4 points K = – A , L = – B , M = – C , N = – D.

$$AL + CN = L - A + N - C = -B - A - C - D = 0$$

So vectors AL and NC are equal, and ALCN is a parallelogram.
Similarly, vector AL coincides with vector MD and BK.
Therefore ALDM and MDKB and BKCN are parallelograms. Hence points A, B, C, D, K, L, M, N are the vertexes of parallelepiped.

The common perpendicular to AB and CD is orthogonal to two faces of the parallelepiped. Therefore a is the distance between faces AMBN and CKDL. Similarly, b and c are distances between other pairs of parallel faces.

Lemma. If a, b, c are distances between parallel faces of parallelepiped then the volume $V \geq abc$.

The parallelepiped consists of tetrahedrons ABCD, ABCN, BCDK, ACDL, ABDM. It is easy to see that the last 4 have volume $V/6$ each, so volume of ABCD is $V/3$. It remains to prove the lemma.
Proof of lemma. Volume = altitude of parallelepiped on AMBN times altitude of face AMBN on AM times AM. First factor is a, second is greater than b, third is greater than c.

3. We have a system of L lamps and B buttons. Each button has one of two states: "on" or "off". Each button is connected to several lamps. A lamp may be connected to more than one button. Pressing a button toggles all connected lamps to the opposite state.
(a) Prove that number of potential states of lamps is a power of 2.
(b) Suppose that for every subset S of lamps there is a button switching state of odd subset of S. Prove that all lamps can be switched off.

Solution. (a) Consider linear space over a field of 2 elements of dimension L. Each button specifies a vector in that space (coordinated 1 if the lamp is connected, coordinate 0 otherwise). Combination of several buttons corresponds to the sum of their vectors. Sums of given vectors form a linear subspace. Since it is still linear space over the field of two elements, number of possible changes that can come from combinations of buttons is a power of two.
(b) All possible changes that might be applied to the system form a linear subspace in the space of all conceivable changes. If this subspace would not be everything, it would lie in a hyperplane. That means all possible changes would have zero scalar product with a given nonzero vector. That means there's a subset of lamps, such that every possible change flips even number of lamps in that subset. This is specifically forbidden by the condition.

4. Compute $\int\limits_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$.

Solution. Denote $f(x)=\frac{x\sin x}{1+\cos^2 x}$.

$$\int\limits_0^{\pi} f(x)dx=\frac{1}{2}\int\limits_0^{\pi} f(x)+f(\pi-x)dx=\frac{1}{2}\int\limits_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx=\frac{\pi}{2}\int\limits_0^{\pi}\frac{-d(\cos x)}{1+\cos^2 x}=$$

$$=\frac{\pi}{2}\int\limits_1^{-1}\frac{-dy}{1+y^2}=\frac{\pi}{2}\int\limits_{-1}^{1}\frac{dy}{1+y^2}=\frac{\pi}{2}\arctan\Big|_{-1}^{1}=\frac{\pi}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right)=\frac{\pi}{2}\cdot\frac{\pi}{2}=\frac{\pi^2}{4}$$

5. A is a 2×2 matrix with integer coefficients. Absolute values of all entries of A are less than 10. Absolute values of all entries of A^{1000} are less than 10^9. Prove that they are actually less than 10^6.

Solution. First consider determinant. det $A = d$ which is an integer number.
det $A^{1000} = d^{\,1000}$
If $|d| \geq 2$, then det $A^{1000} \geq 2^{1000} \geq 1000^{100} = 10^{300}$ and that is definitely not a determinant of a matrix having absolute values of entries less than 10^9.
Therefore, the d is 1, 0, or -1.
We shall use the following lemma.

Lemma. Absolute values of eigenvalues of A are less than 1.2.

Proof of lemma. Assume A has an eigenvalue k such that $|k| \geq 1.2$ and it corresponds to an eigenvector v. Then $k^2 = 1.44 > \sqrt{2}$, hence $k^4 > 2$, and $k^{40} > 2^{10} > 1000$.
Therefore $k^{1000} > 1000^{25} = 10^{75}$. But $A^{1000}v = k^{1000}v$.
The vector is multiplied by a number, of absolute value greater than 10^{75}; it wouldn't happen if all entries of A^{1000} are less than 10^9. QED of lemma.

Therefore, absolute value of trace(A) can't be 3 or greater: then absolute value of one eigenvalue would be above 1.5 (since trace is the sum of eigenvalues) and that can't happen. Hence trace(A) can be 0, ±1, or ±2. Multiplying the whole matrix A by -1 doesn't change A^{1000}, so we may assume that trace(A) is nonnegative.

To summarize: WLOG det(A) is -1, 0 or 1 and trace(A) is 0, 1 or 2.
This gives us 9 possibilities. Let us check all of them:

a) det(A) = -1 and trace(A) = 2.

Eigenvalues satisfy $x^2 - 2x - 1 = 0$ which is $x^2 - 2x + 1 = 2$ so they are $\frac{1 \pm \sqrt{2}}{2}$. The greater eigenvalue is $\frac{1+\sqrt{2}}{2} > \frac{1+1.4}{2} = 1.2$, and this can't happen because of the lemma.

b) $\det(A) = 0$ and $\operatorname{trace}(A) = 2$.

One eigenvalue is 0 (since the product is 0), another is 2 (since the sum is 2) and it can't happen because of the lemma.

c) $\det(A) = 1$ and $\operatorname{trace}(A) = 2$.

This is a hard case. Characteristic polynomial is $x^2 - 2x + 1 = 0$ so the eigenvalues are equal 1. Let $\mathbf{E}$ be the unit matrix. Then $A - \mathbf{E} = \mathrm{N}$ is a nilpotent matrix. Since the dimension is 2, $\mathrm{N}^2 = 0$. Therefore $A = \mathbf{E} + \mathrm{N}$, and

$$A^{1000} = (\mathbf{E} + \mathrm{N})^{1000} = \mathbf{E} + 1000\mathrm{N}$$

Since all entries of N are less than 11 by absolute values, all entries of 1000N are less than 11000 and all entries of A^{1000} are less than 11001.

d) $\det(A) = -1$ and $\operatorname{trace}(A) = 1$.

Eigenvalues satisfy $x^2 - x - 1 = 0$ so they are $\frac{1 \pm \sqrt{5}}{2}$. The greater eigenvalue is bigger then 1.5, so this case is ruled out by the lemma.

e) $\det(A) = 0$ and $\operatorname{trace}(A) = 1$.

By Cayley-Hamilton, $A^2 - A = 0$, hence $A^2 = A$. By induction. $A^{K+1} = A^K$ and therefore $A^{1000} = A$ and its entries have absolute values no greater than 10.

f) $\det(A) = 1$ and $\operatorname{trace}(A) = 1$.

$A^2 - A + \mathbf{E} = 0$. We multiply that by $A + \mathbf{E}$, and we get $A^3 + \mathbf{E} = 0$.

Therefore $A^3 = -\mathbf{E}$ and $A^{999} = -\mathbf{E}$ and $A^{1000} = -A$. It has the same entries as A, maybe with different signs, but absolute values are still < 10.

g) $\det(A) = -1$ and $\operatorname{trace}(A) = 0$.

$A^2 - \mathbf{E} = 0$. So $A^2 = \mathbf{E}$. Therefore $A^{1000} = \mathbf{E}$.

h) $\det(A) = 0$ and $\operatorname{trace}(A) = 0$.

The matrix is nilpotent. $A^2 = 0 = A^{1000}$.

i) $\det(A) = 1$ and $\operatorname{trace}(A) = 0$.

$A^2 + \mathbf{E} = 0$. So $A^2 = -\mathbf{E}$. Therefore $A^4 = \mathbf{E}$. And $A^{1000} = \mathbf{E}$.

Targil 1 - determinants.

1. All entries of a 10×10 matrix A belong to the interval [-1,1].
Can it happen that det $A > 5770$?

2. Matrix B has zeroes on the main diagonal and ones at all other places.
a. Prove that B is non-degenerate.
b. Compute det B.

3. a. Compute $u = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 2^3 & 3^3 & 4^3 & 5^3 \end{pmatrix}$

b. Prove that $v = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2^4 & 3^4 & 4^4 & 5^4 \\ 2^7 & 3^7 & 4^7 & 5^7 \end{pmatrix}$ is divisible by u.

4*. The inverse and the determinant of the following matrix: $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$.

5.** Consider an anti-symmetric ($A = -A^T$) matrix with integer coefficients.

Show that the determinant is a perfect square.

Targil 1 - determinants.

1. All entries of a 10×10 matrix A belong to the interval [-1,1].
Can it happen that det $A > 5770$?

Solution. We shall use a following lemma:

Lemma. If $N = 2^k$, we can construct a square $N\times N$ matrix which consists of numbers 1 and -1 and its columns are mutually orthogonal.

Historical remark. A matrix of that kind, with mutually orthogonal columns and made of ones and minus ones, is called Hadamard matrix after a famous French (Jewish) mathematician Hadamard (pronounced [adamAr]).
It is easy that the size of Hadamard matrix, if it is bigger than 2, should be divisible by 4; Hadamard conjecture is that for any N divisible by 4 there exists an Hadamard matrix of that size. The first nontrivial examples, of size 12 and 20 were constructed by Hadamard in 1893; Wikipedia lists several other constructions for different sizes, the last of those is the construction in 2004 by Hadi Kharaghani and Behruz Tayfeh-Rezaie of size 428; existence for size 668 is still an open.
http://en.wikipedia.org/wiki/Hadamard_matrix

Suppose we have proved the lemma; let's solve our problem in this case.
Determinant of Hadamard matrix is $\pm N^{N/2}$, and we may choose the sign: if we want to change it, we multiply one of the rows by -1. Indeed, determinant is an oriented volume of a parallelepiped (מקבילון) spanned by the vectors of the columns; our parallelepiped is a cube, sides are of length $\sqrt{N}$. So the volume, up two a sign is a product of sides.

So the determinant of Hadamard 2×2 matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is 2, and the determinant of Hadamard 2×2 matrix is $8^4 = 2^{12} > 4000$ and if we put those two matrixes as blocks along the diagonal we get a 10×10 matrix with det $> 8000 > 5770$.
Now it remains to prove the lemma.

Proof of lemma. In our case, orthogonality means that each two columns match in precisely half of the places (and mismatch in the other half). We shall construct a matrix of zeroes and ones with the same combinatorial properties; to turn it into Hadamard matrix, it will be enough to replace zeroes by -1.
Since $N = 2^k$, the rows, as well as the columns, will be numbered by a different vectors of k-dimensional space over the field of two elements $\{0, 1\}$.
The number in the column that corresponds to vector v and row that corresponds to vector u will be the scalar product of v and u.
Consider two different columns corresponding to binary vectors v and w. The numbers in these columns in row corresponding to vector u match, iff (if and only if) u is orthogonal to $v - w$. Orthogonality to $v - w$ is a condition, that specifies a hyperplane in the linear space over the field of two elements. Every hyperplane in that space contains precisely ½ of its vectors; adding any vector outside the hyperplane is a bijection (העתקה חח"ע ועל) between the hyperplane and its complement. QED.

Remark. Another way to do this construction is by induction.
A shorter, but also a less elementary way, for those who know tensor product, is to take n'th tensor power of the 2×2 Hadamard matrix,

2. Matrix B has zeroes on the main diagonal and ones at all other places.
a. Prove that B is non-degenerate.
b. Compute det B.

Solution. a. Sum of all columns divided by something is a vector of ones.
Difference between that and some column is an element of the standard basis.
So, any element of a standard basis can be spanned by the columns of our matrix; hence the whole space is spanned by the columns of our matrix.
b. If I add a unit matrix to B, I get a matrix of rank 1, therefore -1 is an eigenvalue of multiplicity $n - 1$ at least. If I subtract n – 1 times unit matrix, I get the following matrix: 1 – n on the diagonal, and 1 elsewhere. Sum of all columns is a zero vector, hence the matrix is degenerate. Hence $n - 1$ is also an eigenvalue.
We have found n eigenvalues ($n - 1$ times -1, and once $n - 1$); that's the complete list of eigenvalues; the determinant = product of all eigenvalues = $(-1)^{n-1}(n - 1)$.

3. a. Compute $u = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 2^3 & 3^3 & 4^3 & 5^3 \end{pmatrix}$

b. Prove that $v = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2^4 & 3^4 & 4^4 & 5^4 \\ 2^7 & 3^7 & 4^7 & 5^7 \end{pmatrix}$ is divisible by u.

3. a. We shall compute the more general determinant:

$$u = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$$

(sometimes a more general question is easier than a special case).
The determinant is 0 when $b = a$, so it is divisible by $b - a$.
For similar reasons, it is also divisibly by $c - a$, by $d - a$, … .
So, we have a polynomial in a, b, c, d, which is divisible by

$$(b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$$

The last determinant, as well as u, is a polynomial of degree 6 (in each permutation, we have something of degree 1 times something of degree 2 times something of degree 3). Hence

$$u = K(b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$$

where K is a constant. The only thing that we have to do yet is to guess a constant. When we compute u by a sum of permutation, we have only one way to get highest (third) power of d, and second power of c, and first power of b; by the diagonal permutation. When we open brackets in the product, the only way to get this (highest powers of the latest variables) is to take all plusses out of brackets.
In both cases, we get 1 as the coefficient, hence $K = 1$.

$$u = (b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$$

In our case, we get $u = 1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 3 = 12$.

Remark. The same thing that was done for the 4×4 matrix can be done for $N\times N$ matrix in the same way. If the k'th row from matrix k contains powers of x_i, from 0 power = 1 to the $N-1^{\text{st}}$ power, the determinant is $\prod_{i<j}\left(x_j - x_i\right)$.

The proof is precisely the same: notice that it is divisible by $x_j - x_i$, compare degrees (which are equal), and find the coefficients.
This nice determinant is called Vandermonde.

b. By similar reasons, $v = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^4 & b^4 & c^4 & d^4 \\ a^7 & b^7 & c^7 & d^7 \end{pmatrix}$ is divisible by (b – a), (c – a), etc.

hence it is divisible by the product $u = (b - a)(c - a)(d - a)(c - b)(d - b)(d - c)$.
The ratio is a symmetric polynomial in a, b, c, d.

Well, actually in the above proof we have used here some non-obvious facts, which I told this yom revii in the classroom (when we talked of resultants).
I shall not prove them now in a detailed way, but I shall split it into a sequence of very easy (but not completely obvious) lemmas:

Lemma 1. Consider a polynomial of one variable over an infinite field. Suppose its value at every point is zero. Then it's the zero polynomial (all coefficients are 0).
Lemma 2. Consider a polynomial of several variable over an infinite field. Suppose its value at every point is zero. Then it's the zero polynomial.
Lemma 3. Suppose a polynomial of several variables $x_1, x_2, \ldots, x_n$ has zero values at all points of a hyperplane $\{ x_1 = 0 \}$. Then the polynomial is divisible by x_1 (it can be written as x_1 times another polynomial).
Lemma 4. Suppose a polynomial of several variables $x_1, x_2, \ldots, x_n$ has zero values at all points of a hyperplane $\{ l(x_1, x_2, \ldots, x_n) = 0 \}$, where l is a linear function. Then the polynomial is divisible l.
Lemma 5. Suppose a polynomial of several variables $x_1, x_2, \ldots, x_n$ with integer coefficients has zero values at all points of a hyperplane $\{ l(x_1, x_2, \ldots, x_n) = 0 \}$, where l is a linear function with integer coefficients, and coefficient of x_k is 1. Then the polynomial is l as a polynomial with integer coefficients.

4*. The inverse and the determinant of the following matrix: $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$.

Solution. Once again, consider a more general problem N×N matrix, in column i row j we have the number $\frac{1}{a_i + b_j}$. Our case is $N = 4$, $a_i = i$, $b_j = j - 1$.

Unfortunately, the entries of the matrix and its determinant art not polynomials. To make everything polynomial, we should multiply each column by all denominators there, thus making a matrix terrifying, but polynomial. The determinant will be multiplied by the common denominator $\prod_{i=1}^{N}\prod_{j=1}^{N}\left(a_i + b_j\right)$ which is a polynomial of degree N^2. The determinant originally was a rational function of degree $-N$, so now it is a polynomial of degree $N\cdot(N-1)$.

The determinant is 0 when $a_i = a_j$, and when $b_i = b_j$, hence it is divisible by $\prod_{i<j}\left(a_j - a_i\right)\left(b_j - b_i\right)$. The degrees coincide, so that is an answer up to a constant factor. So the original determinant is $\frac{\prod_{i<j}\left(a_j - a_i\right)\left(b_j - b_i\right)}{\prod_{i=1}^{N}\prod_{j=1}^{N}\left(a_i + b_j\right)}$ up to coefficient.

To compute the coefficient, let us take a special case: $a_i = i, b_i = \varepsilon - i$, where ε is a very small positive number. All entries of the matrix outside the main diagonal are decent bounded numbers, while on main diagonal we have $\frac{1}{\varepsilon}$. The determinant is a very big number: $\left(\frac{1}{\varepsilon}\right)^N + O\left(\left(\frac{1}{\varepsilon}\right)^{N-2}\right)$. Now estimate the second expression:

$$\frac{\prod_{i<j}\left(a_j - a_i\right)\left(b_j - b_i\right)}{\prod_{i=1}^{N}\prod_{j=1}^{N}\left(a_i + b_j\right)} = \frac{\prod_{i<j}(j-i)(i-j)}{\prod_{i=1}^{N}\prod_{j=1}^{N}(\varepsilon + j - i)} \approx \frac{\prod_{i\neq j}(j-i)}{\varepsilon^N \times \prod_{i\neq j}(j-i)} = \frac{1}{\varepsilon^N}$$

So, both expressions are of asymptotically $\approx \frac{1}{\varepsilon^N}$ therefore the coefficient between them is 1. Hence the determinant is $\dfrac{\prod\limits_{i<j}(a_j - a_i)(b_j - b_i)}{\prod\limits_{i=1}^{N}\prod\limits_{j=1}^{N}(a_i + b_j)}$.

The matrix made from $\dfrac{1}{a_i + b_j}$ is called Cauchy matrix (after a famous French mathematician). The special case of $\dfrac{1}{i+j-1}$ is called Hilbert matrix (after a famous German mathematician). In our special case of 4×4 matrix of size 4 we get:

$$\det = \frac{(1\cdot 1\cdot 1\cdot 2\cdot 2\cdot 3)^2}{1\cdot 2^2\cdot 3^3\cdot 4^4\cdot 5^3\cdot 6^2\cdot 7}$$

Now about inverse matrix. There is a formula for that: the element row *k*, column *l* of inverse matrix is

$$r_{k,l} = (-1)^{k+l}\frac{\det A_{l,k}}{\det A}$$

where $A_{l,k}$ (aka minor) is a matrix without row *l*, and column *k*.

But minors of a Cauchy matrix are again Cauchy matrix:

$$r_{k,l} = (-1)^{k+l}\frac{\det A_{l,k}}{\det A} = (-1)^{k+l}\frac{\prod\limits_{\substack{i<j\\ i,j\neq l}}(a_j - a_i)\prod\limits_{\substack{i<j\\ i,j\neq k}}(b_j - b_i)}{\prod\limits_{\substack{i=1\\ i\neq l}}^{N}\prod\limits_{\substack{j=1\\ j\neq k}}^{N}(a_i + b_j)}\cdot\frac{\prod\limits_{i=1}^{N}\prod\limits_{j=1}^{N}(a_i + b_j)}{\prod\limits_{i<j}(a_j - a_i)(b_j - b_i)} =$$

$$= \frac{\prod\limits_{i\neq l}(a_i + b_k)\prod\limits_{j=1}^{N}(a_l + b_j)}{\prod\limits_{i\neq l}(a_l - a_i)\prod\limits_{j\neq k}(b_k - b_j)}$$

Specifically for Hilbert matrix we get $r_{k,l} = \dfrac{\prod\limits_{i\neq l}(i+k)\prod\limits_{j=1}^{N}(l+j)}{\prod\limits_{i\neq l}(l-i)\prod\limits_{j\neq k}(k-j)}$.

5.** Consider an anti-symmetric ($A = -A^T$) matrix with integer coefficients.

Show that the determinant is a perfect square.

Remark. $\det A = \det A^T = (-1)^n \det A$, so it is nonzero (and non-obvious) only for even dimension.

First solution. Determinant is integer, so it is enough to prove the it is a square of rational number, then we shall know it is a square of integer. If we apply a certain permutation on rows and the same permutation on columns, matrix will remain anti-symmetric and will keep the same determinant.

So we may assume that unless the matrix consists of zeroes only, then cells near the left-top corner (1,2) and (2,1) are non-zero: one is a, another is $-a$. Then by adding linear combinations of first and second rows to all other rows, we can eliminate all numbers in the first and second columns after the second row. These Gauss method operations are equivalent to multiplying the matrix from the left by an invertible matrix.

If A is anti-symmetric, then it is easy to see that BAB^T is also anti-symmetric. Let B be the matrix that is doing Gauss method operation to eliminate the first two columns under the top-left 2×2 block. Then B^T does the same operations on the columns. Obviously, both B and B^T are rational, so determinant is multiplied by a square of rational number. That number is nonzero, since B is invertible.

But now we get a block matrix, that consists of 2 anti-symmetric blocks, so the statement follows by induction over dimensions.

Second proof. It is known, that over anti-symmetric multi-linear forms the wedge product is defined, that makes a k+m-form out of k-form and m-form.

$$(\kappa \wedge \mu)(v_1, v_2, \dots, v_m) = \frac{1}{k!m!} \sum_{\sigma \in \mathbb{S}_{k+m}} \operatorname{sgn}(\sigma) \kappa\left(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}\right) \cdot \mu\left(v_{\sigma(m+1)}, v_{\sigma(m+2)}, \dots, v_{\sigma(m+k)}\right)$$

(here we divide by $k!m!$ to kill ambiguity – no need to sum equivalent summands several time, so this formula is actually integer).

This product is super-commutative and associative.

Any anti-symmetric 2-form can be represented in a general form as $\sum_{i<j} a_{ij} x_i \wedge x_j$,

where x_i are basic linear functionals corresponding to "taking i'th coordinate", or, when suitable basis is chosen, in a canonic form:
$\omega = k_1 x_1 \wedge x_2 + k_2 x_3 \wedge x_4 + k_3 x_5 \wedge x_6 + \ldots + k_n x_{2n-1} \wedge x_{2n}$.

Actually, that was what we have proven in the first solution.

But since the definition of the wedge product doesn't use coordinates, as well as some definitions of determinant, if we prove certain equality between those in the canonical basis, we shall know it for any basis.

Consider the product $\dfrac{\omega \wedge \omega \wedge \ldots \wedge \omega}{n!}$, where ω is multiplied by itself n times.

When we open brackets, all products with similar factors cancel out. So we get $n!$ equivalent products, so after dividing by $n!$ we get an expression which is integer and not fractional in the coefficients, and that is $(k_1 k_2 k_3 \ldots k_n) x_1 \wedge x_2 \wedge x_3 \wedge \ldots \wedge x_{2n}$, product of all coefficients time standard volume form.

The determinant of the anti-symmetric matrix is $k_1^2 k_2^2 k_3^2 \cdot \ldots \cdot k_n^2$. It is the square of the coefficient before the volume form of $\dfrac{\omega \wedge \omega \wedge \ldots \wedge \omega}{n!}$. So it will be not necessarily in the canonical basis.

Example. Consider $n = 4$. Matrix $A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$ is represented by a form $\omega = a_{12} x_1 \wedge x_2 + a_{13} x_1 \wedge x_3 + a_{14} x_1 \wedge x_4 + a_{24} x_2 \wedge x_4 + a_{23} x_2 \wedge x_3 + a_{34} x_3 \wedge x_4$.

Then $\dfrac{\omega \wedge \omega}{2} = (a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23}) x_1 \wedge x_2 \wedge x_3 \wedge x_4$.

(When computing this things, just multiply each couple of terms once and don't divide by 2).

So $\det A = \left(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\right)^2$.

Outline of third solution (Ofir Gorodetzky)

We know (either by guessing or from previous solution) the formula for the expression whose square is the determinant: it is a sum over all ways to decompose the set of all indices into pairs, of product of cells corresponding to that pairs (one index is of row, another of column), signs are chosen by the sign of a permutation which is formed when we write down all those pairs in a row, pair after pair.

So, we can prove combinatorially, that the square of that expression is the determinant. The determinant is a sum of all products over all permutations (or maximal rook arrangements). Some of those permutations contain odd cycles, others only even cycles. We can show that any permutation containing at least one odd cycle will cancel out with another permutation because the matrix is anti-symmetric (by transposing only that specific cycle).

So, we remain with permutations having even cycles only. Sides of even circle might be colored into black and white. That splits the permutation into two perfect matchings. Each of those perfect matchings can be considered as a summand in the polynomial we described, so the determinant is what we get after multiplying that expression by itself (since each time we unite 2 pair decompositions, we get a permutation with even cycles). Working out the signs is left as an exercise ☺.

Targil 2.

(Questions about existence of weird analytical objects)

1. A $\{a_n\}$ is a decreasing sequence of positive numbers, $\sum_{n=1}^{\infty} a_n$ is a divergent series (טור מתבדר). Consider the sequence $\{b_n = \min(a_n, 1/n)\}$.

Can we claim that $\sum_{n=1}^{\infty} b_n$ diverges?

2.* Is it possible to construct $f:[0,1]\to\mathbb{R}$ which is continuous, monotone non-decreasing and satisfies the following two equations for every $x\in[0,1]$:

$$f(x/3)=f(x)/2$$
$$f(x)+f(1-x)=1$$

3.* Derivative of a continuous function $f:\mathbb{R}\to\mathbb{R}$ at all rational points exists and equals zero. Is f necessarily a constant?

4. $\{a_n\}$ is a sequence of positive numbers, $\sum_{n=1}^{\infty} a_n = 1$. Prove that it is possible to insert into the open interval (0,1) a countable set of closed mutually disjoint intervals, indexed by positive integers, so that a closed interval number n is of length a_n.

5. $f:\mathbb{R}\to\mathbb{R}$ such that for any positive real x, y the sequence $f(x+ny)$, for $n\in\mathbb{N}$, tends to infinity.

a. Can we claim that that $f(x)\underset{x\to\infty}{\longrightarrow}\infty$?

b.** Can we claim that $f(x)\underset{x\to\infty}{\longrightarrow}\infty$, if it is given that f is continuous?

Targil 2.

(Questions about existence of weird analytical objects)

1. A $\{a_n\}$ is a decreasing sequence of positive numbers, $\sum_{n=1}^{\infty} a_n$ is a divergent series (טור מתבדר). Consider the sequence $\{b_n = \min(a_n, 1/n)\}$.

Can we claim that $\sum_{n=1}^{\infty} b_n$ diverges?

Answer. Yes.

Solution. If we have finite number of $b_n = 1/n$ then except for the finite number of elements $b_n = a_n$ and $\sum_{n=1}^{\infty} b_n$ behaves the same way as $\sum_{n=1}^{\infty} a_n$.

It remains to consider the case when for infinite number of indexes $b_n = 1/n$.
Take the increasing sequence of all such indexes. Choose an infinite subsequence, $n_1, n_2, n_3, \ldots$ such that $n_{k+1} > 2n_k$.

The series $\sum_{n=1}^{\infty} b_n$ might be divided into sub-segments $\sum_{n=n_{k-1}+1}^{n_k} b_n$. We shall show that the sum in each sub-segment is at least ½, and that will imply $\sum_{n=1}^{\infty} b_n = \infty$.

Indeed, for any n between n_{k-1} and n_k, we have

$$b_n = \min\left(a_n, 1/n\right) \geq \min\left(a_{n_k}, 1/n_k\right) = 1/n_k \ .$$

Hence $\sum_{n=n_{k-1}+1}^{n_k} b_n \geq \left(n_k - n_{k-1}\right) \frac{1}{n_k} > \frac{1}{2}$. QED

2.* Is it possible to construct $f : [0,1] \to \mathbb{R}$ which is continuous, monotone non-decreasing and satisfies the following two equations for every $x \in [0,1]$:

$$f\left(x/3\right) = f\left(x\right)/2$$
$$f\left(x\right) + f\left(1-x\right) = 1$$

Answer: yes.

Solution. The idea comes from Cantor set. Cantor set is made as follows:

Take [0,1] interval. Exclude the open interval (1/3 , 2/3) in the middle, now you get two intervals. Then exclude an open interval of length 1/9 in the middle of each closed interval, you get 4 intervals. Repeat this operation infinite number of times (excluding open interval in which is thrice shorter and located in the middle of each remaining closed interval). What remains after infinite number of steps, is by definition the Cantor set.
Another way to define Cantor set: take all numbers between 0 and 1, that can be written by ternary fraction (base 3, unlike standard decimal or well-known to computer programmers binary system) using only digits 0 and 2 (not using digit 1). For instance 1 = 0.22222… base 3, ¼ = 0.0202020202…. base 3, etc.

Cantor set has many counter-intuitive properties. It is big in the sense of cardinality (as big as the set of all real numbers) but small in the sense of "length" (set of measure 0).

By the same idea we construct Cantor function, aka Cantor's ladder. First we take $f(0) = 0$, $f(1) = 1$. On the interval [1/3, 2/3] we take $f(x) = ½$.
On each step of construction of Cantor set, we exclude an open interval in the middle of a closed interval; by that moment, we have already defined values a and b at the ends of that interval; the values on the closure of the open interval which is being excluded will be $(a + b)/2$. This way we shall define the function at all points of the Cantor set and at some points in it; the values at the rest of the points of the Cantor set, such as ¼ , is defined by monotonicity.

Another way to define this function is as follows: for every x take the ternary fraction representing x, which is $0.x_1x_2x_3\ldots$ (when representing 1, write 0.222… and not 1.000…). Find the first appearance of digit 1 in the fraction (if it exists) and erase both that digit and all digits coming after it. After that replace all the digit 2 by digit 1 at all places; that will be the binary fraction of $f(x)$.

After the example is constructed, it is not hard to verify that it satisfies the condition; we leave it as an exercise to the reader.

3.* Derivative of a continuous function $f : \mathbb{R} \to \mathbb{R}$ at all rational points exists and equals zero. Is f necessarily a constant?

Answer. No.

Solution. First, it is enough to do it for a closed interval. Indeed, we smoothly map a real line into a closed interval, while rational numbers will go to rational numbers, for instance as follows:

$$q:\mathbb{R}\to[-1,1]$$

$$q(x)=\frac{1}{1+x^2}$$

If we build a function g on $[-1,1]$ satisfying all conditions, then function

$$f(x) = g(q(x))$$

will satisfy all our conditions.

The set of rational numbers is countable. Therefore, we can assign a natural index to each rational number.

We shall build some kind of Cantor's ladder, but irregular (unlike the previous Cantor's ladder where we used precisely the middle 1/3 of each interval).

First define $g(-1)=-1, g(1)=1$.

Each time, when we have a closed interval, we shall choose on it a rational number with the smallest index inside it; for that rational number we shall take an open neighborhood inside that interval, with irrational ends. On the closure of this neighborhood, the functions will be given a constant values, equal to the average of the values at the ends of the closed interval. The missing values after infinite number of steps are completed by monotonicity.

The Cantor function constructed in that way, for each rational number it is a constant in a neighborhood, so it has a zero derivative at each rational number. However, it isn't constant.

4. $\{a_n\}$ is a sequence of positive numbers, $\sum_{n=1}^{\infty} a_n = 1$. Prove that it is possible to insert into the open interval (0,1) a countable set of closed mutually disjoint intervals, indexed by positive integers, so that a closed interval number n is of length a_n.

Solution. We shall insert the intervals iteratively. First of all, we shall insert interval of length a_1, all the other intervals with odd indexes will be assigned to the subinterval before a_1, and all the intervals with even indexes will be assigned to the

subintervals after a_1. The location of a_1 interval will be chosen so that the length of the before interval is equal to sum of lengths of all odd indexes > 1, and the length of the after interval will be equal to sum of the lengths of all even indexes.
We shall continue the process inductively. On each stage, we have several open intervals, to each an infinite set of closed intervals is assigned. From the closed intervals assigned to each open interval, we take one the interval with the smallest index, and separate the rest of them into two infinite subsets. The first subset is assigned to the "before subinterval", the second subset is assigned to the "after subinterval", the location of the chosen closed interval is chosen so that the lengths if the before and the after subinterval will be precisely sufficient to cover the intervals assigned to them.
At stage n, the interval of index n is already inserted, and all closed intervals are disjoint; therefore this process will insert all intervals, and all will be disjoint.

5. $f:\mathbb{R}\to\mathbb{R}$ such that for any positive real x, y the sequence $f(x+ny)$, for $n\in\mathbb{N}$, tends to infinity.

a. Can we claim that that $f(x)\underset{x\to\infty}{\longrightarrow}\infty$?

b.** Can we claim that $f(x)\underset{x\to\infty}{\longrightarrow}\infty$, if it is given that f is continuous?

Answer. a. No. b. Yes

Solution. a. Let $a > 1$ be a transcendent number, and consider the following function: $f(a^n) = 0$ for every natural n, and $f(x) = x^2$ for all other points.

Any arithmetic progression can have no more than two common points with the sequence a^n, since if it would have 3 common points, a would be a root of a polynomial with rational coefficients. Hence any sequence $f(x+ny)$ tends to infinity, and $f(x)$ doesn't.

b. Suppose it doesn't. Then for some M there is a sequence x_k, converging to infinity, such that $f(x_k) < M$. Then, since f is continuous, for $|x - x_k| < \varepsilon_k$, we have $f(x_k) < 2M = N$, where ε_k are small numbers, chosen separately for different k. So, to get the contradiction we need to do one thing: build an arithmetic sequence which intersects infinite subset of these small intervals.

Assume we have built a sequence $\{ny\}$ which intersects K intervals:

$n_k y \in \left(x_{m_k} - \varepsilon_{m_k}, x_{m_k} + \varepsilon_{m_k}\right)$, for some indices n_k , m_k for $k \le K$.

We can move the y in certain interval so that the conditions $n_k y \in \left(x_{m_k} - \varepsilon_{m_k}, x_{m_k} + \varepsilon_{m_k}\right)$ still hold for the same n_k , m_k, because intersection of open intervals is still an open interval, if it is nonempty.

We shall find such y that satisfies this condition for as large K as we want by induction over K. For $K = 1$ it is obvious.

Assume that $n_k y \in \left(x_{m_k} - \varepsilon_{m_k}, x_{m_k} + \varepsilon_{m_k}\right)$, for given n_k , m_k for $k \le K$, in the interval $y \in (\alpha Y, Y)$, where $\alpha < 1$.

Possible values of ny will cover the interval between ny and $(n+1)y$ if $(n+1)\alpha Y < nY$ i. e. $1 + \frac{1}{n} = \frac{n+1}{n} < \frac{1}{\alpha}$, or $n > \frac{\alpha}{1-\alpha}$.

So, all numbers above $Y\frac{\alpha}{\alpha - 1}$ will be covered by possible values of ny.

But x_k tends to infinity, so we can choose x_m such that it can be equal to ny for a certain value of y in the interval. This completes the induction.

By this inductive procedure we shall build an infinite set of indices m_k and a nested system of intervals $(\alpha_k Y_k, Y_k)$ such that if we choose y in interval number K then $\{ny\}$ intersects intervals $\left(x_{m_k} - \varepsilon_{m_k}, x_{m_k} + \varepsilon_{m_k}\right)$ for $k \le K$, the intersection of all those intervals has at least on point y, and for that y sequence $\{ny\}$ intersects infinite number of intervals. Hence $f(ny)$ doesn't tend to infinity, contradiction, QED.

Targil 3.

Reminder: a set is convex, if for any two points inside the set, it contains the interval connecting these two points.

1. Consider a subset in $\mathbb{R}^K$. At each step, we add to our subset all the interior points of all intervals having both ends in that subset. The process is stopped when the set is already convex. What is the maximal possible number of steps in this process?

2. A family of N convex sets in $\mathbb{R}^K$ is given, $N > K$. Each $K + 1$ sets of the family have a common point. Prove that all sets have a common point.

3. Matrix S is square and has the following 3 properties:

(a) All entries are nonnegative.
(b) Sum of numbers in any row is 1.
(c) Sum of numbers in any column is 1.

Prove that this matrix is a weighted average of permutation matrixes.

4. Given k balls of radius 1 in $\mathbb{R}^3$ a point on the boundary of a ball is called "isolated" if it doesn't see any other ball (the balls are not transparent). What is the area of the set of isolated points?

5. Consider a bounded convex shape of area S in plane with smooth boundary of length l. Find the area of R-neighborhood given R, S and l. (By R-neighborhood we mean the set of all points that have distance at most R from the original shape.)

6. We are given $N > 2$ circles of radius 1. Every straight line meets less than 3 circles. Their centers are O_1, O_2, … O_N. Prove that

a. $$\sum_{i<j} \frac{1}{O_iO_j} < \frac{n\pi}{4}$$

b.* $$\sum_{i<j} \frac{1}{O_iO_j} < \frac{(n-1)\pi}{4}$$

Targil 3.

Reminder: a set is convex, if for any two points inside the set, it contains the interval connecting these two points.

1. Consider a subset in $\mathbb{R}^K$. At each step, we add to our subset all the interior points of all intervals having both ends in that subset. The process is stopped when the set is already convex. What is the maximal possible number of steps in this process?

Answer. $\lceil \log_2 (K+1) \rceil$, where $\lceil x \rceil$ denote ceiling of x, which is the least integer number not smaller than x.

Solution.

Definition. For vectors $v_1, v_2, \ldots, v_n$ a ***convex combination*** is a sort of linear combination, $a_1v_1 + a_2v_2 + \ldots + a_nv_n$ with two additional conditions: the coefficients are nonnegative and sum of coefficients is 1.

The physical meaning of convex combination is the mass center with some masses.

Definition. The minimal convex set containing the original set is called a ***convex hull*** (in Hebrew קמור) of the original set.

So, first of all, we had some original set A. Take all (finite) convex combinations of elements of A.

Lemma 1. All convex combinations form a convex set.

Lemma 2. WLOG any convex combination is of no more than K+1 points. If there are more points, we can write the same point as a convex combination of fewer points from A.

Now denote A_1 union of all intervals with endpoints at A. Similarly A_{m+1} union of all intervals with endpoints at A_m.

Lemma 3. A_m is a set of all convex combinations of no more than 2^m points.

From the lemmas we get our claim in one direction. That is if $m = \lceil \log_2(K+1) \rceil$, in other words m is the smallest integer such that $2^m \geq K + 1$, then by lemmas 2 and 3, A_m contains all convex combinations of points from A. That is a convex set by lemma 1, so the process will stop.

For any K we can construct many examples in which the process won't stop earlier. Indeed, assume the original set A was $K + 1$ points $v_0, v_1, \ldots, v_K$ in general position. That is, they were not on a hyperplane.
The center of mass $(v_0 + v_1 + \ldots + v_K) / (K + 1)$ is a convex combination of $K + 1$ points but not of K points, otherwise a point of A would belong to the hyperplane spanned by the other points. Therefore A_{m-1} doesn't contain it, since it contains only convex combinations of at most 2^{m-1} points, and by definition m is the smallest integer such as … .

It remains to prove the lemmas. All of them are based on the expression from analytic geometry to the interval connecting points P and Q. The interval PQ is described as $\{\alpha P + \beta Q \mid \alpha + \beta = 1 \ , \ 0 \leq \alpha, \beta\}$.
That formula is an exercise to the reader. Direct application of this formula makes lemmas 1 and 3 obvious,

Proof of lemma 2. Consider convex combination of N vectors, $N > K + 1$.

$$C = a_1 v_1 + a_2 v_2 + \ldots + a_N v_N$$

We want to prove that we can get C as a convex combination of fewer points. Construct vectors w_m in $\mathbb{R}^{K+1}$: first K coordinates of w_m coincide with those of v_m, and the last coordinate of w_m is 1. Now we have N vectors in $\mathbb{R}^{K+1}$, hence they are linearly dependent. Hence there is a nontrivial zero linear combination of those which is $\sum_{m=1}^{N} b_m w_m = 0$. This condition can be written as two conditions: when you look at the last coordinate, you see that $\sum_{m=1}^{N} b_m = 0$, and when you look at the first K coordinates, you get $\sum_{m=1}^{N} b_m v_m = 0$. We have both positive and negative coefficients,

since the combination is non-trivial, and sum of coefficients is zero. Let i_1, i_2, i_3, … be the indexes of positive coefficients, j_1, j_2, j_3, … be the indexes of negative coefficients, and $c_m = |b_m|$, Then after moving negative coefficients to the RHS (right hand side), we get $\sum c_{i_k} v_{i_k} = \sum c_{j_k} v_{j_k}$, and here all the coefficients are positive. Take $q = \max\limits_{c_k \neq 0}(a_k / c_k)$. WLOG, we may assume that the maximal value was achieved on the RHS (otherwise we shall revert the inequality).
We shall take the original convex combination $a_1v_1 + a_2v_2 + \ldots + a_Nv_N$ and add to it the expression $q \cdot \left(\sum c_{i_k} v_{i_k} - \sum c_{j_k} v_{j_k}\right)$. Sum of coefficients will remain the same (i.e. 1), one coefficient will cancel out, the others will remain positive. So we get a convex combination with fewer points. QED.

2. A family of N convex sets in $\mathbb{R}^K$ is given, $N > K$. Each $K + 1$ sets of the family have a common point. Prove that all sets have a common point.

Proof. The proof goes by induction. Assume we have proven that each M sets have a common point, $M > K + 1$. It remains to prove that each $M + 1$ sets, for instance $A_0, A_1, \ldots, A_M$, have a common point.
Consider sets $B_i = A_i \cap A_0$, for $i = 1, 2, \ldots, M$. It is easy to see that these sets are convex, since an intersection of two convex set is obviously convex.
Any $M - 1$ of sets B_i have a common point (since their intersection is actually an intersection of M sets A_i). So, intersection of any $K + 1$ among sets B_i is non-empty, and by induction assumption the intersection of M of them is non-empty.
Therefore $B_1 \cap B_2 \cap \ldots \cap B_M = A_0 \cap A_1 \cap A_2 \cap \ldots \cap A_M$ is nonempty.

It remains to build a base for this induction. It has to be based on a different idea. Because, for example, intersection of any two sides of triangle is nonempty, but intersection of all 3 is empty.
During the proof of lemma 2 in the solution of the previous problem we have noticed the following fact: for more than $K + 1$ points $v_1, \ldots, v_N$ in K – dimensional space, we can choose some two subsets of indexes and positive coefficients c_k, such that $\sum c_{i_k} v_{i_k} = \sum c_{j_k} v_{j_k}$ and $\sum c_{i_k} = \sum c_{j_k}$. If we divide all coefficients by $s = \sum c_{i_k} = \sum c_{j_k}$ we arrive to the following conclusion: there is a point which is a

convex combination of any of the two disjoint subsets of given points (assuming the number of points is at least K+2).
The base of induction, that we have to prove, is the following: if any K+1 sets intersect, then any K+2 sets intersect. Take K+2 sets $A_1, A_2, \ldots, A_{K+2}$. Intersection of all those sets except A_m contains at least one point P_m.
Now we have K+2 points. We can choose two disjoint subsets of indexes $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$ such that a certain point P in space is a convex combination both of $P_{i_1}, P_{i_2}, \ldots$ and of $P_{j_1}, P_{j_2}, \ldots$. We shall prove that the point P belongs to any A_m, therefore it belongs to their intersection and hence it is nonempty.
Indeed, assume that m is not one of $i_1, i_2, i_3, \ldots$. Then A_m contains $P_{i_1}, P_{i_2}, \ldots$ and because it is convex, it contains all their convex combinations, and P in particular. If is not one of $i_1, i_2, i_3, \ldots$, then it is not one of the $j_1, j_2, \ldots$, then A_m contains $P_{j_1}, P_{j_2}, \ldots$ and all their convex combinations, hence it contains P.

Remarks. I think problem 2 was also called Haley theorem, but I can't find it in google. In order to solve such a problem, a good approach is to solve 2- or 3-dimensional case first (worked for me), even if when you write it down you find out you don't use

3. Matrix S is square and has the following 3 properties:

(a) All entries are nonnegative.
(b) Sum of numbers in any row is 1.
(c) Sum of numbers in any column is 1.

Prove that this matrix is a weighted average of permutation matrixes.

Remark. The matrixes described in the question are called bistochastic or doubly stochastic. Stochastic is a scientific word for probabilistic. Nonnegative numbers with sum 1 can mean probabilities; bistochastic means both rows and columns can be probabilities.

Solution. The cells of the matrix which are zeroes or ones will be called good cells. The other cells will be called bad cells.
Lemma. If we have a matrix which has some bad cells, it is a convex combination (weighted averaged) of matrixes with fewer bad cells.

Proof of lemma. Choose a bad cell and mark it with black. There is another bad cell in the same row, mark it with white and move to that cell. There is another bad cell in the same column, mark it with black and move to that cell and so on. At some moment, we shall be able to move to the cell which was already marked with the opposite color.
In that way we shall create a closed loop of even length, all even steps are vertical from a black cell to a white cell, and all odd steps are horizontal from a white cell to the left cell. If we add ε to all white cells of that loop, and subtract ε from all white cells of that loop, then the sum in every row and column remains the same. So, when ε is close to 0, the matrix remains bistochastic, when it is far enough for 0, the matrix stops being bistochastic, because some numbers in the matrix become negative. When we substitute different values of ε we get a line in the linear space of all matrixes. When ε is 0, we get the original matrix A. When ε is minimal possible/maximal possible so that matrix is still bi-stochastic, we get matrixes A_0 and A_1 which have fewer bad cells than A, and A is on the interval connecting A_0 to A_1.
So, we can replace each matrix by a convex combination of matrixes with less bad cells; those can be replaced by convex combinations of matrixes with even smaller number of bad cells etc. With each step in our convex combination maximal number of bad cells in a matrix will be reduced, and after n^2 steps at most we get the same matrix as convex combination of matrixes with no bad cells.
Matrixes which have only zeroes and ones and sum in each column and in each row is 1 are precisely the permutation matrixes.

Remark. There's a more general statement called Krein-Milman theorem. In finite dimensional case, it states that a compact convex set is a convex hull of its extreme points (where extreme points are such points of the set that are not interior points of the intervals joining points of that set). In the infinite dimensional case, convex hull is replaced by the closure of convex hull.

4. Given *k* balls of radius 1 in $\mathbb{R}^3$ a point on the boundary of a ball is called "isolated" if it doesn't see any other ball (the balls are not transparent).
What is the area of the set of isolated points?

Answer. The area of the unit sphere, that is 4π.

Solution. For every isolated point of any sphere, consider a unit normal vector to its sphere, looking outside the sphere.
That defines a map from the set of isolated points to the unit sphere (since a unit vector always belongs to the unit sphere). We shall prove that this map, up to a set of measure zero, is bijective (חח"ע ועל).
Injectivity (חח"ע): at any isolated point its sphere has a tangent plane. By definition of isolated point, all the other points of all the spheres are on one side of that plane. Therefore, scalar product of corresponding normal vector with the given isolated point is greater than for any other point. Hence it is unique.
Surjectivity (על) up to measure 0: consider any unit vector v, and consider of all points on given spheres point p which has highest scalar product with v.
It might happen that we have two or more points of that kind, from different spheres, but probability of it is 0. It would mean that scalar product of v with two centers of different spheres is 0, and that means that v belongs to one of the finite number of arcs, defined by orthogonality to interval connecting two centers of spheres. There is a finite number of those intervals.
Outside those cases with probability 0, we get just one point with highest scalar product, and that is the isolated point with given normal. QED.

5. Consider a bounded convex shape of area S in plane with smooth boundary of length l. Find the area of R-neighborhood given R, S and l. (By R-neighborhood we mean the set of all points that have distance at most R from the original shape.)

Solution. Consider a convex polygon, that is inscribed into the shape and approximates it (it can be done by walking around along the boundary in small steps). The R neighbourhood of the polygon consists of:

a. Rectangles of height R and bases = sides of the polygon.
b. Sectors of disc of radius R (angles are 180° – internal angle of polygons).

The rectangles can be glued together into a rectangle of area $R \times$ perimeter.
The sectors can be glued together into a disc of radius R and are πR^2.
In addition, we have the internal area of the polygon.

When the polygon is close to the shape, the R-neighborhood of the polygon is close to (but slightly smaller than) R-neighborhood of the original shape.
So the limit area is $\pi R^2 + lR + S$.

Remark. Of course, similar thing happens in higher dimensions (for almost the same reason): the answer is a polynomial in R of degree = dimension, the first coefficient = volume of the unit ball, linear coefficient = area of surface (n-1 dimensional) and the free coefficient is the volume. Other coefficients are more complicated.

6. We are given $N > 2$ circles of radius 1. Every straight line meets less than 3 circles. Their centers are O_1, O_2, … O_N. Prove that

a. $$\sum_{i<j} \frac{1}{O_iO_j} < \frac{n\pi}{4}$$

b.* $$\sum_{i<j} \frac{1}{O_iO_j} < \frac{(n-1)\pi}{4}$$

Solution. *a.* Consider angles of view of all circles other than number j from the point O_j. By condition, those angles don't intersect. Even if we add to each angle the opposite angle, the angles still don't intersect (otherwise there would be a line through O_j cutting circle j and two more circles). So their sum is less than 2π. Denote by $\alpha_{i,j}$ half the angle of view of circle number i from point O_j.

We actually proved that $\sum_{i \neq j} \alpha_{i,j} < \frac{\pi}{2}$ for any j. But $\frac{1}{O_iO_j} = \sin\alpha_{i,j} < \alpha_{i,j}$.

Therefore $\sum_{i \neq j} \frac{1}{O_iO_j} < \frac{\pi}{2}$ for any j. Summing over j gives $\sum_{j=1}^{n} \sum_{i \neq j} \frac{1}{O_iO_j} < n\frac{\pi}{2}$.

Hence $\sum\sum_{i<j} \frac{1}{O_iO_j} < \frac{n\pi}{4}$.

b. Consider angle $O_jO_kO_i$. The distance from O_i to the line O_jO_k is at least 2.

Therefore $O_iO_k \cdot \sin\left(O_jO_kO_i\right) \geq 2$. Hence the angle $\sin\left(O_jO_kO_i\right) \geq \frac{2}{O_iO_k}$.

Hence both the angle $O_jO_kO_i$ and $180° - O_jO_kO_i$ are greater than $\frac{2}{O_iO_k}$ (or than $\frac{2}{O_jO_k}$ for the same reason).

Let n be the total number of points. For given k, consider lines via O_k and all other O_j this line split plane into $n-1$ pairs of symmetric angles. Angle bounded (from the clockwise direction) by O_jO_k is greater than $\frac{2}{O_jO_k}$. There are two symmetric angles of that kind, so for each k we get $2\sum_{j\neq k}\frac{2}{O_jO_k}<2\pi$.

Summing these things will lead to the same conclusion once again, so we need yet another idea. The idea is: consider the convex hull of all O_j. Not all O_j are the vertexes of the convex hull, just m of them.

Sum of all angles of the convex hull is not $m\pi$ but $(m-2)\pi$.

Fix vertex of the convex hull O_k and denote the angle of the convex hull at it α_k. Then rays O_kO_j split the angle α_k into $n-2$ parts. The part $O_iO_kO_j$ is greater than by both $\frac{2}{O_jO_k}$ and $\frac{2}{O_iO_k}$.

Summing inequalities of that kind we can write $\sum_{j\neq i,k}\frac{2}{O_jO_k}<\alpha_k$. In this sum $\frac{2}{O_iO_k}$ is omitted. We can omit any we choose, so we shall omit the longest O_iO_k.

Therefore $\sum_{j\neq k}\frac{2}{O_jO_k}\leq\frac{n-1}{n-2}\sum_{j\neq i,k}\frac{2}{O_jO_k}<\frac{n-1}{n-2}\alpha_k$.

Sum it over all vertexes of the convex hull:

$$\sum_{O_k\in hull}\sum_{j\neq k}\frac{2}{O_jO_k}<\sum_{O_k\in hull}\frac{n-1}{n-2}\alpha_k=\frac{n-1}{n-2}\sum_{O_k\in hull}\alpha_k=\frac{n-1}{n-2}(m-2)\pi\leq(m-1)\pi$$

Hence the sum over all vertexes

$$\sum_{k}\sum_{j\neq k}\frac{2}{O_jO_k}=\sum_{O_k\in hull}\sum_{j\neq k}\frac{2}{O_jO_k}+\sum_{O_k\neq hull}\sum_{j\neq k}\frac{2}{O_jO_k}\leq(m-1)\pi+(n-m)\pi=(n-1)\pi$$

Each pair here appears twice. Therefore sum over pairs $2\sum_{i<j}\frac{2}{O_iO_j}<(n-1)\pi$. QED.

Targil 4.

(Taking the extreme).

1. On a plane, there are $2n$ points in general position (no 3 are on the same line). Half of them are blue, others are red. Prove that it is possible to divide them into pairs, each pair consisting of one blue point and one red point in each pair, so that the straight intervals connecting these pairs won't intersect.

2.* Inside a regular N-gon N points are marked. Consider N pairs: in each pair there is one side of the N-gon and one marked points. Each marked point and each side of the triangle is used in one pair precisely. From each pair, a triangle is formed (as a convex hull of the side and the point of that pair).
Show that the pairing can be chosen in such a way, that the triangles won't overlap.

3. Prove that $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}$ isn't integer.

4.* A finite number of points are chosen on a plane. A line through any two of the chosen points contains at least 3 chosen points. Prove that all points are collinear.

5. Consider a connected graph. At some vertexes, real numbers are written. Prove that you can write real numbers at all the other vertexes, so that every number written by you will be equal to the average of its neighbors.

6.* A billiard table is convex and has a smooth boundary. Prove that there an infinite number of different closed trajectories of the billiard ball (in other words, infinite number of closed broken lines inside the given table, such that at every vertex is on the boundary, and the sides at this vertex are symmetric w.r.t. the normal).

Targil 4.

Taking the extreme.

The general idea is to consider the extreme object/case/state.
The question that remains is: extreme in what sense?

1. On a plane, there are $2n$ points in general position (no 3 are on the same line). Half of them are blue, others are red. Prove that it is possible to divide them into pairs, each pair consisting of one blue point and one red point in each pair, so that the straight intervals connecting these pairs won't intersect.

Solution. Consider all possible red-blue pairings. There is finite number of those. For each pairing, consider the total sum of lengths of all its intervals. We take the pairing, in which the sum of lengths is minimal.
Let us prove it satisfies the condition. Suppose not: intervals R_1B_1 and R_2B_2 intersect. If we would connect R_1 to B_2, and R_2 to B_1, we would get smaller sum of lengths (due to the triangular inequality, sum of two diagonals in the convex quadrilateral is greater than the sum of two opposite sides). Hence our pairing doesn't give the minimal sum of all intervals, contradiction.

2.* Inside a regular N-gon N points are marked. Consider N pairs: in each pair there is one side of the N-gon and one marked points. Each marked point and each side of the triangle is used in one pair precisely. From each pair, a triangle is formed (as a convex hull of the side and the point of that pair).
Show that the pairing can be chosen in such a way, that the triangles won't overlap.

Solution. Choose the pairing, for which the product of all areas of triangles is minimal. We shall prove that this pairing has no overlaps. Assume two triangles overlap. One is formed by side s_i of the polygon and inner point A_i, another by side s_k and inner point A_k. What happens if we flip them, assign the polygon side to the inner point of the second and vice versa?
There are two cases. If s_i is parallel to s_k, then the distances from both sides to corresponding vertexes decrease, and both areas decrease.
Now consider the case when s_i and s_k intersect at point O. For each point P, denote by d_j(P) distance from line P to line s_j.

Consider the angle with vertex at O formed by the continuations of sides s_i and s_k. Inside this angle, the level sets of a function $f(X) = d_i(X)/d_k(X)$ are the rays starting at O. Value of f goes monotonically from 0 to plus infinity as the ray rotates from s_i to s_k. Therefore, $f(A_k) > \mathrm{f}(A_i)$ (otherwise the triangles would be in disjoint angles and wouldn't overlap) in other words $d_i(A_k) / d_k(A_k) > d_i(A_i) / d_k(A_i)$ or

$$d_i(A_k)\, d_k(A_i) > d_i(A_i)\, d_k(A_k)$$

If we multiply that by $s_i s_k /4$, we see that the product of areas will be greater after the flip. That is a contradiction, since we have chosen a pairing with the greatest product of areas. Hence our pairing is non-overlapping.

3. Prove that $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}$ isn't integer.

First solution. Let p be the greatest prime $\leq n$. By Chebyshev theorem, there is a prime number between n and $n/2$, hence $p > n/2$. So between 1, 2, …, n there is no number divisible by p except p. Hence if we multiply our number by all numbers between 1 and n except p, all terms except one will be integer. Hence the number is not integer.

Remark. Many people might not accept this solution, since they don't know Chebyshev theorem. But there is also another solution.

Second solution. Take the greatest $2^k \leq n$. The first number which is divisible by 2^k other than itself is 2^{k+1} and it is $> n$. Hence if we multiply the original number by 2^{k-1} and by the product of all odd numbers $\leq n$, and we get it.

4.* A finite number of points are chosen on a plane. A line through any two of the chosen points contains at least 3 chosen points. Prove that all points are collinear.
Remark. This problem is called Sylvester problem, after famous 19th century Jewish (English) mathematician James Joseph Sylvester.

Solution. Suppose not. Take a triangle with minimal altitude (גובה) the altitude from A to BC in a triangle ABC is minimal. There is one more chosen point D on BC. So the altitude A to the line BCD is minimal. WLOG, C is between B and D. Either angle ACD or angle ACB is non-acute. WLOG, ACB is non-acute. So in the triangle ABC the minimal Altitude is from C and not from B.

5. Consider a connected graph. At some vertexes, real numbers are written. Prove that you can write real numbers at all the other vertexes, so that every number written by you will be equal to the average of its neighbors.

First solution. The conditions are a linear system of equations. The matrix is quadratic: number of equations = number of unknowns. If for one set of given values in given points there is a unique solution, then matrix is non-degenerate and for any set of given values there's a unique solution.
So, it is enough to show existence and uniqueness of solutions in the case when all given values are 0. One obvious solution exists in that case: write 0 in every vertex. It remains to show there is no other solution.
Assume there is a solution where at least one number is positive. Take the maximal number in the graph. It is average of the neighbors, so its neighbors are also maximal numbers in the graph. Same for neighbors of neighbors, etc. but since graph is connected all numbers in it should be positive. Contradiction, so there are no positive numbers in the graph.
For similar reasons there are no negative numbers.
Hence the only solution is of zeroes, QED.
Second solution. Take the minimum of sum over all edges of squares of differences at between the values at the ends of the edge.
Well, first one could ask why the minimum exists.
The minimum in any compact set, for instance when absolute values of all numbers are not greater than 1000000, exists. Outside that compact set the values are large enough, so we shouldn't look there (to be precise, if M is the maximal absolute value among given numbers, and N is number of vertexes in the graph, and E number of edges, then we can assume that all numbers are less than 2EMN , otherwise we have an edge with absolute value of difference at least 2EM, and the square of that is greater than the whole sum if we would write zeroes at all empty vertexes).
In the minimal situation, the derivative of the "energy" by every number we wrote is 0 (since we are at the minimal point. That gives precisely the condition we wanted.
Remark. This is a discrete version of Dirichlet principle (invented by Riemann). http://en.wikipedia.org/wiki/Dirichlet's_principle In our "finite" situation, there are no difficulties with justification of minimum existence.

6.* A billiard table is convex and has a smooth boundary. Prove that there an infinite number of different closed trajectories of the billiard ball (in other words, infinite number of closed broken lines inside the given table, such that at every vertex is on the boundary, and the sides at this vertex are symmetric w.r.t. the normal).

Solution. For every N, take the longest closed line $A_1A_2A_3\ldots A_N$, such that all A_i are on the boundary. Small perturbations of any single vertex along the boundary are not enlarging the sum of distances. So the derivative there is 0, which gives the billiard law of the equality of angles.
Of course, all these polygons are different (the larger the N, the longer the line) and each is a billiard trajectory.

Gal found a mistake in the above. It may happen (and can really happen, for instance for the circle) that the longest trajectory of $N = KM$ vertexes is actually the longest trajectory of M vertexes repeated K times. Of course, if N is prime, it cannot happen, and there is an infinite number of prime numbers, so we still get an infinite number of different trajectories.

Another, but similar way to solve this problem is to consider the convex polygon of greatest perimeter with N vertexes inscribed in the billiard table. In this case, when perimeter is greatest, vertexes won't coincide (otherwise we would disregard one of the vertexes and add another vertex that would enlarge perimeter, because of triangle inequality).

Targil 5.

(Integrals – everybody loves them)

1. Compute $\int_0^{2\pi} \frac{x^2 \sin x}{1+\cos^2 x} dx$.

2. A center of the disc of radius *R* is on distance *d* from the axis *α*, which is parallel to in the plane of the disc. The disc rotates around its center with angular velocity 2 and simultaneously around revolves around axis *α* with angular velocity 3.
A red point on the boundary of the disc goes along a closed trajectory.
Compute the length of that trajectory.

3.** A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ will be called "nice of order *m*", if $f(0,0,...,0) = 0$ and for any positive integer $k \le m$, for any i_1, i_2, …, i_k we have $\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_k}} f(0,0,...,0) = 0$.
Consider a function f which is smooth and nice of order *m*. Prove that there are *n* continuous functions $f_1, f_2, ..., f_n : \mathbb{R}^n \to \mathbb{R}$, smooth and nice of order *m* – 1, such that $f(x_1, x_2, ..., x_n) = \sum_{k=1}^{n} x_k f_k(x_1, x_2, ..., x_n)$.

4.** Let *n* be a positive integer, and $f : [0,1] \to \mathbb{R}$ a continuous function such that

$$\int_0^1 x^k f(x) dx = 1$$

for every $k \in \{0,1,...,n-1\}$. Prove that $\int_0^1 (f(x))^2 dx \ge n^2$.

5. Let $f, g : [a,b] \to (0,\infty)$ be continuous, non-decreasing functions, such that for every $x \in [a,b]$ we have $\int_a^x \sqrt{f(t)} dt \le \int_a^x \sqrt{g(t)} dt$

and
$$\int_a^b \sqrt{f(t)} dt = \int_a^b \sqrt{g(t)} dt$$

Prove that
$$\int_a^b \sqrt{1+f(t)} dt \ge \int_a^b \sqrt{1+g(t)} dt .$$

Targil 5.

1. Compute $\int_0^{2\pi} \frac{x^2 \sin x}{1+\cos^2 x} dx$.

Solution.

$$\int_0^{2\pi} \frac{x^2 \sin x}{1+\cos^2 x} dx = \int_0^{\pi} ...dx + \int_{\pi}^{2\pi} ...dx = \int_0^{\pi} \frac{x^2 \sin x}{1+\cos^2 x} - \frac{(x+\pi)^2 \sin x}{1+\cos^2 x} dx =$$

$$= \int_0^{\pi} \frac{\left(-2\pi x - \pi^2\right)\sin x}{1+\cos^2 x} dx = \int_0^{\pi/2} ...dx + \int_{\pi/2}^{\pi} ...dx$$

In the second part, take $y = \pi - x$.

$$\int_0^{2\pi} \frac{x^2 \sin x}{1+\cos^2 x} dx = \int_0^{\pi} ...dx + \int_{\pi}^{2\pi} ...dx = \int_0^{\pi} \frac{x^2 \sin x}{1+\cos^2 x} - \frac{(x+\pi)^2 \sin x}{1+\cos^2 x} dx =$$

$$= \int_0^{\pi} \frac{\left(-2\pi x - \pi^2\right)\sin x}{1+\cos^2 x} dx = \int_0^{\pi/2} ...dx + \int_{\pi/2}^{\pi} ...dx$$

$$\int_0^{\pi} \frac{\left(-2\pi x - \pi^2\right)\sin x}{1+\cos^2 x} dx = \int_0^{\pi/2} ..dx + \int_0^{\pi/2} \frac{\left(-2\pi(\pi - y) - \pi^2\right)\sin y}{1+\cos^2 y} dy =$$

$$= \int_0^{\pi/2} \frac{\left(-2\pi x - \pi^2\right)\sin x}{1+\cos^2 x} dx + \int_0^{\pi/2} \frac{\left(2\pi x - 3\pi^2\right)\sin x}{1+\cos^2 x} dx =$$

$$= \int_0^{\pi/2} \frac{\left(-2\pi x - \pi^2 + 2\pi x - 3\pi^2\right)\sin x}{1+\cos^2 x} dx = \int_0^{\pi/2} \frac{\left(-4\pi^2\right)\sin x}{1+\cos^2 x} dx =$$

$$= \int_1^{0} \frac{4\pi^2}{1+u^2} du = 4\pi^2 \arctan u\Big|_1^0 = 4\pi^2 \cdot \left(-\frac{\pi}{4}\right) = \pi^3$$

2. A center of the disc of radius *R* is on distance *d* from the axis *α*, which is parallel to in the plane of the disc. The disc rotates around its center with angular velocity 2 and simultaneously revolves around axis *α* with angular velocity 3.
A red point on the boundary of the disc goes along a closed trajectory.
Compute the length of that trajectory.

Solution. Consider rotating plane with internal coordinates (x, y) in which a point goes by a circle with center (d, 0) and radius R and angular velocity 2. The parametric description of point's trajectory is $(d + R\cos(2t), R\sin(2t))$.

Now consider our red point. It is easy to see that in time $2\pi/3$ the circle rotating with angular velocity 3 will return to the original position. The red point within the plane of the circle rotates with period π (and only one moment of each period it is at the maximal distance from the axis α). So, the trajectory becomes closed after the period of time which is the least common multiple of $2\pi/3$ and π, which is 2π.

So, the time interval for the limit is $[0, 2\pi]$. But what is in the integral?

The length of the curve can be computed as an integral of the length of velocity vector times dt. To make the life easier, we won't go to the Cartesian coordinates, but rather decompose the velocity vector v into sum of two: v_1 is inside the plane of the circle, and it comes from the rotation of disc in its plane around its center, and v_2 which is orthogonal to the plane of the disc, and comes from the revolution around axis α. Since v_1 and v_2 are orthogonal, $|v| = \sqrt{|v_1|^2 + |v_2|^2}$.

$|v_1|$ is constant and equals $2R$. $|v_2|$ is not constant, and equals $3r$, where r is the distance between the axis α and the red point, which is $|d + R\cos(2t)|$.

Totally, we get $L = \int_0^{2\pi} \sqrt{4R^2 + (d + R\cos(2t))^2}\,dt$. Same as

$$L = 2\int_0^{\pi} \sqrt{4R^2 + (d + R\cos(2t))^2}\,dt$$

Substitute $x = 2t$:

$$L = \int_0^{\pi} \sqrt{4R^2 + (d + R\cos(x))^2}\,dx$$

Denote $a = d / 2R$.

$$L = 2R\int_0^{\pi} \sqrt{1 + \left(a + \tfrac{1}{2}\cos(x)\right)^2}\,dx = 2R\int_0^{\pi} \sqrt{1 + a^2 + a\cos x + \tfrac{1}{4}\cos^2(x)}\,dx$$

Well, this integral is not elementary. Even for $a = 0$ we get an elliptic integral.
(Of course, if I would know it at the beginning I wouldn't suggest the problem.
Bu I thought it would be something short and simple.)

3.** A smooth function $f:\mathbb{R}^n \to \mathbb{R}$ will be called "nice of order *m*", if $f(0,0,...,0)=0$ and for any positive integer $k \le m$, for any i_1, i_2, …, i_k we have $\frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_k}} f(0,0,...,0)=0$.

Consider a function f which is smooth and nice of order *m*. Prove that there are *n* continuous functions $f_1, f_2,..., f_n:\mathbb{R}^n \to \mathbb{R}$, smooth and nice of order $m-1$, such that $f(x_1,x_2,...,x_n)=\sum_{k=1}^{n} x_k f_k(x_1,x_2,...,x_n)$.

Solution.

$$f(x_1,x_2,...,x_n)=\int_0^1\left(\frac{d}{dt}f(tx_1,tx_2,...,tx_n)\right)dt=$$

$$=\int_0^1\left(\sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k} f(tx_1,tx_2,...,tx_n)\right)dt=\sum_{k=1}^{n} x_k \int_0^1\left(\frac{\partial}{\partial x_k} f(tx_1,tx_2,...,tx_n)\right)dt$$

Denote $f(k)=\int_0^1\left(\frac{\partial}{\partial x_k} f(tx_1,tx_2,...,tx_n)\right)dt$; it is easy to see that it has the necessary properties.

4.** Let *n* be a positive integer, and $f:[0,1]\to\mathbb{R}$ a continuous function such that

$$\int_0^1 x^k f(x)\,dx=1$$

for every $k\in\{0,1,...,n-1\}$. Prove that $\int_0^1 (f(x))^2\,dx \ge n^2$.

First solution. First idea: projection to the space of polynomial of low degree.
Let $p(x)$ be a polynomial of degree less than *n,* satisfying the conditions of *f*. Define $g = f - p$. Then $\int_0^1 x^k g(x)\,dx=0$, moreover $\int_0^1 q(x)g(x)\,dx=0$ for *q* polynomial of degree less than *n*.

$$\int_0^1 f^2dx=\int_0^1 (g+p)^2\,dx=\int_0^1 g^2dx+2\int_0^1 g\cdot p\,dx+\int_0^1 p^2dx=\int_0^1 g^2dx+\int_0^1 p^2dx\ge\int_0^1 p^2dx$$

Therefore, if such p exists then it is the function that minimizes $\int_0^1 f^2 dx$.

Of course, it exists, since the 1, x, x^2, … , x^{n-1} are linearly independent and the scalar product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ is non-degenerate on polynomials (since if the scalar square of a polynomial is zero, then the polynomial itself is zero), therefore the polynomial exists and is unique.

Let $p(x) = \sum_{i=0}^{n-1} a_i x^i$ be the polynomial satisfying the conditions.

Then $\int_0^1 (p(x))^2 dx = \int_0^1 p(x) \sum_{i=0}^{n-1} a_i x^i dx = \sum_{i=0}^{n-1} a_i \int_0^1 p(x) x^i dx = \sum_{i=0}^{n-1} a_i$.

So we have to prove that $\sum_{i=0}^{n-1} a_i = n^2$.

Our scalar product, if we write it as a matrix in the basis 1, x, x^2, … , x^{n-1} is the Hilbert matrix: at row i, column j you have $\frac{1}{i+j-1}$ (that is the scalar product of x^i and x^j).

The conditions of p: scalar products of p with 1, x, x^2, … , x^{n-1} are ones.

So we actually get a problem of linear algebra when we attempt to compute its coefficients a_i:

$$\sum_{i=0}^{n-1} \frac{a_i}{k+i}, \qquad k = 1, 2, \ldots, n$$

Actually, we don't have to compute a_i, only their sum.

This is done by a trick. Consider the function $r(x) = \sum_{i=0}^{n-1} \frac{a_i}{x+i} - 1$.

This function has n roots 1, 2, …, n.

So if we make a common denominator, the numerator will be a polynomial of degree not higher than n, we shall have precisely these roots in the numerator.

$$r(x) = \sum_{i=0}^{n-1} \frac{a_i}{x+i} - 1 = \frac{q(x) - x(x+1)(x+2)\cdot\ldots\cdot(x+n-1)}{x(x+1)(x+2)\cdot\ldots\cdot(x+n-1)}$$

It is easy to see that $q(x)$ is a polynomial of degree $n-1$ and its highest coefficient is $\sum_{i=0}^{n-1} a_i$ (which we want to compute so much).

So, the numerator is the polynomial of degree n, highest coefficient -1 and we know its roots, so it is

$$q(x)-x(x+1)(x+2)\cdot\ldots\cdot(x+n-1)=-(x-1)(x-2)\cdot\ldots\cdot(x-n)$$

Hence

$$q(x)=x(x+1)(x+2)\cdot\ldots\cdot(x+n-1)-(x-1)(x-2)\cdot\ldots\cdot(x-n)$$

It remains to compute the coefficient of x^{n-1} in both products to finish it.

We get $\sum_{i=0}^{n-1} a_i = (1+2+\ldots+(n-1))-(-(1+2+\ldots+n))=n^2$, QED.

Second solution. Like in the first solution, we start with the orthogonal projection to the space of polynomials of degree less than n. Let $p(x)=\sum_{i=0}^{n-1} a_i x^i$ be the polynomial satisfying the conditions.

$$(*)\ \int_0^1 p^2 dx = \int_0^1 p(x)\cdot\left(\sum_{i=0}^{n-1} a_i x^i\right)dx = \sum_{i=0}^{n-1} a_i\left(\int_0^1 x^i p(x)dx\right) = \sum_{i=0}^{n-1} a_i = p(1)$$

Define the following sequence of polynomials.

$$p_0(x)=p(x)$$

$$p_{i+1}=\int_0^x p_i(t)dt$$

Claim. $\int_0^1 x^k p_i(t)dt = 0$ for $0 < i < n-k$.

Proof of the claim. By induction.

Base of induction:

$$1=\int_0^1 t^k p_0(t)\,dt = t^k p_1(t)\Big|_0^1 - \int_0^1 kt^{k-1} p_1(t)\,dt$$

$$t^k p_1(t)\Big|_0^1 = p_1(1) = \int_0^1 p(t)\,dt = 1 \qquad .$$

$$\int_0^1 t^{k-1} p_1(t)\,dt = 0$$

Step of induction:

$$0=\int_0^1 t^k p_i(t)\,dt = t^k p_{i+1}(t)\Big|_0^1 - \int_0^1 kt^{k-1} p_{i+1}(t)\,dt$$

$$t^k p_{i+1}(t)\Big|_0^1 = p_{i+1}(1) = \int_0^1 p_i(t)\,dt = 0$$

$$\int_0^1 t^{k-1} p_{i+1}(t)\,dt = 0$$

QED of claim.

Reminder: Leibnitz formula. Leibnitz formula allows to open brackets in the derivative of the product.

In the simplest form (base of induction) it is $\frac{d}{dx}(fg) = \left(\frac{df}{dx}\right)g + f\left(\frac{dg}{dx}\right)$.

In the general form it is $\left(\frac{d}{dx}\right)^n (fg) = \sum_{k=0}^{n} \binom{n}{k} \left(\left(\frac{d}{dx}\right)^k f\right) \times \left(\left(\frac{d}{dx}\right)^{n-k} g\right)$.

The general form is proven by induction, and the simplest form is the base of that induction.

By definition, $p_i(0)=0$. Also, $p_i(1)=0$ for $i \le n$ (by the claim).

It follows that p_n is divisible by $x^n(x-1)^{n-1}$.

Since $\deg(p_0)<n$, $\deg(p_n)<2n$.

Hence $p_n = \alpha x^n (x-1)^{n-1}$ where α is constant. We shall compute α.

$$p_1 = \left(\frac{d}{dx}\right)^{n-1} p_n = \left(\frac{d}{dx}\right)^{n-1}\left(\alpha x^n (x-1)^{n-1}\right) = q(x)(x-1) + (n-1)!\alpha x^n$$

(the last equality follows from Leibnitz formula)

Substitute 1 :
$1 = p_1(1) = (n-1)!\alpha$

Hence $p_n(x) = \frac{x^n(x-1)^{n-1}}{(n-1)!}$.

By (*) all we need to prove is $p(1) \geq n^2$.

$$p = \left(\frac{d}{dx}\right)^n p_n = q(x)(x-1) + n \cdot \frac{d(x^n)}{dx} \cdot \left(\alpha \left(\frac{d}{dx}\right)^{n-1} (x-1)^{n-1}\right)$$

$$p(1) = \left(n \cdot nx^{n-1} \cdot 1\right)\Big|_{x=1} = n^2$$

Outline of third solution. As in solution 1, we arrive to the linear algebra problem. We have to solve Ax = b, where A is the Hilbert matrix, and b is a vector of ones. The Hilbert matrix is a special cases of the Cauchy matrix: the matrix in which at row *i* column *j* we have $\frac{1}{s_i + t_j}$.

All the minors of Hilbert matrix are Cauchy matrixes.
If you know how to compute the determinant of Cauchy matrix (and if you don't, read the solution of targil 1 problem 4) you can invert the Hilbert matrix by Leibnitz-Cramer formula. With some patience and luck, you can finish it.

Fourth proof (Shahar Papini). Handling scalar product in some weird basis (such as 1, x, x^2, x^3, …) is messy and unpleasant. Better to switch to an orthogonal basis.

For our scalar product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, there is a well known polynomial orthogonal basis called ***shifted Legendre polynomials***:

$$P_n(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n \left(\left(x^2 - x\right)^n\right).$$

(usual Legendre polynomials are an orthogonal for scalar product $\int_{-1}^1 f(x)g(x)dx$, and are related to the shifted by the substitution of $2x - 1$ instead x.)

It is not orthonormal, but orthogonal: $\langle P_n, P_n\rangle = \frac{1}{2n+1}$, and $m \neq n$ for $\langle P_n, P_m\rangle = 0$.
(Both facts can be obtained using integration by parts.)

Another nice fact is $P_n(1)=1$. This can be proven by Leibnitz formula (which was mentioned in the second solution).

The polynomial $p(x)$ of degree less than n and having unit scalar products with $1, x, x^2, \dots, x^{n-1}$ can be described as follows.
For any polynomial $q(x)=b_0+b_1x+b_2x^2+\dots+b_{n-1}x^{n-1}$ of degree less than n,

$$\langle p,q\rangle = b_0+b_1+b_2+\dots+b_{n-1}=q(1)$$

This allows to compute expansion of p in the shifted Legendre basis:

$$\langle p,P_k\rangle = P_k(1)=1$$

Therefore $p=\sum_{k=0}^{n-1}(2k+1)P_n$.

Hence $\langle p,p\rangle = \sum_{k=0}^{n-1}(2k+1)^2\langle P_n,P_n\rangle = \sum_{k=0}^{n-1}(2k+1)=n^2$.

5. Let $f,g:[a,b]\to(0,\infty)$ be continuous, non-decreasing functions, such that for every $x\in[a,b]$ we have $\int_a^x\sqrt{f(t)}dt \le \int_a^x\sqrt{g(t)}dt$

and $\int_a^b\sqrt{f(t)}dt = \int_a^b\sqrt{g(t)}dt$

Prove that $\int_a^b\sqrt{1+f(t)}dt \ge \int_a^b\sqrt{1+g(t)}dt$.

Solution. Take $F(x)=\int_a^x\sqrt{f(t)}dt$, $G(x)=\int_a^x\sqrt{g(t)}dt$.

Graph of F lies below graph of G on $[a, b]$ but they coincide at the ends of an interval. Also, both graphs are convex, because derivatives are non-decreasing.

We actually need to prove $\int_a^b\sqrt{1+\left(\frac{dF}{dt}\right)^2}dt \ge \int_a^b\sqrt{1+\left(\frac{dG}{dt}\right)^2}dt$.

In other words, $\int_a^b\sqrt{dt^2+dF^2} \ge \int_a^b\sqrt{dt^2+dG^2}$.

These are the expressions for the lengths of graphs of F and G.

Consider a tangent line to G at point (c, G(c)) where $c \in [a,b]$. This line cuts the curve of graoh of F in both direction. If we replace in graph of F the part of the curve by the interval of this line, we shall still get a graph which is below the graph of G and shorter (because we replace a curve by a line). We shall do these replacements many times at different points. As all the interval between points decrease, both the graph of F gets shorter and shorter and converges to the graph of G (together with derivative), and the length converges to the length of G.

Remark. This can be generalized as follows: if a convex body A is inside another convex body B, than the surface area of A is smaller than the surface area of B.

Targil 6.

This targil is inspired by SEEMOUS 2010.

1. a.* Question from the last SEEMOUS: given a real 2×2 matrix A, prove that there are two 2×2 real matrixes B and C such that $A = B^2 + C^2$ (this was the nicest problem in the competition).

b.** The natural generalization: for real matrixes $n \times n$, is it possible to represent each matrix as a sum of squares, and if yes, how many squares are required?

2. a. Prove that a real matrix which is sufficiently close to the unit matrix is a square of real matrix.
b.* Let D be a diagonal matrix with positive numbers at the diagonal. Prove that a matrix which is sufficiently close to D is a square of real matrix. (If you don't know what is "sufficiently close" stop complaining and invent a definition.)

3. Is it true that every complex $n \times n$ matrix is a square of a complex matrix?

4. A determinant of a 2×2 real matrix is positive. Is it true that this matrix is a square of a real matrix?

5. Is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ a sum of two squares of real matrixes?

Targil 6.

This targil is inspired by SEEMOUS 2010.

1. a.* Question from the last SEEMOUS: given a real 2×2 matrix A, prove that there are two 2×2 real matrixes B and C such that A = $B^2 + C^2$ (this was the nicest problem in the competition).

Solution. The nicest solution was invented by Ohad Livne (under the influence of Minkowski) during the competition. Any matrix has the following decomposition:

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} k & m \\ 0 & l \end{pmatrix}$$

Where k and l are positive. The first matrix correspond to a complex numbers when you consider it as a linear transformation of $\mathbb{R}^2$. Complex number has a root as a complex number, so that matrix is a square of a 2×2 matrix.

The second matrix is a square of $\begin{pmatrix} \sqrt{k} & x \\ 0 & \sqrt{l} \end{pmatrix}$, where x is easily computed:

$$\begin{pmatrix} \sqrt{k} & x \\ 0 & \sqrt{l} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{k} & x \\ 0 & \sqrt{l} \end{pmatrix} = \begin{pmatrix} k & \left(\sqrt{k} + \sqrt{l}\right)x \\ 0 & l \end{pmatrix}$$

So it is enough to take $x = \dfrac{m}{\sqrt{k} + \sqrt{l}}$. QED.

b.** The natural generalization: for real matrixes n×n, is it possible to represent each matrix as a sum of squares, and if yes, how many squares are required?

The answer is the following:
For n = 1 it doesn't work (there are negative numbers which are not sums of squares).
For other odd n: three squares are required.
For even n: two squares are required.
The solution will be explained in the end.

2. a. Prove that a real matrix which is sufficiently close to the unit matrix is a square of real matrix.

Solution. Use Newton's binomial formula:

$$(1+x)^\alpha = f_\alpha(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

The radius of convergence for $\alpha = \frac{1}{2}$ is 1. Therefore, for a matrix of norm < 1 the series converge. The identity $\left(f_{1/2}(x)\right)^2 = 1+x$ can be verified directly, and it is true for linear transformation whenever the series converge (since the powers of the same matrix commute).

b.* Let D be a diagonal matrix with positive numbers at the diagonal. Prove that a matrix which is sufficiently close to D is a square of real matrix. (If you don't know what is "sufficiently close" stop complaining and invent a definition.)

Solution. By $\sqrt{D}$ we shall denote the diagonal matrix, which has square roots of D matrix elements at the respective diagonal cells.
Consider the transformation $X \mapsto X^2$ in the neighborhood of $\sqrt{D}$. By the inverse function theorem, the transformation is invertible in the neighborhood if the differential is non-degenerate. Which means it is enough to verify that the derivation in every direction is nonzero: $\left.\frac{d}{d\varepsilon}\left(\sqrt{D}+\varepsilon A\right)^2\right|_{\varepsilon=0} \neq 0$ for each $A \neq 0$.
The computation yields: $\frac{d}{d\varepsilon}\left(\sqrt{D}+\varepsilon A\right)^2 = \sqrt{D}\cdot A + A\cdot\sqrt{D}$.
In both terms of this sum, entries of A are multiplied by positive numbers (in one the rows are multiplied by respective diagonal entries of $\sqrt{D}$, in another the columns). So if A isn't zero, the directional derivative isn't 0. QED.

3. Is it true that every complex $n\times n$ matrix is a square of a complex matrix?

Solution. No. The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not a square. It is nilpotent of order 2, so the "square root" would be nilpotent of order > 2 (which is impossible for matrixes 2×2).

4. A determinant of a 2×2 real matrix is positive. Is it true that this matrix is a square of a real matrix?

Answer. No.

Solution. Assume that $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = B^2$ (clearly, A has a positive determinant). Then the complex eigenvalues of B are $\alpha = \pm 1, \beta = \pm 2$ so they are not conjugate to each other hence the characteristic polynomial of B is not a polynomial with real coefficients.

5. Is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ a sum of two squares of real matrixes?

Answer. No.

Solution. Assume $C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = A^2 + B^2$. The matrix C is scalar, so it has the same form in any basis. We shall choose (a complex) basis in which A is upper triangular (for example Jordan) than also A^2 is upper triangular and $B^2 = C - A^2$ is upper triangular. The diagonal elements of A^2, B^2 are their eigenvalues, and they are squares of eigenvalues of A, B. Denote a_1, a_2, a_3 eigenvalues of A, b_1, b_2, b_3 eigenvalues of B. There are 3 equations:

$$a_1^2 + b_1^2 = -1$$
$$a_2^2 + b_2^2 = -1$$
$$a_3^2 + b_3^2 = -1$$

One of eigenvalues of both A and B is real, so it must be in pair with imaginary eigenvalue of the other matrix, so that sum of the squares will be -1. But imaginary values come in pairs, so both A and B have two imaginary values. So one of these equations is a sum of squares of two imaginary values.
WLOG, let us assume that a_1, b_2 are real and others are imaginary.
Hence $b_1^2 = b_3^2 \le -1$, and $a_2^2 = a_3^2 \le -1$, hence $a_3^2 + b_3^2 \le -2$, which is impossible.

And, finally:
Solution for 1b.
For even case. Let m be the maximum of absolute values of the entries of A.
In ε-neighborhood of **1**, square root is defined (see problem 2).
Take a huge number M such that $\frac{1}{\mathrm{M}^2} < \frac{\varepsilon}{m}$.
Write $A = D + (-\mathrm{M}^2\mathbf{1})$, where **1** is the unit matrix.
Then $D = \mathrm{M}^2(\mathbf{1} + A / \mathrm{M}^2)$. But $\mathbf{1} + A / \mathrm{M}^2$ is in ε-neighborhood of **1**, so it has square root, hence D has square root.
The second summand, $-\mathrm{M}^2\mathbf{1}$ is a square also, because in even dimensions $-\mathbf{1}$ is a square (for $n = 2$ it is rotation by 90°, for greater dimension it can be constructed from blocks). So, we have constructed decomposition into sum of two squares.

Not every matrix is a square; indeed, every negative-determinant matrix is not a square. So 2 is the minimal number.

For odd case. We shall prove that the sum of two squares can't give $-\mathbf{1}$, therefore in some cases at least 3 squares are needed, and construct decomposition into 3 squares for every matrix.

The construction of decomposition is similar to the even case. Let V be the diagonal matrix such that right bottom corner is 1 and all other elements are –9.
Let U be the diagonal matrix s. t. top left corner is 1 and all other elements are –9.
These matrixes obviously have square root (take square root of –1 in dimension less by 1, multiply it by 3, and add 1-block).

Every matrix A can be written as A = MV + MU + W, for any huge positive number M. But if M is sufficiently huge, W/M is very close to a specific diagonal matrix with positive numbers at the ends (8 in both corners and 18 elsewhere) hence it will have square root by problem 2.

It remains to prove that sum of two squares can't be $-\mathbf{1}$. Assume $A^2 + B^2 = -\mathbf{1}$, where A, B are real matrixes. Choose a basis over complex numbers, in which A would be upper triangular. In that basis, A^2 and $B^2 = A^2 - 1$ are upper triangular. On the diagonals we have squares of eigenvalues of A and B. That yields a system of equations: $a_k^2 + b_k^2 = -1$.

At least one eigenvalue of A and at least one eigenvalue of B is real (polynomial of odd degree has a real root), others are either real or complex conjugate.

Assume a_k is real, $a_k^2 \geq 0$. So $b_k^2 \leq -1$, therefore b_k is imaginary. Hence it has a conjugate, b_j. Hence $a_j^2 \geq 0$ and a_j is real.

Therefore, if we fix pairing between conjugate imaginary roots of characteristic polynomial of B, we get a pairing between real roots of A. But it is impossible, because polynomial of odd degree has odd number of real roots (considered with multiplicity). Contradiction, QED.

Targil 7.

Polynomials and Vieta.

1. Prove that for every polynomial $p(x)$, the polynomial $p(x + p(x))$ is divisible by $p(x)$.

2. Prove that a polynomial $p(p(p(x))) - x$ is divisible by $p(x) - x$.

3. Six real numbers satisfy:

$$x_1 < x_2 < x_3$$
$$y_1 < y_2 < y_3$$
$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$
$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$$
$$x_1 > y_1$$

Prove that $x_3 > y_3$.

4. Three rational nonzero numbers a, b, c are such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$ are integers. Prove that $|a|=|b|=|c|$.

5. The complex roots of polynomial $p(x)$ of degree n are $\alpha_1, \alpha_2, \ldots, \alpha_n$, the complex roots of its derivative are $\beta_1, \beta_2, \ldots, \beta_{n-1}$.
(***a***) Prove that the mass center of $\alpha_1, \alpha_2, \ldots, \alpha_n$ coincides with the mass center of $\beta_1, \beta_2, \ldots, \beta_{n-1}$.
(***b***) Prove that $\beta_1, \beta_2, \ldots, \beta_{n-1}$ are in the convex hull of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Targil 7.

Polynomials and Vieta.

Generic idea. There are two ways to look at the polynomial: either look at the coefficients or look at the roots. People often look from the coefficient viewpoint, and think that the problem is difficult, but sometimes the trick is to look from the root viewpoint.

1. Prove that for every polynomial $p(x)$, the polynomial $p(x + p(x))$ is divisible by $p(x)$.

First solution. The idea is that the roots of $p(x)$ are also the roots of $p(x + p(x))$, hence the second polynomial is divisible by the first. Of course, to do it carefully, we must take multiplicity into account.

If x is a root of multiplicity n of p, then $p(x+h)=O(h^n)$.

Then $p((x+h)+p(x+h))=p(x+h+O(h^n))=p(x+O(h))=O(h^n)$.

Therefore, x is also a root of the second polynomial of degree n at least.

Second solution. Substitute $z=x+p(x)$ into $p(z)=\sum_{k=0}^{n} a_k z^k$.

After opening brackets, some terms will contain $p(x)$ and hence will be divisible by $p(x)$. Other terms will contain only powers of x with coefficients, but sum of those other terms will be precisely $p(x)$.

2. Prove that a polynomial $p(p(p(x))) - x$ is divisible by $p(x) - x$.

Solution. Same idea as in the first solution of the first problem. If $p(x)=x$ then $p(p(p(x)))=p(p(x))=p(x)=x$.

So the roots of $p(x) - x$ are also roots of $p(p(p(x))) - x$.

But to make it precise we need to count multiplicities.

Which can be done the same way: root x of degree n means $p(x+h)=x+O(h^n)$.

Then $p(p(p(x+h)))=p(p(x+O(h^n)))=p(x+O(h^{n^2}))=x+O(h^{n^3})$ so x is a root of multiplicity at least n^3 of polynomial $p(p(p(x)))-x$ which. So the polynomial has the same roots with the same or bigger multiplicities.

3. Six real numbers satisfy:

$$x_1 < x_2 < x_3$$
$$y_1 < y_2 < y_3$$
$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$
$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$$
$$x_1 > y_1$$

Prove that $x_3 > y_3$.

Consider polynomials:

$p(x)=(x-x_1)(x-x_2)(x-x_3)$ and $q(x)=(x-y_1)(x-y_2)(x-y_3)$.

According to the conditions (and Viete) all coefficients except the free coefficient coincide. Therefore $p(x)=q(x)+C$.

The derivative of $p(x)$ has precisely two roots $A < B$ (of degree 2, Rolle theorem). Hence $p(x)$ is monotone increasing on $(-\infty, A]$, monotone decreasing on $[A,B]$ monotone increasing on $[B,\infty)$. Hence, if $C > 0$, we have $x_1 > y_1$, $x_2 < y_2$, $x_3 > y_3$, and if $C < 0$ we have $x_1 < y_1$, $x_2 > y_2$, $x_3 < y_3$. The last condition that was given shows it is the first case.

4. Three rational nonzero numbers a, b, c are such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ and $\frac{a}{c}+\frac{c}{b}+\frac{b}{a}$ are integers. Prove that $|a|=|b|=|c|$.

Solution. Consider the polynomial

$$\left(x-\frac{a}{b}\right)\left(x-\frac{b}{c}\right)\left(x-\frac{c}{a}\right)=x^3-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)x^2+\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)-1.$$

It is a polynomial with integer coefficients and rational roots. There is a generic way to find all rational roots of a polynomial with integer coefficients:

Theorem. If a polynomial $a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ has integer coefficients and a rational root $\frac{p}{q}$, where p and q are coprime integers, then a_0 is divisible by p, and a_n is divisible by q.

Since there are finite number of divisors for any integer number, the search for rational roots is now reduced to only a finite number of verifications.
Corollary. When the leading coefficient of the polynomial is 1 (and all other are still integer), then any rational root has to be integer.

In our specific case, when both leading coefficient and free coefficient are ones, the only possible rational roots are ± 1. And roots $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ are rational, which completes the solution. The theorem is well-known and might be used without proof, but we'll still prove it.

Proof of the theorem. Substitute the root and multiply the expression by q_n:

$$a_n p^n + a_{n-1}p^{n-1}q + ... + a_2 p^2 q^{n-2} + a_1 p q^{n-1} + a_0 q^n = 0$$

All terms in the sum except $a_0 q^n$ are divisible by p. Therefore $a_0 q^n$ is also divisible by p. But p and q are co-prime, hence a_0 is divisible by p.
On the other hand all the terms in the sum except $a_n p^n$ are divisible by q. So $a_n p^n$ is also divisible by q. But p and q are co-prime, hence a_n is divisible by q.

5. The complex roots of polynomial $p(x)$ of degree n are $\alpha_1, \alpha_2, \dots, \alpha_n$, the complex roots of its derivative are $\beta_1, \beta_2, \dots, \beta_{n-1}$.
(***a***) Prove that the mass center of $\alpha_1, \alpha_2, \dots, \alpha_n$ coincides with the mass center of $\beta_1, \beta_2, \dots, \beta_{n-1}$.
(***b***) Prove that $\beta_1, \beta_2, \dots, \beta_{n-1}$ are in the convex hull of $\alpha_1, \alpha_2, \dots, \alpha_n$.

Solution. (***a***) Assume that the polynomial is $c_n x^n + c_{n-1}x^{n-1} + ...$. By Vieta's theorem, the sum of the roots is $-\frac{c_{n-1}}{c_n}$ and their mass center of the roots is $-\frac{c_{n-1}}{nc_n}$.

The derivative is $nc_n x^{n-1} + (n-1)c_{n-1}x^{n-2} + ...$, sum of its roots is $-\frac{(n-1)c_{n-1}}{nc_n}$ and their mass center is $-\frac{c_{n-1}}{nc_n}$ which is the same as before.

(***b***) Let us start with some generic remarks about derivatives of complex function. A function $\mathbb{C} \to \mathbb{C}$ can be considered as a function $\mathbb{R}^2 \to \mathbb{R}^2$. One can consider a 2×2 matrix of partial derivatives (Jacobian). However, for a polynomial (or any other complex analytic function) the form of this matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a+bi$ is the value of complex derivative at the same point.
This gives two non-trivial relations between the partial derivatives of real and imaginary part of any complex analytic function (called Cauchy-Riemann equations), and to some geometric conclusion: complex analytic function is corresponds to a conformal mapping (i. e. preserves angles) when derivative is non-zero, and to a mapping with zero partial derivatives in all direction when complex derivative is zero. We shall use only the last part.

Another thing which is useful for this problem is the following simple fact: a compact convex set can be separated by a straight line from any point Z outside it. Indeed, take a point X in that set which is closest to Z. The perpendicular bisector to XZ is such a line (otherwise X wouldn't be the closest to Z). It is easy to replace the word "compact" by the word "closed" in that statement, and it is also possible to generalize it to infinite dimensions (then it will be called Hahn-Banach theorem), but in our case we have a polygon anyway.

So, let z be a root of p', which is outside the convex hull of the roots of polynomial p. The partial derivatives of p at z in all directions are zeroes.
The partial derivatives of $|p|$ in all directions are also zeroes at point z.
The polynomial can be written as a product over roots: $p(x) = a\prod_{k=1}^{n}(x - x_k)$.
Therefore $|p(x)| = |a| \cdot \prod_{k=1}^{n}|x - x_k|$. Partial derivative in some direction of Euclidean distance $|x - x_k|$ is positive, if scalar product of that vector with a vector from x_k to x is positive. If we consider point z, there's a line separating z from all x_k. A normal vector v to that line, pointing to the z half-plane, will have positive scalar products with all vectors pointing from x_k to z. Hence the partial derivative in direction v at point z of $|p(z)| = |a| \cdot \prod_{k=1}^{n}|z - x_k|$ is strictly positive. This contradicts our former conclusion (that all partial derivatives there are zeros).

Targil 8.

This targil is about 3d geometry, but mostly about cubes, as you could have guessed by its number.

1. What is the radius of the largest planar disc inside the unit cube?

2. A box $\{0 \leq x \leq 1 \ , 0 \leq y \leq 1 \ , 0 \leq z \leq A \ \}$, where A is a positive real, is intersected by a family planes $\{x + y + z = n + \alpha\}$, where α is a real number, and n are all possible integer numbers. The intersections of this family of planes and the box gives a family of planar polygons.
a. Prove that sum of areas of these polygons does not depend on α.
b. Prove that mass center of all these polygons is the center of the box.

3. What is the greatest possible area of the orthogonal projection of the unit cube? (among all possible directions)

4. a. What is the greatest triangular planar section of a tetrahedron (not necessarily regular)?
b*. What is the greatest possible planar section of a tetrahedron?

5. a**. What is the greatest possible area of a planar section of a unit cube?
b**. Same question for the box $a \times b \times c$.

Targil 8.

1. What is the radius of the largest planar disc inside the unit cube?

Answer. $\frac{\sqrt{3}}{2}$.

Solution. Unless the plane is parallel to one of the faces, the intersection might be defined as follows. Each pair planes of parallel faces define a strip on that plane – a part of plane between two parallel lines. The planar section is the intersection of 3 such strips. Each strip has its width, and the diameter of the disc is bounded by that width. If normal vector to the plane is a unit vector (a, b, c), then the width of parallel strips are $\frac{1}{|a|},\frac{1}{|b|},\frac{1}{|c|}$ (easy exercise to the reader). The largest among $|a|,|b|,|c|$ must be at least $\frac{1}{\sqrt{3}}$, since sum of their squares is 1. Therefore the thinnest of the three strips is of width $\sqrt{3}$ at most, and the radius is bounded by $\frac{\sqrt{3}}{2}$ anyway.

The equality can be achieved: when the plane is the perpendicular bisector of cube's diagonal, the section is a regular hexagon, and the disc is tangent to all its sides, hence its diameter equals the width of all three strips. But in this case $|a|=|b|=|c|=\frac{1}{\sqrt{3}}$, therefore the diameter is $\sqrt{3}$.

2. A box $\{0 \le x \le 1\ ,\ 0 \le y \le 1\ ,\ 0 \le z \le A\ \}$, where A is a positive real, is intersected by a family planes $\{x + y + z = n + \alpha\}$, where α is a real number, and n are all possible integer numbers. The intersections of this family of planes and the box gives a family of planar polygons.

a. Prove that sum of areas of these polygons does not depend on α.

b. Prove that mass center of all these polygons is the center of the box.

Solution. By shifting the polygons in x direction by integer numbers let us move all of them to the plane $x + y + z = \alpha$.

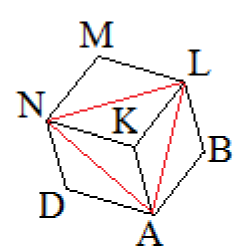
We shall get the intersection of plane $x + y + z = \alpha$ and infinite box $\{0 \le y \le 1\ , 0 \le z \le A\ \}$. The intersection is a parallelogram.
Its area obviously doesn't depend on a, and its mass center is the same as its center of symmetry. Therefore, its mass center has $y = ½$, $z = A/2$.
The same can be said about the original system of polygons, because we moved everything in x direction, so y and z of the mass center are unchanged.
But the original system is symmetric w. r. t. the plane $x = y$, so the center of mass also has $x = ½$. So it is in the center of the box, regardless of α.

3. What is the greatest possible area of the orthogonal projection of the unit cube? (among all possible directions)

Solution. We shall prove a nice lemma.
Lemma. Area of the orthogonal projection of the unit cube to a plane equals length of orthogonal projection of that cube to the line, which is orthogonal to the plane.

From this lemma it directly follows that the greatest projection is to the plane, which is orthogonal to cubes diagonal, and its area is equal to the length of the diagonal, which is $\sqrt{3}$.

Proof of lemma. Projection of each face is a parallelogram. These parallelograms are congruent. One of projected parllelograms is ABCD (we may assume it is non-degenerate), another is KLMN (shifting by vector AK moves A to K, B to L, C to M, D to N).



We may assume WLOG that when vector AK is expressed as a linear combination of vectors AB and AD, the coefficients are nonnegative. In this case, the projection is a hexagon ABLMND (this hexagon is convex though in some cases, it degenerates into rectangle).
The intervals AL, LN, NA cut parallelograms BAKL, KLMN, DAKN into pairs of equal triangles; hence the area of triangle LAN is half the area of the entire hexagon.
If we have a planar polygon P in space, and an interval I which is orthogonal to the plane of that polygon, then the area of projection from P to the plane is orthogonal

to the length of projection of I to the line orthogonal to that plane, because both the length and the area are multiplied by cosine of the same angle.
Hence in our case, we see that the area of projection of the cube is twice the area of projection of a triangle, which is proportional to the length of projection of the diagonal of the cube to the orthogonal line. This diagonal is formed by two opposite vertexes, both of which are projected inside the hexagon, so one of them is the upmost, and another downmost with respect to the plane of projection. Therefore, projection of diagonal to the orthogonal line coincides with the projection of the cube to the same line.
So, the length of the projection of the cube to the line is proportional to the area of its projection to the orthogonal plane. It only remains to compute the proportionality coefficient. This is easy – just consider a plane which is parallel to a face of the cube.

4. a. What is the greatest triangular planar section of a tetrahedron (not necessarily regular)?
b*. What is the greatest possible planar section of a tetrahedron?

Answer. The greatest face.
Solution. a. Take some triangular section: its vertexes are 3 points A, B, C on three different edges. While keeping A and B stable, move C along its edge.
The basis AB of the triangle ABC is stable; so the only thing that influences the area is the distance from line AB to C. Since distance from a line to a point is a convex function of the point, the maximum will be achieved in one of the endpoints, which is a vertex of the tetrahedron.
Therefore, in the greatest triangular section C will be a vertex; similarly, A and B will be vertexes. So it will be a face. QED.
b. In the previous section, we saw that the greatest triangular section is a face. However, there are also quadrilateral sections. So, now we shall consider quadrilateral section *KLMN* of a tetrahedron *ABCD*. The cutting plane splits between two non-adjacent edges: WLOG, those are *AB* and *CD*, so we shall assume that *K* is on *AC*, *L* is on *CB*, *M* is on *BD*, *N* is on *DA*.
Lemma. (three-dimensional version of Menelaus theorem)
$$\frac{AK}{KC}\cdot\frac{CL}{LB}\cdot\frac{BM}{MD}\cdot\frac{DN}{NA}=1.$$

Proof of lemma. Let l be a line, orthogonal to the plane $KLMN$.
We shall take the orthogonal projection of the whole picture to the line l.
The points A, B, C, D will be projected to A',B',C',D', the point K, L, M, N will all be projected to the same point O.

$$\frac{AK}{KC}=\frac{A'O}{OC'}\ ,\ \frac{CL}{LB}=\frac{C'O}{OB'}\ ,\ \frac{BM}{MD}=\frac{B'O}{OD'}\ ,\ \frac{DN}{NA}=\frac{D'O}{OA'}$$

Multiplication of these 4 fractions gives the result.

Now, we shall prove that the quadrilateral section is smaller than one of the faces.
Project orthogonally the tetrahedron to the plane $KLMN$: the vertexes A, B, C, D of will go to the points A_1, B_1, C_1, D_1 on the plane. We shall prove that the area of $KLMN$ is not greater than projection of one of the faces, so before the projection it was greater that face was greater.
Assume that we know the ratios in which K, L, M N divide the sides:

$$\frac{A_1K}{KC_1}=\frac{AK}{KC}=\frac{\alpha}{1-\alpha}\ ,\ \frac{CL}{LB}=\frac{C_1L}{LB_1}=\frac{\beta}{1-\beta}\ ,\ \frac{BM}{MD}=\frac{B_1M}{MD_1}=\frac{\gamma}{1-\gamma}\ ,\ \frac{D_1N}{NA_1}=\frac{DN}{NA}=\frac{\delta}{1-\delta}$$

Denote $S=S_{B_1C_1A_1D_1}$, $S_A=S_{C_1A_1D_1}$, $S_C=S_{B_1C_1A_1}$, $S_B=S_{C_1B_1D_1}$, $S_D=S_{B_1D_1A_1}$.
The maximum between S_A, S_B, S_C, S_D will be denoted $S_{\min}$, the maximum $S_{\max}$.
Of course, $S=S_A+S_B=S_C+S_D=S_{\min}+S_{\max}$.
We want to prove $S_{KLMN}\le S_{\max}$. To compute S_{KLMN}, we shall compute the rest of the area inside S:

$$S_{NKA_1}=\alpha(1-\delta)S_A\ ,\ S_{KLC_1}=\beta(1-\alpha)S_C\ ,\ S_{LMB_1}=\gamma(1-\beta)S_B\ ,\ S_{MND_1}=\delta(1-\alpha)S_D$$

Hence

$$S-S_{KLMN}=\alpha(1-\delta)S_A+\beta(1-\alpha)S_C+\gamma(1-\beta)S_B+\delta(1-\alpha)S_D\ge$$
$$\ge S_{\min}\left(\alpha(1-\delta)+\beta(1-\alpha)+\gamma(1-\beta)+\delta(1-\alpha)\right)$$

By the lemma, we have the condition: $\alpha\beta\gamma\delta=(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$.
Therefore,

$$0=1-\alpha-\beta-\gamma-\delta+\alpha\beta+\beta\gamma+\alpha\gamma+\alpha\delta+\beta\delta+\gamma\delta-\alpha\beta\gamma-\alpha\gamma\delta-\alpha\beta\delta-\beta\gamma\delta\ .$$
$$\alpha(1-\delta)+\beta(1-\alpha)+\gamma(1-\beta)+\delta(1-\alpha)=1+\alpha\gamma+\beta\delta-\alpha\beta\gamma-\alpha\gamma\delta-\alpha\beta\delta-\beta\gamma\delta$$

We shall prove that the last expression is at least 1. From this we get directly:

$$S-S_{KLMN}\ge S_{\min}\left(\alpha(1-\delta)+\beta(1-\alpha)+\gamma(1-\beta)+\delta(1-\alpha)\right)\ge S_{\min}$$

$S_{\max}\ge S_{KLMN}$, QED.

So it remains to prove: $1+\alpha\gamma+\beta\delta-\alpha\beta\gamma-\alpha\gamma\delta-\alpha\beta\delta-\beta\gamma\delta\ge 1$.
In other words, $\alpha\gamma+\beta\delta\ge\alpha\beta\gamma+\alpha\gamma\delta+\alpha\beta\delta+\beta\gamma\delta$.

Recall that $0<\alpha,\beta,\gamma,\delta<1$.

So the required result is: $\frac{1}{\beta\delta}+\frac{1}{\alpha\gamma}\geq\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$.

In other words, $\left(\frac{1}{\alpha}-1\right)\left(\frac{1}{\gamma}-1\right)+\left(\frac{1}{\beta}-1\right)\left(\frac{1}{\delta}-1\right)\geq 2$

Denote $U=\left(\frac{1}{\alpha}-1\right)\left(\frac{1}{\gamma}-1\right)$, $V=\left(\frac{1}{\beta}-1\right)\left(\frac{1}{\delta}-1\right)$

We already know $\alpha\beta\gamma\delta=(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$, if we divide both sides of this condition by $\alpha\beta\gamma\delta$ we get: $UV = 1$.

Hence by AMGM (Cauchy inequality) we get $\frac{U+V}{2}\geq\sqrt{UV}=1$.

That means $U+V\geq 2$, QED.

5. a**. What is the greatest possible area of a planar section of a unit cube?
b**. Same question for the box $a\times b\times c$.

Answer. a. $\sqrt{2}$

b. If $a < b,c$ it is $\sqrt{b^2+c^2}$

Anyway, the greatest section is "diagonal" rectangle.

Solution. Let us start with easy steps:

Lemma 1. We may assume that the greatest section passes through the center.

Lemma 2. The section through center is a central-symmetric polygon: quadrilateral or hexagon.

Lemma 3. The greatest quadrilateral section is the diagonal.

Proof of lemma 1. Let P be a section which doesn't go through the center. Let Q be another section, symmetric to P w.r.t. the center of the box (Q is symmetric, and hence congruent to P). Consider a family of straight parallel lines in the plane of P, that intersect P. The extreme two lines in this family will be denoted l_1, l_2.

Line k_2, k_1 are symmetric to l_1, l_2 respectively with respect to the center of the box. Consider the family F of parallel planes, among which are the plane through l_1 and k_2, the plane through l_2 and k_1, and all the planes between them.

Let M be the planar section of the same box, is parallel to both P and Q which and passes through the center of the box.
Consider intersection of any plane from *F* with P, Q, and M. All three intersections are intervals; intervals of intersection with P and Q form a trapezoid, which is entirely contained in the box (since the box is convex) so the mid-segment of the trapezoid is entirely contained in M. So, length intersection of the plane with M is greater or equal to the average of intersections with P and Q. Since area of each section can be computed as integral of intersection lengths with all planes, parallel to a certain direction, we see that are of section M is greater or equal than the area average area of P and Q which is equal to the area of P. So, if instead of P we consider parallel section through the center, it will have greater or equal area.

Proof of lemma 2. Each face of a box defines a half-space, and the box itself is intersection of these half-spaces. In every plane, this half-spaces define half-planes (though sometimes they give an empty set or the entire plane). So, we have a polygon with no more than 6 sides (because we start with no more than 6 half-spaces). The plane and the box are symmetric w. r. t. the center of the box, hence the polygon is also symmetric w. r. t. the same center. Hence number of sides is even (there are pairs of opposite sides). Therefore number of sides is 4 or 6 (2 is impossible).

Proof of lemma 3. If we have quadrilateral, it doesn't intersect a pair of opposite faces. WLOG these are horizontal faces (if not, rotate the box). Therefore it can be projected orthogonally onto each of those two faces. Hence the area of the section is the area of the face divided by the cosine of the slope. We want that cosine to be minimal.
Unless the cutting plane is horizontal (but then the cosine is maximal), let us walk along the steepest descent direction along the cutting plane, until we reach the plane of the horizontal face. We shall cover y = half the height of the box vertically, and distance x horizontally. The tangent of the slope angle will be y/x: we want it to be maximal, then the slope would be maximal, the cosine minimal, and the area maximal. So, we want x to be minimal. It is the distance in horizontal projection to a point which is in the plane of the horizontal face but not inside the face itself. The closest point in that set is obviously the center of the longest edge. But in this case we have the "diagonal" section, QED.

Now it remains to study the hexagonal case, and to prove it cannot be of greater area than the "diagonal" rectangle.

Lemma 4. Consider a polytope. For each face, consider an outside normal vector of this face which is of length equal to the area of that face. Then sum of those vectors is a zero vector.

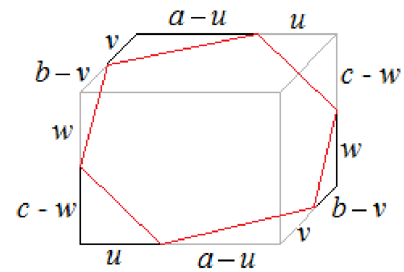
Lemma 5. Let consider the closed broken line $A_1A_2\ldots A_{2n}$, each interval of which intersects certain plane: $A_{2n}A_1$ intersects that plane at B_1, A_1A_2 at B_2, A_2A_3 at B_3 etc. Then $\frac{A_{2n}B_1}{B_1A_1}\cdot\frac{A_1B_2}{B_2A_2}\cdot\frac{A_2B_3}{B_3A_3}\cdot\ldots\cdot\frac{A_{2n-1}B_{2n}}{B_{2n}A_{2n}}=1$.

Lemma 5 is very similar to the lemma from the solution of problem 4, so we shall leave the proof as an exercise to the reader.

Lemma 4 has and extremely beautiful physical proof, though some of you might not accept it as a proof, hence we shall give two proofs.

First proof of lemma 4. Consider a large part of space of air (or water) standing still with no wind (no current). Consider inside a smaller part of space in the form of that polytope. It doesn't move. Hence the sum of forces applied to it is zero. Forces applied to it are forces of air pressure (לחץ אוויר) – they are proportional to the areas of the faces and directed into the faces. Reverse the signs, QED.

Second proof of lemma 4 (which is just a translation of the first proof to a rigorous language). We shall prove that z coordinate of the sum of normal vectors is zero, it is the same for x and y coordinates. The z coordinate of normal vector equals to the oriented area of projection of that face to xy plane: it is considered positive, it the original face was facing up, negative if down. So, the z coordinate is sum of areas of xy-projections facing up minus sum of xy-projections facing down. Since above each point there's the same number of both kinds, they cancel out.



So, back to our problem. We cut the box through the center in two halves by a hexagon. This hexagon cuts 6 edges into 12 sub-intervals of lengths:
u, *a* – *u*, *v*, *b* – *v*, *w*, *c* – *w*, *u*, *a* – *u*, *v*, *b* – *v*, *w*, *c* – *w*
(recall that lengths of edges are a, b, c, and last six repeat first six because of the symmetry).

By lemma 5 we get $\frac{u}{a-u}\cdot\frac{v}{b-v}\cdot\frac{w}{c-w}\cdot\frac{u}{a-u}\cdot\frac{v}{b-v}\cdot\frac{w}{c-w}=1$.

Or simply $\frac{u}{a-u}\cdot\frac{v}{b-v}\cdot\frac{w}{c-w}=1$.

We shall find the expression for the area using lemma 4. Consider one of the halves of the box as a polytope with 7 faces. Sum of normal vectors is zero.
Take the two vertical vectors. One corresponds to $a\times b$ rectangle with right-angled triangle (with legs $a-u$ and v) cut out, another is a triange with opposite orientation with legs $a-u$ and v.
The contribution of these two sides together is the same as we would get from $a\times b$ rectangle if rectangle $(a-u)\times v$ would be cut out. So, total of these two vectors is of length and looking down $ab-(a-u)v$.
Similar things can be said about other pairs of parallel faces. So the total of six vectors (all of them except the hexagon) is $\begin{pmatrix} \pm\left(ab-(a-u)v\right) \\ \pm\left(bc-(b-v)w\right) \\ \pm\left(ca-(c-w)u\right) \end{pmatrix}$.

The total of all seven vectors is zero, so the vector that corresponds to the hexagon can be described by the same expression with opposite sign in each coordinate.

For the case of cube, finding the hexagonal section of greatest are is reformulated as follows: given the condition $\frac{u}{1-u}\cdot\frac{v}{1-v}\cdot\frac{w}{1-w}=1$, find the maximum of
$\left(1-(1-u)v\right)^2+\left(1-(1-v)w\right)^2+\left(1-(1-w)u\right)^2$.
For the case of generic box, if we denote $C=ab, A=bc, B=ca$ (areas of the faces) and change *u*, *v*, *w* by *au*, *bv*, *cw*, the condition will be the same as for the cube, but

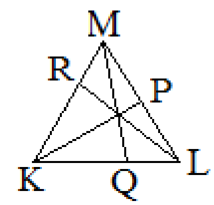
the function to minimize will be

$$C^2\left(1-(1-u)v\right)^2+A^2\left(1-(1-v)w\right)^2+B^2\left(1-(1-w)u\right)^2.$$

This inequality is not easy, because we have to take the condition into account. The condition reminds of

Ceva theorem. Consider triangle KLM, point P on the side LM, point Q on KL, point R on LK. Assume KP, LQ, MR meet in one point. Then $\frac{\mathrm{KR}}{\mathrm{RM}}\cdot\frac{\mathrm{MP}}{\mathrm{PL}}\cdot\frac{\mathrm{LQ}}{\mathrm{QK}}=1.$

We shall give the proof (though it is a standard fact from elementary geometry, but that is one of those things that are too elementary for the university and too hard for the high-school, so many people might not have heard of it).



Proof of Ceva theorem. Put mass k in point K and mass l in point L so that the mass center of these two points will be at R. Put mass m in point M such that the mass center of K and M will be at R. Then the mass center G of all three points would be both on KP and on MR. Also, G will also be on MQ', where Q' is the mass center of K and M. It is easy to see that $\frac{\mathrm{KR}}{\mathrm{RM}}\cdot\frac{\mathrm{MP}}{\mathrm{PL}}\cdot\frac{\mathrm{LQ'}}{\mathrm{Q'K}}=\frac{m}{k}\cdot\frac{k}{l}\cdot\frac{l}{m}=1.$ But since G is intersection of KP and MR, also LQ goes through it, hence Q and Q' coincide. QED.

So, we had the condition $\frac{u}{1-u}\cdot\frac{v}{1-v}\cdot\frac{w}{1-w}=1$. We can say that what we actually have are Ceva picture with equilateral triangle KLM, all sides of which are 1, and (in the notations of Ceva theorem as we formulated it):
KR = w, RM = 1 – w, MP = u, PL = 1 – u, LQ = v, QK = 1 – v.
Our problem was that this parametrization was hard to handle: it comes with an ugly condition. Let us consider reformulating it in terms of masses (like in the proof of Ceva theorem) the intersection point is a mass center of mass x at point K, mass y at point L, and mass z at point R. Then, since points P, Q, R are centers of mass of couples of points we get

$w=\frac{y}{x+y}$, $1-w=\frac{x}{x+y}$, $u=\frac{z}{y+z}$, $1-u=\frac{y}{y+z}$, $v=\frac{x}{x+z}$, $1-V=\frac{z}{y+z}$

Where x, y, z, are any real numbers. Inequalities about arbitrary real numbers are usually easier than inequalities about numbers satisfying some fancy condition.

$$1-(1-w)u=1-\frac{x}{x+y}\cdot\frac{z}{y+z}=\frac{(x+y)(y+z)-xz}{(x+y)(y+z)}=\frac{y(x+y+z)}{(x+y)(y+z)}$$

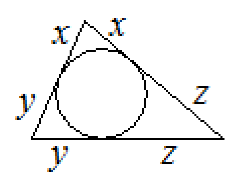
Similar for the other two expressions, hence:

$$C^2\left(1-(1-u)v\right)^2+A^2\left(1-(1-v)w\right)^2+B^2\left(1-(1-w)u\right)^2=$$

$$=\left(A\frac{x(x+y+z)}{(x+y)(x+z)}\right)^2+\left(B\frac{y(x+y+z)}{(x+y)(y+z)}\right)^2+\left(C\frac{z(x+y+z)}{(x+y)(y+z)}\right)^2$$

And we have to find, for which positive x, y, z will that achieve its maximal value.

Consider yet another geometric picture. Consider triangle ABC, its sides AB = $x + y$, BC = $y + z$, CA = $z + x$ (yes, this creature really exists).



In this context, x, y, z have a geometric meaning: the tangency points of the incircle and triangles sides divide the sides of triangle into the intervals of lengths x, y, z.

Denote angles of the triangle BAC = 2α, ABV = 2β, ACB = 2γ.

Easy exercise. $\frac{x(x+y+z)}{(x+y)(x+z)}=(\cos\alpha)^2$.

Hints. (1). Denote $p = x + y + z$. (2) First prove that $S = pr$, where r is the radius of incircle.

So, finding for non-negative x, y, z of the maximum of the function,

$$\left(A\frac{x(x+y+z)}{(x+y)(x+z)}\right)^2+\left(B\frac{y(x+y+z)}{(x+y)(y+z)}\right)^2+\left(C\frac{z(x+y+z)}{(x+y)(y+z)}\right)^2$$

Is the same as for nonnegative α,β,γ such that $\alpha+\beta+\gamma=1$ finding the maximum of the function $A^2\cos^4\alpha+B^2\cos^4\beta+C^2\cos^4\gamma$.

True, we now have the condition again, but not that ugly and also the function is much nicer.

The last lemma. The maximum of that function will be in one of the vertexes of the domain (which is triangle). That is α, β, γ = 0, 0, 1 in some order.

This case corresponds in the original problem to degenerate case, when the hexagon becomes the rectangle. In this case the value of the function (which was the squared are of the hexagon) is something like $A^2 + B^2$, which is squared area of the diagonal rectangle.

So, it remains to prove the last lemma.

Proof of the last lemma. (Alexey Gladkich) Our domain is triangle: $\alpha, \beta, \gamma \geq 0$, $\alpha + \beta + \gamma = 1$. The maximum can be either in the vertex, or on the side, on in the interior.
Assume that the maximum is on the side $\gamma = 0$, $\alpha + \beta = 1$. The function is $A^2 \cos^4 \alpha + B^2 \cos^4 \beta + C^2 = A^2 t^2 + B^2 (1-t)^2 + C^2$, where $t = \cos^2 \alpha \in [0,1]$.
As a function of *t*, it is convex; so maximum is in one of the endpoints of the domain, hence at the vertex of the triangle.

It remains to exclude the possibility of the maximum inside the triangle.
Then we can differentiate:

$$0 = \left. \frac{df(\alpha+\varepsilon, \beta-\varepsilon, \gamma)}{d\varepsilon} \right|_{\varepsilon=0} = A^2 \cos^3 \alpha \sin \alpha - B^2 \cos^3 \beta \sin \beta$$

Hence, $A^2 \cos^3 \alpha \sin \alpha = B^2 \cos^3 \beta \sin \beta$. Doing the same for another couple of coordinates, we get $A^2 \cos^3 \alpha \sin \alpha = B^2 \cos^3 \beta \sin \beta = C^2 \cos^3 \gamma \sin \gamma$.

Therefore, we can apply some scaling to the coefficients (that won't shift the position of the maximum) and assume:

$$A^2 = \frac{1}{\cos^3\alpha\sin\alpha}$$
$$B^2 = \frac{1}{\cos^3\beta\sin\beta}$$
$$C^2 = \frac{1}{\cos^3\gamma\sin\gamma}$$

Hence at the maximal point $f(\alpha,\beta,\gamma) = \cot(\alpha) + \cot(\beta) + \cot(\gamma)$.

At vertexes we have things like $f(0,0,1) = \frac{1}{\cos^3\alpha\sin\alpha} + \frac{1}{\cos^3\beta\sin\beta}$.

We shall assume that $\alpha \le \beta \le \gamma$, and prove that

$$\cot(\alpha) + \cot(\beta) + \cot(\gamma) \le \frac{1}{\cos^3\alpha\sin\alpha} + \frac{1}{\cos^3\beta\sin\beta},$$

and that will complete the proof.

First, notice that $\frac{1}{\cos^3\phi\sin\phi} = \frac{\left(\cos^2\phi + \sin^2\phi\right)^2}{\cos^3\phi\sin\phi} \ge \cot\phi + 2\tan\phi$, at least for $0 \le \phi \le \pi/2$. Hence it is enough to prove that

$\cot(\alpha) + \cot(\beta) + \cot(\gamma) \le \cot(\alpha) + 2\tan(\alpha) + \cot(\beta) + 2\tan(\beta)$

$\cot(\gamma) \le 2\tan(\alpha) + 2\tan(\beta)$

Let's denote $\psi = \frac{\alpha+\beta}{2}$. Then $\cot(\gamma) = \tan(\alpha+\beta) = \tan(2\psi)$.

In the relevant domain tan is a convex function (first derivative is $1/\cos^2$ and it grows monotonically). Hence $\frac{\tan(\alpha) + \tan(\beta)}{2} \ge \tan\left(\frac{\alpha+\beta}{2}\right) = \tan(\psi)$.

We are supposed to prove $\cot(\gamma) \le 2\tan(\alpha) + 2\tan(\beta)$, but it is enough to prove that $\tan(2\psi) \le 4\tan(\psi)$, because $4\tan(\psi) \le 2\left(\tan(\alpha) + \tan(\beta)\right)$.

By assumption $\alpha \le \beta \le \gamma$, hence $\psi = \frac{\alpha+\beta}{2} \le \frac{\alpha+\beta+\gamma}{3} = \frac{\pi}{6}$. So $\tan(\psi) \le \frac{1}{\sqrt{3}}$.

Hence the claim $\tan(2\psi) = \frac{2\tan(\psi)}{1-\tan^2(\psi)} \le 4\tan(\psi)$ is obvious. QED.

Targil 9.

This targil is about eigenvalues.

1. Is it true for any real square matrix A that it is similar to A^T ?
(in other words, is it true that for any A there's invertible Q such that $A^T = QAQ^{-1}$).

2. A is real skew-symmetric matrix (meaning $A = -A^T$).
a. Prove that its eigenvalues are imaginary,
b. Can it have non-trivial Jordan cells (of size > 1)?

3. On a circle, n real numbers are written, not all of them are equal. Each second, all numbers change simultaneously: every number is replaced by the average of itself and its clockwise neighbor.
a. Prove that all numbers on the circle converge to the average of all numbers.
b. Prove it converges exponentially, in other words, the distance of a value at some point after k steps from its limit value is $O(\alpha^k)$ where is a real number in (0, 1).
c. Compute the value of that α (the minimal value for which that claim will hold for any starting configuration).

4. Prove: if the minimal polynomial of A is $(x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2}\cdot\ldots\cdot(x-\lambda_k)^{n_k}$, then the minimal polynomial of $\begin{pmatrix} A & I \\ 0 & A \end{pmatrix}$ is $(x-\lambda_1)^{n_1+1}(x-\lambda_2)^{n_2+1}\cdot\ldots\cdot(x-\lambda_k)^{n_k+1}$.

5. Denote $A^\# = (I + A)^{-1}(I - A)$.
a. Show that $A^{\#\#} = A$.
b. Show that A is an orthogonal matrix iff $A^\#$ is skew-symmetric.

Targil 9.

This targil is about eigenvalues.

1. Is it true for any real square matrix A that it is similar to A^T ?
(in other words, is it true that for any A there's invertible Q such that $A^T = QAQ^{-1}$).

Solution. Let us $J = PAP^{-1}$ be the Jordan form. The transposition of A is similar to J^T, by the means of $(P^T)^{-1}$. Every Jordan block is similar to its transpose, by the means of a permutation matrix (ones on the secondary diagonal, zeros elsewhere). Hence J is similar to J^T, and A is similar to J, and A^T to J^T.
Therefore A is similar to A^T.

2. A is real skew-symmetric matrix (meaning $A = -A^T$).
a. Prove that its eigenvalues are imaginary,
b. Can it have non-trivial Jordan cells (of size > 1)?

Solution. a. Assume $Av = \lambda v$. By * we shall denote the composition of transposition and complex conjugation. So if v is a column vector, v^* is a row of complex conjugates to its coordinates.
Then $\lambda|v|^2 = \lambda v^* v = v^*(Av) = (v^*A)v = -(v^*A^*)v = -(Av)^* v = -\overline{\lambda}v^*v = -\overline{\lambda}|v|^2$.
Hence, if v is nonzero vector, $\lambda = -\overline{\lambda}$. Hence λ is imaginary.

b. No. Matrix iA is Hermitian, hence it can be diagonalized by a unitary matrix. Matrix A can be diagonalized by the same unitary matrix.

3. On a circle, n real numbers are written, not all of them are equal. Each second, all numbers change simultaneously: every number is replaced by the average of itself and its clockwise neighbor.
a. Prove that all numbers on the circle converge to the average of all numbers.
b. Prove it converges exponentially, in other words, the distance of a value at some point after k steps from its limit value is $O(\alpha^k)$ where is a real number in (0, 1).
c. Compute the value of that α (the minimal value for which that claim will hold for any starting configuration).

Solution. Let C be the matrix of cyclic permutation. It has ones on the diagonal above the main diagonal, and one in bottom-left corner, and zeros elsewhere.

Its eigenvalues are roots of $x^n - 1$, and its eigenvectors are multiples of $\begin{pmatrix} 1 \\ \xi \\ \xi^2 \\ \dots \end{pmatrix}$, in other words a geometric progression, with ratio ξ which is a root of $x^n - 1$.

We can diagonalize the matrix by choosing the above vectors as a basis (in fact, switching to this eigenbasis is so often convenient, that it has a special name – discrete Fourier transform).

So, our operator could be written as $A = (\mathbf{1} + C)/2$, where $\mathbf{1}$ is the unit matrix.

In the eigenbasis of C, it becomes diagonal. On the diagonal we have the numbers $\frac{1+\xi}{2}$ where $\xi^n = 1$. Since ξ is on the unit circle, $\frac{1+\xi}{2}$ are on the circle whose diameter is the interval [0, ½].

Therefore, the eigenvalue of A which corresponds to the constant vector is 1, and all other eigenvalues are less than one. We shall denote vector (1, 1, …. 1) by v_1.

Easy exercise sum of coordinates of any other eigenvector is zeroes.

So, assume we represent a given vector as a linear combination of eigenbasis. The coefficient of the v_1 is the average of the coordinates of the original vector (that follows easily from the above easy exercise). When we multiply it by A many times, this coefficient of v_1 remains always the same, and all other coefficients are reduced in geometric progression.

The two geometric progression that reduce in the slowest rate are those that correspond to eigenvalues of highest norm, and these are closest to 1, those are $\alpha_{1,2} = \frac{1+e^{\pm\frac{2\pi i}{n}}}{2}$. The absolute values of these two are

$$\left|\frac{1+e^{\pm\frac{2\pi i}{n}}}{2}\right| = \left|\frac{e^{\mp\frac{\pi i}{n}}+e^{\pm\frac{\pi i}{n}}}{2}e^{\pm\frac{\pi i}{n}}\right| = \left|\cos\left(\frac{\pi}{n}\right)\right|\cdot\left|e^{\pm\frac{\pi i}{n}}\right| = \cos\left(\frac{\pi}{n}\right).$$

So, the deviation from the limit is $s\alpha_1^k + t\alpha_2^k$ + geometrical progressions which descend more rapidly. Therefore, the convergence is $O\left(\cos\left(\frac{\pi}{n}\right)^k\right)$. The

progressions rotate around the circle with a step of full circle/n, so in each n consequent steps there is a step when they don't cancel out.
Of course, for some initial states geometric sequence can converge more rapid: these correspond to the cases when the coefficients which correspond to these two eigenvectors are zero, for instance when the sequence has a sub-period.

4. Prove: if the minimal polynomial of A is $(x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2}\cdot\ldots\cdot(x-\lambda_k)^{n_k}$, then the minimal polynomial of $\begin{pmatrix} A & I \\ 0 & A \end{pmatrix}$ is $(x-\lambda_1)^{n_1+1}(x-\lambda_2)^{n_2+1}\cdot\ldots\cdot(x-\lambda_k)^{n_k+1}$.

Solution.

$$\begin{pmatrix} A & I \\ 0 & A \end{pmatrix}^n = \begin{pmatrix} A^n & nA^{n-1} \\ 0 & A^n \end{pmatrix}$$

This follows by induction: $n = 1$ is obvious, and

$$\begin{pmatrix} A & I \\ 0 & A \end{pmatrix}^{n+1} = \begin{pmatrix} A & I \\ 0 & A \end{pmatrix}\begin{pmatrix} A^n & nA^{n-1} \\ 0 & A^n \end{pmatrix} = \begin{pmatrix} A^{n+1} & (n+1)A^n \\ 0 & A^{n+1} \end{pmatrix}.$$

Therefore for any polynomial, $p\left(\begin{pmatrix} A & I \\ 0 & A \end{pmatrix}\right) = \begin{pmatrix} p(A) & p'(A) \\ 0 & p(A) \end{pmatrix}$. Hence p has to be such that both p and p' are divisible by $(x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2}\cdot\ldots\cdot(x-\lambda_k)^{n_k}$. Therefore λ_i is a root of degree $n_i + 1$ at least, and vice versa – if all each λ_i is a root of degree $n_i + 1$, we get zero. So $(x-\lambda_1)^{n_1+1}(x-\lambda_2)^{n_2+1}\cdot\ldots\cdot(x-\lambda_k)^{n_k+1}$ is the minimal polynomial.

5. Denote $A^{\#} = (I + A)^{-1}(I - A)$.
a. Show that $A^{\#\#} = A$.
b. Show that A is an orthogonal matrix iff $A^{\#}$ is skew-symmetric.

Solution. a. First, notice that $A^{\#}$ is defined only if –1 is not an eigenvalue of A. It doesn't create a problem:

$$A^{\#} = (I + A)^{-1}(I - A) = (I + A)^{-1}(2I - (I + A)) = 2(I + A)^{-1} - I$$

So, if $A^{\#}$ is defined, then $A^{\#}$ is invertible matrix minus unit matrix, hence it doesn't have –1 as an eigenvalue. So, if we can define $A^{\#}$ we can also define $A^{\#\#}$.

This condition can also be rewritten as $(I + A^{\#}) = 2(I + A)^{-1}$ which is the same as

$$(I + A^{\#})(I + A) = 2I$$

Which is certainly symmetric, hence $A^{\#\#}=A$.

b. We shall use the equation $(I + A^{\#})(I + A) = 2I$.

If A is orthogonal, which means $A^TA = AA^T = I$, let us transpose the previous equation:

$$(I + A^T)(I + A^{\#T}) = 2I$$

If product of two matrixes is I, then they commute (since they are mutually inverse). Hence if their product is 2I, they commute anyway. Therefore:

$$(I + A^{\#T})(I + A^T) = 2I$$

Multiply both sides by A from the right:

$$(I + A^{\#T})(A + I) = 2A$$

Sum it with the original equation, $(I + A^{\#})(I + A) = 2I$. We get:

$$(2I + A^{\#} + A^{\#T})(I + A) = 2(I + A)$$

We live under assumption that -1 is not an eigenvalue of A, that is, I + A is invertible. Hence $2I + A^{\#} + A^{\#T} = 2I$. That is, $A^{\#} + A^{\#T} = 0$, QED.

The other direction: assume $A + A^T = 0$.

$$(I + A)A^{\#} = I - A \quad (*)$$

Transpose :

$$A^{\#T}(I + A^T) = I - A^T$$

$$A^{\#T}(I - A) = I + A$$

We knew that I + A is invertible, but now we see that I – A is invertible (they are invertible anyway, since eigenvalues of A are imaginary as we saw in problem 2a above). Multiply the last equality by (*) from the left

$$(I + A)A^{\#}A^{\#T}(I - A) = (I - A)(I + A)$$

Obviously I + A and I – A commute, and we saw they are invertible, hence we can cancel them out, and we get $A^{\#}A^{\#T} = I$.

QED.

Remark. This transformation is called Cayley transform. For complex numbers, Cayley transform is a special Möbius transformation, which is $\frac{1+z}{1-z}$ which moves imaginary axis to the unit circle (and vice versa), thus transforming a circle to a halfplane. It is not surprising that the same transformation for matrices interchanges orthogonal matrixes (whose eigenvalues live on the unit circle) with anti-symmetric (whose eigenvalues live on the imaginary axis).

Targil 10.

This targil is about … well, I bet you'll guess.

1. Prove that for any $0 < p < 1$, and integers m, n > 1,

$$(1-p^n)^m + (1-(1-p)^m)^n > 1.$$

2. The floor is tiled by $l \times l$ squares (so it looks like a lattice). You throw a needle of length n on the floor. What is the chance that when it falls, it won't cut the lines, but will be entirely within one of the squares?

3. For any natural numbers k, m, n compute the integral

$$\int_0^1 \left(\int_0^{1-y} y^k x^m (1-x-y)^n \, dx \right) dy .$$

4.* **a.** Four aliens land on the surface of a spherical planet, randomly and independently (a chance for every alien to land inside a country is proportional to the area of this country). What is the probability that they all landed on the same hemisphere? In other words, what's the probability that there exists a plane through the planets center, such that all landing points are on the same side of that plane?
b. Ten aliens land on the planet (independently, proportional to surface area). What is the probability that they are on the same hemisphere?

5.** A unit cube is orthogonally projected on a random plane (the normal vector to the plane is distributed uniformly over the unit sphere).
What is the expectation of projection area?

Targil 10.

This targil is about probability.

1. Prove that for any $0 < p < 1$, and integers $m, n > 1$,

$$(1-p^n)^m + (1-(1-p)^m)^n > 1.$$

Solution. Consider a $m \times n$ table – it has n rows, m columns.

In every cell, we write 1 with probability p, and 0 with probability $1 - p$.
So, a probability that a given column row doesn't consists of ones is $1 - p^n$ (or, in other words, given column has zero). The probability that every column has zero is $(1-p^n)^m$. Similarly, the probability that every row has one is $(1-(1-p)^m)^n$.
It is not possible that neither of this two events take place: it'd mean we have both a column of zeroes, and a row of ones, and then in the intersection of these two would be a contradiction. This gives a non-strict inequality.
It is easy to find a table where both events take place simultaneously; hence the inequality is strict.

2. The floor is tiled by $l \times l$ squares (so it looks like a lattice). You throw a needle of length n on the floor. What is the chance that when it falls, it won't cut the lines, but will be entirely within one of the squares?

Solution. Let us start with one-dimensional case: a needle of length n is thrown on the real line, which is tiled by intervals of length l. The probability of hitting a joint is 1 if $n \geq l$ and is n / l if $n < l$.
Now we drop a needle on the two-dimensional floor. It is rotated by angle α, which is random (and uniformly distributed). Given α, we can split the needle into two projection: it has length $|n \cos \alpha|$ in x direction, and $|n \sin \alpha|$ in y direction.

The probability of hitting vertical lines is $p_h = \max\left(\left|\frac{n\cos\alpha}{l}\right|, 1\right)$, and of hitting horizontal lines is $p_v = \max\left(\left|\frac{n\sin\alpha}{l}\right|, 1\right)$.

There are 3 cases of different nature.

a. $n \geq l\sqrt{2}$, then the probability of hitting the grid is 1, and nothing to compute.
b. $l\sqrt{2} > n > l$, then at some angles probability is 1, at other angles probability is analytic expression.
c. $l \geq n$, then for each angle probability can be computed as a fraction (without maximum).

We shall start with case c.
The needle doesn't hit the grid if it doesn't hit neither vertical nor horizontal lines, which is $\left(1-\left|\frac{n}{l}\cos\alpha\right|\right)\left(1-\left|\frac{n}{l}\sin\alpha\right|\right)$. This probability must be averaged along the circle:

$$\frac{1}{2\pi}\int_0^{2\pi}\left(1-\tfrac{n}{l}|\cos\alpha|\right)\left(1-\tfrac{n}{l}|\sin\alpha|\right)d\alpha = \frac{4}{2\pi}\int_0^{\pi/2}\left(1-\tfrac{n}{l}\cos\alpha\right)\left(1-\tfrac{n}{l}\sin\alpha\right)d\alpha =$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\left(1-\tfrac{n}{l}\cdot 2\sin\alpha+\tfrac{n^2}{2l^2}\sin(2\alpha)\right)d\alpha = 1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\cos\alpha-\tfrac{n^2}{2l^2}\cos(2\alpha)\right)\Big|_0^{\pi/2} =.$$

$$=1+\frac{2}{\pi}\left(\tfrac{n^2}{2l^2}-\left(\tfrac{n}{l}\cdot 2-\tfrac{n^2}{2l^2}\right)\right)=1+\frac{2}{\pi}\left(\tfrac{n^2}{l^2}-\tfrac{n}{l}\cdot 2\right)$$

Hence the probability of hitting a line is $\dfrac{2}{\pi}\cdot\dfrac{n}{l}\left(2-\dfrac{n}{l}\right)$.

Now for the case b. There's a range of angles, which are close to the integer multiples of $\dfrac{\pi}{2}$, which give 1 as probability for hitting.
These probability one situations should be of length no more than ϕ from either vertical or horizontal direction, where $\phi=\arccos(l/n)$. So, if slope is between 0 and $\dfrac{\pi}{2}$ (it is something we can assume because of symmetry), then the probability of having angle that implies hitting is $\dfrac{2\phi}{\pi/2}=\dfrac{2}{\pi}\arccos(l/n)$.
To compute the rest of the probability in this case, we have to compute integral similar to the previous case, from ϕ to $\dfrac{\pi}{2}-\phi$.

$$\frac{2}{\pi}\int_{\phi}^{\frac{\pi}{2}-\phi}\left(1-\tfrac{n}{l}\cos\alpha\right)\left(1-\tfrac{n}{l}\sin\alpha\right)d\alpha=\frac{2}{\pi}\int_{\phi}^{\frac{\pi}{2}-\phi}\left(1-\tfrac{n}{l}\cdot 2\sin\alpha+\tfrac{n^2}{2l^2}\sin(2\alpha)\right)d\alpha$$

$$=\frac{2}{\pi}\left(\frac{\pi}{2}-2\alpha\right)+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\cos\alpha-\tfrac{n^2}{2l^2}\cos(2\alpha)\right)\Big|_{\phi}^{\frac{\pi}{2}-\phi}$$

Hence the complete probability of hitting is

$$p=1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\cos\alpha-\tfrac{n^2}{2l^2}\cos(2\alpha)\right)\Big|_{\phi}^{\frac{\pi}{2}-\phi}=$$

$$=1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\sin\phi+\tfrac{n^2}{2l^2}\cos(2\phi)-2\cos\phi+\tfrac{n^2}{2l^2}\cos(2\phi)\right)=$$

$$=1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\sin\phi+\tfrac{n^2}{l^2}\cos(2\phi)-\tfrac{n}{l}2\cos\phi\right)$$

Now recall that $\cos\phi=\dfrac{l}{n}$, $\sin\phi=\sqrt{1-\dfrac{l^2}{n^2}}$, $\cos 2\phi=2\dfrac{l^2}{n^2}-1$.

$$p=1+\frac{2}{\pi}\left(\tfrac{n}{l}\cdot 2\sin\phi+\tfrac{n^2}{l^2}\cos(2\phi)-\tfrac{n}{l}2\cos\phi\right)=$$

$$=1+\frac{2}{\pi}\left(\frac{n}{l}\cdot 2\sqrt{1-\frac{l^2}{n^2}}+\frac{n^2}{l^2}\left(2\frac{l^2}{n^2}-1\right)-2\right)=$$

$$=1+\frac{2}{\pi}\left(\frac{2\sqrt{n^2-l^2}}{l}-\frac{n^2}{l^2}\right)$$

In such messy computations, it is recommended ***to verify extreme cases***.

For instance, when $\dfrac{n}{l}=\sqrt{2}$, the result should be 1; for $n = l$ the result should be the same as in the case c.

Remark. Anyway, if the length of the needle < the diagonal of the tile, we get an experimental method of measuring π.

3. For any natural numbers k, m, n compute the integral

$$\int_0^1\left(\int_0^{1-y} y^k x^m (1-x-y)^n\,dx\right)dy.$$

Remark. Actually, this is an analogue of beta-function (of dimension +1).

Beta function is the integral $\int_0^{1-y} x^m (1-x)^n \, dx$ which is classical Euler's example of for integration by parts (and the answer is almost the reciprocal of binomial coefficient, as in our problem the answer is almost the reciprocal of trinomial coefficient), and of course, this exercise follows from it. However, both ideas below can be applied to the beta function.

First solution. We can conclude this integral from another famous Euler's integral, $\int_0^\infty x^n e^{-x} dx = n!$ (a. k. a. gamma function, up to shift by one). The proof of that is a simple exercise in induction and integration by parts.

No consider three-dimensional integral:

$$k! \cdot n! \cdot m! = \int_0^\infty \int_0^\infty \int_0^\infty x^k e^{-x} \cdot y^m e^{-y} \cdot z^n e^{-z} dxdydz = \int_0^\infty \int_0^\infty \int_0^\infty x^k y^m z^n e^{-(x+y+z)} dxdydz$$

Now switch to another variables $t = x + y + z$, $u = x/t$, $v = y/t$,

$$u = xt \text{ , } v = yt \text{ , } z = (1 - x - y)t.$$

The Jacobi matrix is $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} t & 0 & x \\ 0 & t & y \\ -t & -t & 1-x-y \end{pmatrix}$.

Adding first two rows to the last brings us to the matrix $\begin{pmatrix} t & 0 & x \\ 0 & t & y \\ 0 & 0 & 1 \end{pmatrix}$, so the determinant is t^2. Hence the integral that we computed was

$$k! \cdot n! \cdot m! = \int_0^\infty \iint\limits_{\substack{u,v \geq 0 \\ u+v<1}} (ut)^k (vt)^m \left(t(1-u-v)\right)^n e^{-t} t^2 dudvdt =$$

$$= \iint\limits_{\substack{u,v \geq 0 \\ u+v<1}} u^k v^m (1-u-v)^n \, dudv \cdot \int_0^\infty t^{k+m+n+2} du = X \cdot (k+m+n+2)!$$

Where X denotes the integral we wanted to know from the beginning.

Therefore, $X = \dfrac{k!\cdot n!\cdot m!}{(k+m+n+2)!}$.

Remark. In the definition of both beta function and gamma function the exponents are always $n - 1$ and not n. I think the aesthetic reason for that definition was that people prefer to get the formula $\mathrm{B}(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ and not m + n + 1 below. There are also some ideological reasons for this (they say it is Mellin transform, which is version of Fourier for multiplicative group of positive numbers, but these reasons appeared much later. The next solution is even nicer than the previous, but works only for integer number (while the previous is easily generalizable for complex numbers).

Second solution. We put on the [0,1] interval, randomly (with uniform probability measure) and independently, $N + 2$ dots: two red and N blue dots.
The first red dot is located at x, the second at $1 - y$, on distance y from the right end; the distance between them is $1 - y - x$. The distribution of x and y is uniform on the triangle $\{x, y \geq 0 , x + y \leq 1\}$, so the probability measure is $2dxdy$ supported on that triangle (the two red dots are distributed uniformly over the square, and then we impose the condition that first is before the second, we fold the square).
Now, when the red dots divided the segment into 3 sub-segments, So, let us compute the probability that m first blue points are in the left sub-segment, n next blue points in the middle sub-segment, and k last blue points are in the right sub-segment (here $k + m + n = N$).
If the red points are already placed, the probability is $y^k x^m (1-x-y)^n$.
So if red points are not yet placed, we get $2\int_0^1 \left(\int_0^{1-y} y^k x^m (1-x-y)^n \, dx \right) dy$.
Now let us compute the same probability in another way.
Each point-placing experiment gives us a random order of points, and each order has the same probability: $\dfrac{1}{(N+2)!}$. Out of this orders some are considered good we have m points to place on m first places, other n points for n places, k more

points for k places, and 2 more red points for 2 splitting places, so 2!k!m!n! orders are good, and the probability is $\frac{k!m!n!2}{(k+m+n+2)!}$. Therefore:

$$2\int_0^1\left(\int_0^{1-y} y^k x^m (1-x-y)^n \, dx\right)dy = \frac{k!m!n!2}{(k+m+n+2)!}$$

$$\int_0^1\left(\int_0^{1-y} y^k x^m (1-x-y)^n \, dx\right)dy = \frac{k!m!n!}{(k+m+n+2)!}$$

4.* a. Four aliens land on the surface of a spherical planet, randomly and independently (a chance for every alien to land inside a country is proportional to the area of this country). What is the probability that they all landed on the same hemisphere? In other words, what's the probability that there exists a plane through the planets center, such that all landing points are on the same side of that plane?
b. Ten aliens land on the planet (independently, proportional to surface area). What is the probability that they are on the same hemisphere?

Solution. For simplicity, we assume that the radius is 1. Also, we shall talk about open hemispheres – the chance of landing on the edge of a hemisphere is 0.
We can choose n random points of the sphere with the following procedure: first choose n random diameters of the sphere, then we shall choose one end of each diameter. If we would choose a given hemisphere, the chance that all points are in it would be $1/2^n$, for almost any choice of diameters.
There is a natural one-to-one correspondence between hemispheres and points of the sphere – in spherical geometry, hemisphere is a disc of radius $\frac{\pi}{2}$, and hence it is specified by its center. Another way to say the same thing – hemisphere is a set of vectors on sphere, which have positive scalar product with a given vector; and that vector corresponds to the hemisphere.
So, assume we have n diameters. To each diameter we draw a perpendicular bisector. This gives us n planes; each of this planes cuts the sphere along a big circle. The big circles cut the sphere into K parts, we shall compute K later. If we choose a center of hemisphere in one of these parts, probability that all will land in

it will be $1/2^n$. Events of landing on the given hemisphere are the same if centers are in the same part, and disjoint if they are in different parts. Therefore the probability of being in the same hemisphere is $K/2^n$.

Now it is enough to compute K. First big circle cuts the sphere in two. When we have already m circles, any new circle cuts other circles in $2m$ different points (with probability 1), so we add $2m$ new arcs, each splitting existing region into two new regions; therefore number of regions is increased by 2.

Therefore, if there are 4 points, number of regions is 2 + 2 + 4 + 6 = 14, and the probability is $14 / 2^4 = 7/8$.

For 10 points, number of regions is $2+2+4+...+18=2+\frac{0+18}{2}\cdot 10=2+90=92$, and the probability is $92 / 2^{10} = 23 / 256$.

5.** A unit cube is orthogonally projected on a random plane (the normal vector to the plane is distributed uniformly over the unit sphere).

What is the expectation of projection area?

Answer. 3/2.

Solution. Because the cube is a convex polytope, area of its projection to any plane is half sum of areas of projections of all faces. So, the expectation of the area is 3 times the expectation of projection area of a unit square.

The area of projection of a planar shape to another plane is the original area before the projection times the absolute value of the cosine. In our case, the original area is 1, so we have only the cosine.

Since the area of spherical hat is proportional to its height, we get that expectation of the area of a unit square projection is ½.

Therefore the expectation of area of cube projection is $^3/_2$.

Targil 11.

This targil is about identities. It is much harder to find a nice non-trivial identity than a nice non-trivial inequality; still there are some.

1. Prove that for 2×2 matrixes $\det(A) = \frac{(\operatorname{tr}(A))^2 - \operatorname{tr}(A^2)}{2}$ and, more generally, for $n\times n$ matrixes $\det(A)$ can be expressed as a polynomial in $\operatorname{tr}(A), \operatorname{tr}(A^2), \ldots, \operatorname{tr}(A^n)$.

2.** Prove that for arbitrary $2n$ matrixes $A_1, A_2, \ldots, A_{2n}$ of size $n\times n$,

$$\sum_{\sigma\in S_{2n}} (-1)^{\operatorname{sgn}(\sigma)} A_{\sigma(1)}A_{\sigma(2)}A_{\sigma(3)}\cdot\ldots\cdot A_{\sigma(2n)} = 0.$$

(Here S_{2n} is a group of permutations of 2n numbers).

3. Prove: $\sum_{i=1}^{n} \frac{x_i^m}{\prod_{j\neq i}(x_i - x_j)} = \begin{cases} 0, & \text{for } m = 0,1,\ldots,n-2 \\ 1, & \text{for } m = n-1 \end{cases}$.

4.* We are given an orthogonal matrix $n\times n$; let d_1 be the determinant of its $k\times k$ upper left corner, and d_2 be the determinant of its of its $(n-k)\times(n-k)$ right bottom corner. Prove that $|d_1| = |d_2|$.

5.* Prove: $(1-x)(1-x^2)(1-x^3)\cdot\ldots = 1 + \sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{3k^2-k}{2}} + x^{\frac{3k^2+k}{2}}\right)$

6.** Prove $p\sum_{k=0}^{n}\binom{n}{k}(p-kq)^{k-1}(r+kq)^{n-k} = (p+r)^n$.

Targil 11.

This targil is about identities.

1. Prove that for 2×2 matrixes $\det(A) = \frac{(\mathrm{tr}(A))^2 - \mathrm{tr}(A^2)}{2}$ and, more generally, for $n\times n$ matrixes $\det(A)$ can be expressed as a polynomial in $\mathrm{tr}(A), \mathrm{tr}(A^2), \ldots, \mathrm{tr}(A^n)$.

Solution. Assume we brought our matrix A to diagonal, or at least triangular form. The guys on the diagonal $\alpha_1, \ldots, \alpha_n$ are the eigenvalues. The trace, the determinant and all other coefficients of the characteristic polynomial can be easily expressed as elementary symmetric polynomials of $\alpha_1, \ldots, \alpha_n$.
On the diagonal of A^k we have $\alpha_1^k, \ldots, \alpha_n^k$.

So, for instance for 2×2 we get:
$\det A = \alpha_1 \alpha_2$
$\mathrm{tr}(A) = \alpha_1 + \alpha_2$
$\mathrm{tr}(A^2) = \alpha_1^2 + \alpha_2^2$
Hence $\frac{(\mathrm{tr}(A))^2 - \mathrm{tr}(A^2)}{2} = \frac{(\alpha_1 + \alpha_2)^2 - (\alpha_1^2 + \alpha_2^2)}{2} = \alpha_1 \alpha_2$.

The proof for general case follows directly from the known theorem: if we have any symmetric polynomial of $x_1, x_2, \ldots, x_n$ we can (uniquely) write it as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_n$, and as a polynomial in $s_1, s_2, \ldots, s_n$. Here σ_k is a sum of all products of k different x's (a. k. a. elementary symmetric polynomials), and $s_k = \sum_i x_i^k$. Actually, what we need is to show that one of elementary symmetric polynomial is a polynomial in sums of powers.
Indeed, in the story about matrixes, if $x_1, x_2, \ldots, x_n$ are the eigenvalues of A, then is $s_k = \mathrm{tr}(A^k)$, and σ_k are, up to a sign, coefficients of the characteristic polynomial of A.

There are several proves of this claim. My favorite (and the most fitting for this targil) is based on Newton's identities. From Newton's identities we see immediately that σ_k is a polynomial in $\sigma_1, \sigma_2, ..., \sigma_{k-1}, s_1, ..., s_k$, so the claim follows by induction. Newton identities look as follows.

$\sigma_1 - s_1 = 0$

$2\sigma_2 - \sigma_1 s_1 + s_2 = 0$

$3\sigma_3 - \sigma_2 s_1 + \sigma_1 s_2 + s_3 = 0$

...

$n\sigma_n - \sigma_{n-1} s_1 + ... \mp \sigma_1 s_{n-1} \pm s_n = 0$

$\sigma_n s_1 - \sigma_{n-1} s_2 + ... \mp \sigma_1 s_n \pm s_{n+1} = 0$

...

Their proof of the identities is an exercise to the reader – it is nice, and not complicated, so nobody would like it if I would write it downs right away.

2.** Prove that for arbitrary $2n$ matrixes $A_1, A_2, \ldots, A_{2n}$ of size $n\times n$,

$$\sum_{\sigma\in S_{2n}} (-1)^{\mathrm{sgn}(\sigma)} A_{\sigma(1)} A_{\sigma(2)} A_{\sigma(3)} \cdot ... \cdot A_{\sigma(2n)} = 0 .$$

(Here S_{2n} is a group of permutations of $2n$ numbers).

Remark. This is called Amitsur-Levitzky identity; it is named after two mathematicians from the Hebrew university of Jerusalem who found it.

Solution. We shall show the solution based on the trick of ***linearization***.

That is a procedure, given a polynomial, to obtain linear combinations of its values can give us a linear polynomial in several matrixes.

Simple example: if a polynomial is $p(x) = x^2$ we can take

$p(\mathrm{A} + \mathrm{B}) - p(\mathrm{A}) - p(\mathrm{B}) = \mathrm{AB} + \mathrm{BA}$, which is a linear polynomial in A and B.

If $p(x)$ is a polynomial of degree n, we consider:

$$L_p(\mathrm{A}_1, \mathrm{A}_2, ..., \mathrm{A}_n) = \sum_{S\subseteq P(I_n)} (-1)^{n-|S|}\, p\left(\sum_{i\in S} \mathrm{A}_i\right)$$

Here $I_n = \{1, 2, \ldots, \mathrm{n}\}$ and $P(I_n)$ is a set of its subsets, so we sum over all subsets of indexes. Consider the monomial $\mathrm{A}_{i_1}\mathrm{A}_{i_2}\mathrm{A}_{i_3}...\mathrm{A}_{i_k}$ in this L_p.

If there are less than n different indexes among $i_1, \ldots, i_k$, then there's an index j which is different from $i_1, \ldots, i_k$. Subsets that produce this monomial can be divided into pairs – two subsets of the same pair differ only by containing j / not containing j. The monomial $\mathrm{A}_{i_1}\mathrm{A}_{i_2}\mathrm{A}_{i_3}\ldots\mathrm{A}_{i_k}$ will be produced by the subsets of the same pair with the same coefficient, but of opposite signs (because of the sets in the same pair cardinalities are of different parity), so they cancel out.
Therefore, all terms of degree less than n will die, and from x^n only
$\sum_{\sigma\subseteq S_n} \mathrm{A}_{\sigma(1)}\mathrm{A}_{\sigma(2)}\mathrm{A}_{\sigma(3)}\ldots\mathrm{A}_{\sigma(n)}$ will remain.

The same thing can be done for more general polynomials, for instance polynomials with matrix coefficients or polynomial in matrix coefficients, but the idea is the same.

Now, consider Hamilton-Cayley identity: $\mathrm{A}^n - (\mathrm{tr}\,\mathrm{A})\mathrm{A}^{n-1} + \ldots \pm \det \mathrm{A} = 0$.
After that, in the same way as in problem 1, we shall rewrite all the coefficients as polynomials in $\mathrm{tr}(\mathrm{A}), \mathrm{tr}(\mathrm{A}^2), \ldots, \mathrm{tr}(\mathrm{A}^n)$. We shall get and identity of equality to zero of polynomial in $\mathrm{A}, \mathrm{tr}(\mathrm{A}), \mathrm{tr}(\mathrm{A}^2), \ldots, \mathrm{tr}(\mathrm{A}^n)$. Now we shall linearize it.
We shall get an expression in n matrixes: $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$.
A^n will turn into $\mathrm{X}_1 \ldots \mathrm{X}_n$ and all other terms will contain traces of products of X's.
Then we shall substitute $\mathrm{X}_1 = A_1A_2$, $\mathrm{X}_2 = A_3A_4$, … , $\mathrm{X}_n = A_{2n-1}A_{2n}$.
If we take the sum of linearized identities with permuted $A_1, \ldots, A_{2n}$ multiplied by sign of permutation. The leading term A^n will turn into Amitzur-Levitzky expression; all other terms will vanish.
Indeed, all other terms contain traces of even products of A_k's. Rotating every even product inside trace (for instance replacing tr($KLMN$) by tr(LMNK)) keeps the value unchanged; however, an even cycle is always an odd permutation, so this terms will come with opposite signs, and thus cancel out. QED.

Remark. It is obvious that in Amitzur-Levitzky we can replace $2n$ by any greater number; however we cannot replace it by a smaller number, so this inequality is tight. Indeed, imagine we write this expression for $2n - 1$ matrixes; now take A_{2k-1} to be a matrix that has 1 at place (k,k) and zeroes elsewhere, and A_{2k} has 1 at place

($k+1$, k) and zeroes elsewhere. Out of products of $A_1, \ldots, A_{2n-1}$ only one will be a nonzero matrix, therefore Amitzur-Levitzky expression with $2n-1$ matrixes can be nonzero.

3. Prove: $\sum_{i=1}^{n} \frac{x_i^m}{\prod_{j\neq i}(x_i - x_j)} = \begin{cases} 0, \text{ for } m = 0,1,\ldots,n-2 \\ 1, \text{ for } m = n-1 \end{cases}$.

Remark. These are called Jacobi identities.

First proof. Let us write the following matrix: first $n-1$ rows are the same as in Vandermonde (first row consists of ones, second row is $x_1, x_2, \ldots, x_n$, third row is $x_1^2, x_2^2, \ldots, x_n^2$ etc.) and the last row is $x_1^m, x_2^m, \ldots, x_n^m$.
Then if $m < n-1$ this matrix is degenerate, because two rows coincide; and if m is $n-1$ this matrix is the Vandermonde, and its determinant is $\prod_{i<j}(x_j - x_i)$.

Now compute the determinant by expansion along the last line. All minors are Vandermondes of order $n-1$. Divide it by the big Vandermonde, and you get QED.

Second solution. Integrate $\int \frac{z^m dz}{\prod_{i\leq n}(z - x_i)}$ along a very big circle in complex plane.
If $m < n-1$, integral will be quite close to zero on the large circle; if $m = n - 1$, the integral will be quite close to $\int \frac{dz}{z} = 2\pi i$.
Now compute the same integral by counting residues; poles are at x_i, and residues are precisely the summands in the LHS of the identity.
(Details are easy exercise. Really.)

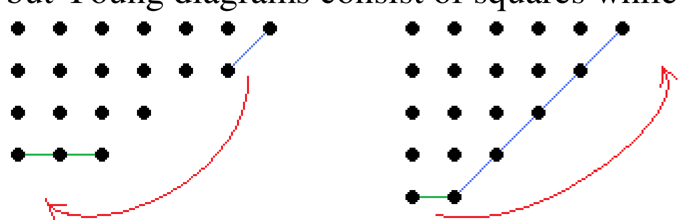
4.* We are given an orthogonal matrix $n\times n$; let d_1 be the determinant of its $k\times k$ upper left corner, and d_2 be the determinant of its of its $(n-k)\times(n-k)$ right bottom corner. Prove that $|d_1| = |d_2|$.

Solution. The original orthogonal matrix will be called A. When we replace its last $n - k$ columns by the corresponding columns of unit matrix, we get matrix B. Obviously $\det B = d_1$.
Consider matrix $A^{\mathrm{T}}B$. It is easy to see that its first k columns are equal to corresponding columns of the unit matrix, and it last $n - k$ columns are equal to corresponding columns of A^{T}. Therefore its determinant is $\det A^{\mathrm{T}}B = d_2$.
Hence $d_2 = \det A^{\mathrm{T}}B = \det A^{\mathrm{T}} \det B = (\det A)\cdot d_1$.
But A is orthogonal, hence $\det A = \pm 1$, hence $d_2 = \pm d_1$ QED.

5.* Prove: $(1-x)(1-x^2)(1-x^3)\cdot\ldots = 1+\sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{3k^2-k}{2}}+x^{\frac{3k^2+k}{2}}\right)$

Remark. This is Euler's identity. Which in this case is hardly an identifier – too many things are called Euler's formula or Euler's identity ☺.

Solution. We open brackets on the left hand side. Monomial $\pm x^n$ appears, each time we have a partition of n into sum of different natural numbers: $n = k_1 + \ldots + k_m$, where $k_1 < \ldots < k_m$. When m is even, we get this monomial with sign plus, when m is odd, we get it with sign minus. The claim is that almost everything cancels out; that is, we can almost build a bijective correspondence between partitions into odd number of summands and partitions into even number of summands.
To describe the correspondence, we shall represent the partitions by Ferre diagrams. Each number is a row of points, in our case each row will be shorter than the previous (there are also equivalent Young diagrams, that are almost the same, but Young diagrams consist of squares while Ferret diagrams consist of points).

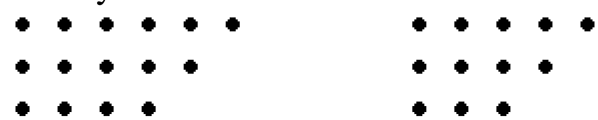


We shall denote the length of the last (and shortest) row by p, and by q length of right diagonal, which consist of right point of the upper row, and of all points which can be reached from that point by a single move of chess bishop (רץ).

So, if $p \le q$, the lower line is distributed between first p rows, one point to each row. But if $p > q$, the right diagonal is taken out and forms a new row below.

It is easy to see that this operation is inverse to itself, and it changes the parity of number of rows; however, in two cases this operation is not well defined.

In the case $p > q$, we usually can put the right diagonal under the right row, because it is smaller, unless $p = q + 1$, and by taking out the right diagonal we reduce the last line. So here a problem can happen when the last row intersects with right diagonal, and they are almost the same size.



In the case when we try to do the opposite, the last row is distributed between the first and largest rows, and there's sufficient number of those because $p \le q$, unless one of these rows is decomposed in the process, and $p = q$.

So, there are precisely two types of diagrams for which our correspondence is not defined: in both cases we have trapezoid (the right diagonal intersects the lowest row), and p equals either q or $q + 1$. The number of points in these cases will be either $\frac{3q^2-q}{2}$ or $\frac{3q^2+q}{2}$. Therefore, precisely for these numbers there is precisely one partition into sum of q numbers which doesn't cancel out.
So, when we open brackets in Euler's identity and cancel things, we shall get zeroes usually, and only for these powers we shall have coefficient of plus or minus 1, where signs are given by the parity of q.

6.** Prove $p\sum_{k=0}^{n}\binom{n}{k}(p-kq)^{k-1}(r+kq)^{n-k}=(p+r)^n$.

Remark. This is called Abel identity; I first heard of this type of things on 2009's SEEMOUS; one of our students on that SEEMOUS (Lior Yanovski) rediscovered a particular case of this identity by combinatorial counting of some trees.
Solution. We shall use two ideas, both of each might be useful olympiad tips.
First idea – if you have some ***natural number*** in your problem (it can be recognized because it might be denoted by n, it might be dimension of the linear space, number of the points, etc.) ***then induction might solve the problem***.
Second idea – use derivative.

So, derive both sides of this identity by r. We get

$$p\sum_{k=0}^{n}\binom{n}{k}(p-kq)^{k-1}(n-k)(r+kq)^{n-k-1}=n(p+r)^{n-1}$$

$$p\sum_{k=0}^{n-1}\frac{n!}{k!(n-k)!}(p-kq)^{k-1}(n-k)(r+kq)^{n-k-1}=n(p+r)^{n-1}$$

$$p\sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-1-k)!}(p-kq)^{k-1}(r+kq)^{n-k-1}=(p+r)^{n-1}$$

$$p\sum_{k=0}^{n-1}\binom{n-1}{k}(p-kq)^{k-1}(r+kq)^{n-1-k}=(p+r)^{n-1}$$

So, derivative is of Abel's identity of degree n is Abel's identity of degree $n-1$.
To make a proof of it, it remains to check Abel's identity for one specific value of r, and to prove it for $n = 0$ (base of induction).
So, let us finish the step of induction first. Take $r = -p$. Then it becomes

$$p\sum_{k=0}^{n}\binom{n}{k}(p-kq)^{k-1}(-p+kq)^{n-k}=0$$

$$p\sum_{k=0}^{n}\binom{n}{k}(p-kq)^{n-1}(-1)^{n-k}=0$$

Consider the following operator "discreet derivative": function $f(x)$ is turned into $\Delta f(x)=f(x+1)-f(x)$. It is really easy to see that if f is a polynomial, then Δf is a polynomial of lower degree. Applying this operator n times results it $\Delta^n f(x)=\sum_{k=0}^{n}(-1)^k f(x+k)$. Therefore if f is a polynomial of degree less then n then $\Delta^n f$ is constant zero. Now apply it to $f(x)=(p+xq)^{n-1}$ and we get what we need in order to finish the step of induction.
It remains to do the base of induction (well, if I'd write it for the jury I would write everything in the reversed order): when $n = 0$,

$$p\sum_{k=0}^{0}\binom{0}{k}(p-kq)^{k-1}(r+kq)^{0-k}=(p+r)^{0}$$

$$p\binom{0}{0}(p-0)^{0-1}(r+0)^{0}=(p+r)^{0}$$

$$p\cdot 1\cdot\frac{1}{p}\cdot 1=1$$

QED.

First stage of Israeli students competition, 2011.

14/1/2011

Duration: 4 hours

1. Find all possible values of $\lim_{x\to\infty} x^{(\ln x)^{\lambda}}$ for real λ.

2. Is it possible to draw a pentagon with integer coordinates of vertices and equal sides?

3. Compute $1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\frac{1}{10}+\frac{1}{11}-\frac{2}{12}+\ldots$.

4. Michal and Ohad play a game in which Michal marks points and arcs in the plane, and Ohad assigns colors to the points. Michal makes the first move. In each of her moves, Michal marks one point, and she can also join it by arcs to some of the existing points, provided that the arcs do not intersect (except possibly in endpoints). Ohad, in turn, paints the last marked point in some color, which must be different than colors of endpoints connected to this point by an arc. Michal wins, if Ohad will use more than 5771 colors. Does Michal have a winning strategy?

5. An infinite sequence of positive real numbers satisfies:

$$\det\begin{pmatrix} 1 & a_i & 0 & a_i^2 \\ 2 & \sqrt{3}\cdot a_{i+1} & a_{i+1} & 2a_{i+1}^2 \\ 2 & a_{i+2} & \sqrt{3}\cdot a_{i+2} & 2a_{i+2}^2 \\ 1 & 0 & a_{i+3} & a_{i+3}^2 \end{pmatrix} = 0$$

Prove that it is periodic.

Good luck!

First stage of Israeli students competition, 2011.

1. Find all possible values of $\lim\limits_{x\to\infty} x^{(\ln x)^{\lambda}}$ for real λ.

Solution. $\lim\limits_{x\to\infty} x^{(\ln x)^{\lambda}} = \lim\limits_{x\to\infty} e^{(\ln x)(\ln x)^{\lambda}} = \lim\limits_{x\to\infty} e^{(\ln x)^{\lambda+1}} = e^{\lim\limits_{x\to\infty}(\ln x)^{\lambda+1}} = e^{\lim\limits_{y\to\infty} y^{\lambda+1}}$,

where $y = \ln x$. The power $\lim\limits_{y\to\infty} y^{\lambda+1}$ can be:

0 if $\lambda+1<0$,

1 if $\lambda+1=0$,

$+\infty$ if $\lambda+1>0$.

Therefore, the original limit can be either 1, e, or $+\infty$.

2. Is it possible to draw a pentagon with integer coordinates of vertices and equal sides?

Answer: no.

Solution. Consider the pentagon of that kind with minimal side. It is defined if such pentagons exist, because the distance between integer points is a root of an integer number (and each non-empty subset of nonnegative integers has minimal element). The number under the root is even if the endpoints are of the same color, and odd if they are of opposite color (here we use the standard chess coloring, point are black if the sum of its coordinates is even, and white if the sum of its coordinates is odd). So, if the length of each side is a square root of an odd number then each two adjacent vertices have different chess colors, which is not possible since 5 is odd. Therefore the length of each side must be even, and all the vertexes must have the same chess color. So, if we rotate the picture around one of the vertices by 45° and reduce it $\sqrt{2}$ times, we shall get another polygon with integer vertexes and equal sides, but this time the sides are shorter. This contradicts the assumption.

3. Compute $1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\frac{1}{10}+\frac{1}{11}-\frac{2}{12}+\ldots$.

First solution: Denote $S_n = 1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\ldots+\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{2}{3n}$. It is enough to compute $\lim_{n\to\infty} S_n$ (and to prove that it exists). Indeed, S_n is 3n'th partial sum, and the $3n+1^{\text{st}}$ and $3n+2^{\text{nd}}$ partial sums are close to it (the distance is less than $1/n$).

$$S_n = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{3n}\right)-3\left(\frac{1}{3}+\frac{1}{6}+\ldots+\frac{1}{3n}\right)=$$
$$=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{3n}\right)-\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)=\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right)=$$
$$=\frac{1}{n}\left(\frac{1}{(n+1)/n}+\frac{1}{(n+2)/n}+\frac{1}{(n+3)/n}+\ldots+\frac{1}{3}\right)\underset{\substack{\text{Riemann}\\ \text{sum}}}{=}\int_1^3\frac{dx}{x}=\ln x\Big|_1^3=\ln 3$$

Second solution. Consider logarithmic Taylor series: $-\ln(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots$

Substitute $\xi=e^{2\pi i/3}=\frac{-1+i\sqrt{3}}{2}$ (it satisfies $\xi^3=1$). The series $\xi+\frac{\xi^2}{2}+\frac{\xi^3}{3}+\frac{\xi^4}{4}+\ldots$ will converge by Dirichlet criterion (since $\xi+\xi^2+\xi^3+\ldots+\xi^n$ is bounded by 2, and $\frac{1}{n}\to 0$ monotonically). For $|x|<1$ the series converge to $-\ln(1-x)$. The series and the complex continuation of $-\ln(1-x)$ are both continuous in the domain of convergence of the series, therefore $-\ln(1-\xi)=\xi+\frac{\xi^2}{2}+\frac{\xi^3}{3}+\frac{\xi^4}{4}+\ldots$.

In the complex continuation of ln, we have $\operatorname{Re}(\ln(z))=\ln|z|$. Therefore

$$-\ln|1-\xi|=\operatorname{Re}\left(\xi+\frac{\xi^2}{2}+\frac{\xi^3}{3}+\frac{\xi^4}{4}+\ldots\right)=\operatorname{Re}\left(\xi+\frac{\overline{\xi}}{2}+\frac{1}{3}+\frac{\xi}{4}+\frac{\overline{\xi}}{5}+\frac{1}{6}+\ldots\right)=$$
$$=-\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{5}+\frac{1}{6}+\ldots$$

Multiplying the right hand side by -2 will give us the series we want to compute. So, it is equal to the left hand side times -2, which is $2\ln|1-\xi|=\ln|1-\xi|^2=\ln\left((1-\xi)\cdot\overline{(1-\xi)}\right)=\ln\left((1-\xi)\cdot(1-\overline{\xi})\right)=\ln\left(1-\xi-\overline{\xi}+1\right)=\ln 3$.

4. Michal and Ohad play a game in which Michal marks points and arcs in the plane, and Ohad assigns colors to the points. Michal makes the first move. In each of her moves, Michal marks one point, and she can also join it by arcs to some of the existing points, provided that the arcs do not intersect (except possibly in endpoints). Ohad, in turn, paints the last marked point in some color, which must be different than colors of endpoints connected to this point by an arc. Michal wins, if Ohad will use more than 5771 colors. Does Michal have a winning strategy?

Answer: Yes.

Solution. We consider the painting as a graph, where the vertices are the points, and the edges between the vertices are the arcs connecting them. Recall that a graph is called a *forest* if it contains no cycles, and the connected components of a forest are called *trees*.

Basic Lemma: Suppose the painting so far is a forest. Then any two points can be connected by a new arc legally. **Why?** Because the arcs do not form cycles, they cannot split the plane into several parts, meaning that the plane with the arcs removed is still one connected face. In particular any two points on it can be connected by an arc not intersecting those previously marked.

Corollary: Suppose the painting so far is a forest. Then after marking a new point, Michal can always choose one point from each tree in the painting, and connect all those points to the new point marked: Indeed, she may simply draw these arcs one after the other, as every arc only joins together two connected components, and does not form cycles – meaning the graph remains a forest after every step, and thus she can draw the next arc by the basic lemma.

Claim: For any k, Michal can construct a forest F_k with N_k vertices (where N_k is some finite number) which Ohad will be forced to paint with at least k different colors. Michal then wins the game by building a copy of F_{5772}.

Proof: By induction: Basis is k=1, where we may simply mark a new point, and it will be colored in one color and be a forest (so N_1=1). Next we assume that it is true for all numbers up to k, and prove it for k+1: To construct F_{k+1}, Michal should first build a copy of $F_1, F_2, \ldots, F_k$. In F_1, there is at least one color used: Denote such one as c_1, and let v_1 be some point in F_1 with that color. In F_2 at least two colors are used, so there must be at least one used other than c_1, denote it c_2, and v_2 is some point in F_2 with that color. Continue in this fashion: For all values of m up to k, at least m colors are used in F_m, so in particular there must appear a color c_m which is different from $c_1,\ldots,c_{m-1}$, and a vertex v_m with that color. Now, as all vertices v_i are in different trees (as they are in disjoint forests), by the corollary above, Michal can mark a new point and connect it to all vertices v_i. Ohad must then paint the new point in a color different from all colors $c_1,\ldots,c_k$, and it is thus clear that with the new point there will be at least k+1 different colors in the graph. Furthermore, the new graph is also clearly a forest, so it is an example of F_{k+1}, with a clearly bounded number of vertices.

Remarks. It can be seen that the above construction actually yields a tree, not a forest. It is easy to verify that the above construction gives $N_k=2^{k-1}$. It is possible not to demand necessarily that F_k are forests, but to demand that on each step in the construction, the new point marked is accessible from the unbounded component of the plane, and stop as soon as you get the new color wanted.

5. An infinite sequence of positive real numbers satisfies:

$$\det\begin{pmatrix} 1 & a_i & 0 & a_i^2 \\ 2 & \sqrt{3}\cdot a_{i+1} & a_{i+1} & 2a_{i+1}^2 \\ 2 & a_{i+2} & \sqrt{3}\cdot a_{i+2} & 2a_{i+2}^2 \\ 1 & 0 & a_{i+3} & a_{i+3}^2 \end{pmatrix} = 0$$

Prove that it is periodic.

Solution. Let us rewrite the determinant as follows:

$$\det\begin{pmatrix} 1 & a_i & 0 & a_i^2 \\ 1 & \frac{\sqrt{3}}{2}\cdot a_{i+1} & \frac{1}{2}\cdot a_{i+1} & a_{i+1}^2 \\ 1 & \frac{1}{2}\cdot a_{i+2} & \frac{\sqrt{3}}{2}\cdot a_{i+2} & a_{i+2}^2 \\ 1 & 0 & a_{i+3} & a_{i+3}^2 \end{pmatrix} = 0$$

This actually means that the following four points

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a_i \\ 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} a_{i+1} \\ \frac{1}{2} a_{i+1} \end{pmatrix} \quad ; \quad \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} a_{i+2} \\ \frac{\sqrt{3}}{2} a_{i+2} \end{pmatrix} \quad ; \quad \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ a_{i+3} \end{pmatrix}$$

belong to one curve which is described by an equation of the following type: $k + lx + my + n\left(x^2 + y^2\right) = 0$. Such equations can describe either a circle (if n is nonzero) or a line (if n is zero).

The length of the above four vectors are a_i, and each one is 30º counter-clockwise with respect to the previous. Therefore, if we define a sequence of vectors, such that the vector number i has argument 30ºi and length is a_i.

Each 4 consequent points will be either on one circle or one straight line. Each three non-collinear points define a unique circle, therefore all points in a sequence will be either on one line or one circle.

If we will have a line, or a circle which doesn't contain the origin, then one of the first seven rays won't even intersect it. If the line or the circle goes through the origin, one of a_i will have to be zero, and it is given that they are positive. Therefore, the points in the sequence are intersections of the rays with some circle, which goes around the origin. This defines each point uniquely. Rays number i and i +12 coincide, therefore, $a_i = a_{i+12}$. So the sequence is periodic (and the period divides 12).

Second stage of Israeli students' competition, 2011.

Duration: 4 hours

1. In each vertex of a connected simple graph a number is written. The following action is repeated infinitely many times: all numbers are replaced simultaneously by the average of their neighbors. Consider the sequence of numbers which appear at a specific vertex of the graph. Assume that one of those sequences does not converge. Prove that the graph is bipartite (which means that its vertices can be painted in black and white so that neighbors are always of opposite colors).

2. Is it possible to find a planar strictly convex equilateral pentagon, all vertices of which are in $\mathbb{Z}^3$ (integer three-dimensional points)?
Remark. A polygon is called equilateral if all its sides are of the same length. It is possible for a polygon to be equilateral but not regular.

3. There is an urn with 5 balls: 2 blue and 3 white. Every minute, a random ball is chosen from the urn and returned with another ball of the same color. What is the limit of the probability that less than a half of the balls are blue, as the time goes to infinity?

4. We have a hyperbola and two distinct points A and B on it. For any additional point X on the same hyperbola, we define 3 numbers:
α = the distance from X to the straight line which is tangent to the hyperbola at A.
β = the distance from X to the straight line which is tangent to the hyperbola at B.
γ = the distance from X to the straight line AB
Prove that $\frac{\alpha\beta}{\gamma^2}$ doesn't depend on the choice of X.

5. Compute $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$.

Good luck!

Second stage of Israeli students competition, 2011.

1. In each vertex of a connected simple graph a number is written. The following action is repeated infinitely many times: all numbers are replaced simultaneously by the average of their neighbors. Consider the sequence of numbers which appear at a specific vertex of the graph. Assume that one of those sequences does not converge. Prove that the graph is bipartite (which means that its vertices can be painted in black and white so that neighbors are always of opposite colors).

Solution. Let N be a number of vertexes. Choose some numbering for the set of vertexes. Then any set of numbers written by the vertexes corresponds to a vector in $\mathbb{R}^N$. "Averaging" is a linear operator. It consists of subsequent application of two linear operators: first summing of neighbors, then dividing by degrees. In terms of matrixes we get: A = DG, where A is the matrix of averaging, D is the diagonal matrix of reciprocal degrees, and G is the matrix of zeroes and ones in which one appears in square (i, j) iff vertexes i and j are connected in the graph, a.k.a. "the graph matrix". We shall also consider matrix R which is "a square root" of D: it is also a diagonal matrix, but the numbers on diagonal are square roots of respective numbers of D. Obviously, R^2 = D, and R is an invertible matrix.
Applying the averaging procedure k times is multiplying by A^k = DGDG… DG.
This matrix is conjugated to RGDG…GR = B^k, where B = RGR, so A^k and B^k have the same eigenvalues with the same geometric and algebraic multiplicities. It is easier to analyze eigenvalues of B^k, since both B and B^k are symmetric matrixes. Therefore, the eigenvalues are real, and geometric multiplicities are equal to algebraic multiplicities, so B has diagonal form with real numbers $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ on the diagonal, and B^k also has a diagonal form with $\lambda_1^k, \lambda_2^k, \lambda_3^k, ..., \lambda_N^k$ on the diagonal, with the same eigenbasis. Notice that since eigenvalues are real, the eigenbasis can also be chosen to be real. So, A^k also has a diagonal form with $\lambda_1^k, \lambda_2^k, \lambda_3^k, ..., \lambda_N^k$ on the diagonal, and real eigenbasis.
Choose a real eigenbasis for A: it will consist of vectors a_1, a_2, … , a_N. It is also an eigenbasis of A^k. Now we shall show that $|\lambda_j| \le 1$ for every j. Otherwise, if we start with a_j, then after k averagings we get $A^k a_j = \lambda_j^k a_j$, and absolute values at all nonzero coordinates keep growing with every step, but that is impossible: maximal absolute value cannot grow during averaging.

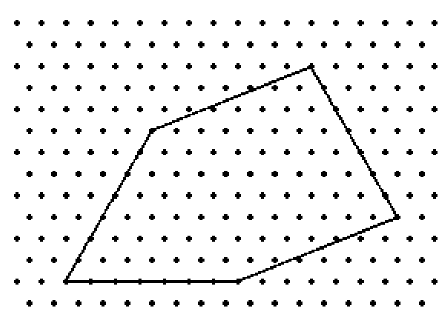
Therefore, $-1 \le \lambda_1, \lambda_2, \lambda_3, ..., \lambda_N \le 1$. Then we ask whether -1 is an eigenvalue of A. Consider both possible answers:

No Then we can easily prove that for any initial vector v, the vectors $\mathrm{A}^k v$ will converge. Indeed, if $v = \sum_j v_j a_j$, then $\mathrm{A}^k v = \sum_j \left(\lambda_j^k v_j\right) a_j$. Coefficient which correspond to eigenvalues equal to 1 remain the same, all other coefficients converge to zero. Therefore $\mathrm{A}^k v$ converges, and all its coordinates converge.

Yes Then consider a nonzero eigenvector. Each number is minus average of its neighbors. Assume that a number m has maximal absolute value. WLOG it is positive (otherwise, multiply all numbers by -1). Then all its neighbors are at least $-m$, and to get m as minus their average they have to be exactly $-m$. F or the same reason all neighbors of $-m$ are m. Since graph is connected, then by induction with this argument we conclude that all numbers in a graph are either m or $-m$, and neighbors are always of opposite signs, so there is "a chess coloring" for the graph.

To conclude: if averagings don't converge, then we have are in the "yes" case (there is a -1 eigenvalue), then there is a chess coloring.

2. Is it possible to find a planar strictly convex equilateral pentagon, all vertices of which are in $\mathbb{Z}^3$ (integer three-dimensional points)?
Remark. A polygon is called equilateral if all its sides are of the same length. It is possible for a polygon to be equilateral but not regular.



Solution. Consider a plane $x + y + z = 0$. Within this plane, we have a triangular lattice (spanned by integer linear combinations of the vectors formed by sides of an equilateral triangle).
On that lattice, we build an equilateral pentagon (see the picture). It is based on the fact that triangle with sides 3, 5, 7 has an angle of 120º (that happens because $3^2 + 3 \cdot 5 + 5^2 = 7^2$).

3. There is an urn with 5 balls: 2 blue and 3 white. Every minute, a random ball is chosen from the urn and returned with another ball of the same color. What is the limit of the probability that more than a half of the balls are blue, as the time goes to infinity?

Solution. Denote that at any time i, there are m_i blue balls and n_i white balls inside the urn. We will compute the probability that at time t there are exactly m_t, n_t blue and white balls in the urn, given that there were m_0 and n_0 time 0 (in our case, $m_0=2$ and $n_0=3$). Denote also $m' = m_t-m_0$, $n' = n_t-n_0$.
For this to happen, we must draw exactly m' blue balls and n' white balls in the first t steps. Denote by I the set of steps up to t on which a blue ball was drawn. It is clear that I may be any subset of {0,…,t-1} of size m', and that for each such set, the probability that it is represents the sequence of draws is exactly

$P_I = \prod_{i\in I} \frac{m_i}{m_i+n_i} \prod_{j\notin j} \frac{m_j}{m_j+n_j}$, as the probability of drawing a blue ball at a single step i is

$\frac{m_i}{m_i+n_i}$, and so on.

We observe that the numerator in the left product comprise all integer numbers sequentially from m_0 to m_t-1, as the number of blue balls increases by exactly one between two blue draws, and similarly the numerators on the right product are all integers from n_0 to n_t-1. Furthermore, all the denominators together are simply the integers m_0+n_0 to m_t+n_t-1, as $m_i+n_i = m_0+n_0+i$. Thus after rearrangement, we obtain

$$P_I = \frac{(m_t-1)!}{(m_0-1)!} \cdot \frac{(n_t-1)!}{(n_0-1)!} \Bigg/ \frac{(m_t+n_t-1)!}{(m_0+n_0-1)!}$$

In particular, this probability is independent of I, and thus to obtain the total probability of reaching m_t and n_t we need only multiply by the number of choices for I, which is $\binom{m'+n'}{m'} = \frac{(m'+n')!}{m'!n'!}$. With more rearrangement, we get

$$P_{m_t,n_t} = \frac{(m'+n')!}{m'!n'!} P_I = \frac{(m_t-1)!}{m'!(m_0-1)!} \cdot \frac{(n_t-1)!}{n'!(n_0-1)!} \cdot \frac{(m'+n')!(m_0+n_0-1)!}{(m_t+n_t-1)!} = \frac{\binom{m_t-1}{m_0-1}\binom{n_t-1}{n_0-1}}{\binom{m_t+n_t-1}{m_0+n_0-1}}$$

This probability can now be considered in a new way: consider an ordered set of size S = m_t+n_t-1, for example {1, 2, …, m_t+n_t-1}. From this set, we choose uniformly a random subset of size m_0+n_0-1. We will denote the subset's elements by $\{x_1, x_2, ..., x_{m_0+n_0-1}\}$, and assume that the sequence of *x*'s is increasing. Consider the element x_{m_0}. It can be immediately computed that the probability of $x_{m_0} = m_t$ is exactly the same expression as (*), i.e. the same as P_{m_t,n_t}. We are interested in the probability that $\frac{m_t}{m_t+n_t} < \frac{1}{2}$, which is therefore equal to the probability that

$$\frac{x_{m_0}}{m_t+n_t} < \frac{1}{2}.$$

As *t* goes to infinity, the limit of the last distribution can be easily computed: it is similar to choosing m_0+n_0-1 points uniformly distributed on the interval [0,1], sorting them as $\{x_1, x_2, ..., x_{m_0+n_0-1}\}$, and then considering only x_{m_0}. It is easy to see that the probability density of the random variable x_{m_0} is $\frac{x^{m_0-1}(1-x)^{n_0-1}}{\mathrm{B}(m_0,n_0)}$, where

$B(m_0,n_0) = \frac{(m_0-1)!(n_0-1)!}{(m_0+n_0-1)!}$ is the Beta function.

Finally, we are interested in the probability that $x_{m_0} < \frac{1}{2}$, which is simply:

$$\frac{1}{\mathrm{B}(m_0,n_0)}\int_0^{1/2} x^{m_0-1}(1-x)^{n_0-1}\,dx = \frac{1}{\mathrm{B}(2,3)}\int_0^{1/2} x(1-x)^2\,dx = \frac{4!}{1!2!}\int_0^{1/2}\left(x-2x^2+x^3\right)dx =$$
$$= 12\left(\frac{1}{2}\cdot\frac{1}{2^2} - 2\cdot\frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{4}\cdot\frac{1}{2^4}\right) = \boxed{\frac{11}{16}}$$

4. We have a hyperbola and two distinct points A and B on it. For any point X on the same hyperbola, we define 3 numbers:
α = the distance from X to the straight line which is tangent to the hyperbola at A.
β = the distance from X to the straight line which is tangent to the hyperbola at B.
γ = the distance from X to the straight line AB

Prove that $\frac{\alpha\beta}{\gamma^2}$ doesn't depend on the choice of X.

Solution. The solution works for any conic (ellipse, parabola, or hyperbola) so from now on we shall talk of conics. We shall denote the tangents to the hyperbola A and B by t_a and t_b respectively. The distance from point (x, y) to t_a can be written as $|l_a(x, y)|$ where $l_a(x, y)$ is a linear function: $kx + my + n$. Linear function l_b is chosen similarly for the line t_b. The third linear function l is such that $|l(x, y)|$ = distance from the line AB to the point (x, y).

Equations of curves of order at most two form a six-dimensional linear space

$$Q = \{\ q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0\ \}$$

Inside that space, the equations of curves that pass through A form a sub-space Q_1; it is strictly smaller (since some curves of order 2 don't pass through A), and is defined by one linear condition – substituting coordinates of A to $q(x, y)$ specifies one linear condition on the coefficients; therefore, equations of order 2 of curves containing A form a five-dimensional space. For similar reasons, since some curves of order 2 contain A but not B, equations of second degree satisfied by A and B form a linear space Q_2 of dimension four.

Inside Q_2, consider such equations, that when we reduce them to t_a we get multiple root at A. In other words, if K is a non-zero vector parallel to t_a, we substitute the coordinates of A + sK to the polynomial q(x, y), we get a polynomial $q_a(s)$ of second degree in t; for all polynomials in Q_2, this polynomial has a root at zero (since the curve goes through A); and we define a subspace Q_3 in Q_2 by the condition that $q_a(t)$ should have a multiple root at zero; thus it should be of form $q_a(s) = hs^2$.

Finally, consider the subspace Q_4 in Q_3 of curves which, when reduced to t_b, have double root at B (similarly to the previous condition, but with B instead of A).

It is easy to find examples of curves of degree two in Q_2 but not in Q_3 (for example the product of two linear equations, one of line AB and another of a line parallel to AB, so it has two distinct roots on the line t_a) so dim $Q_3 = 3$. It is also easy to find an example of something in Q_3 but not in Q_4: for instance $l(x, y) \cdot l_a(x, y)$.

Therefore dim $Q_4 = 2$.

Now we shall show three examples of equations in Q_4.

The first example is the equation of the original conic. The other two obvious examples of curves are $l_b(x, y) \cdot l_a(x, y) = 0$ and $(l(x, y))^2 = 0$.

But dim $Q_4 = 2$, and all three examples define different curves, so neither two of them are linearly dependent. Therefore the equation of the conic can be expressed as a linear combination of the other two. That means it can be written as:

$\lambda \cdot l_a(x, y) \cdot l_b(x, y) + \mu \cdot (l(x, y))^2 = 0$

Where λ, μ are some fixed real numbers. Therefore, for any X on the conic we get $\pm\lambda\alpha\beta \pm \mu\gamma^2 = 0$. Since X doesn't coincide with A or B we are allowed to divide:

$$\frac{\alpha\beta}{\gamma^2} = \left|\frac{\mu}{\lambda}\right|$$

The right hand side obviously doesn't depend on the choice of X.

5. Compute $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$.

Solution. Perform change of variables: $\arctan x = y$, then $\frac{dx}{1+x^2} = dy$

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \int_0^{\pi/4} \ln(1+\tan y)dy = \int_0^{\pi/4} \ln\left(\frac{\sin y+\cos y}{\cos}\right)dy =$$

$$= \int_0^{\pi/4} \ln(\sin y+\cos y)dy - \int_0^{\pi/4} \ln(\cos y)dy$$

Recall that $\sin y+\cos y = \sqrt{2}\left(\cos\frac{\pi}{4}\sin y+\sin\frac{\pi}{4}\cos y\right) = \sqrt{2}\sin\left(\frac{\pi}{4}+y\right)$. Therefore

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \int_0^{\pi/4} \ln(\sin y+\cos y)dy - \int_0^{\pi/4} \ln(\cos y)dy =$$

$$= \int_0^{\pi/4} \ln\sqrt{2}dy + \int_0^{\pi/4} \ln\left(\sin\left(\frac{\pi}{4}+y\right)\right)dy - \int_0^{\pi/4} \ln(\cos y)dy =$$

(here $z = \frac{\pi}{2} - y$)

$$= \int_0^{\pi/4} \ln\sqrt{2}dy + \int_{\pi/4}^{\pi/2} \ln(\sin(y))dy - \int_0^{\pi/4} \ln(\cos y)dy =$$

$$= \int_0^{\pi/4} \ln\sqrt{2}dy + \int_0^{\pi/4} \ln(\cos(z))dz - \int_0^{\pi/4} \ln(\cos y)dy = \frac{\pi}{4}\ln\sqrt{2}$$

Targil 1.

Linear algebra

1. Denote by $\{ f_n \}$ the Fibonacci sequence: $f_0 = 0,\ f_1 = 1,\ f_{n+2} = f_n + f_{n+1}$.
Prove that $f_{n+k} f_{n-k} - f_n^2 = \pm f_k^2$.
(What is the sign in this formula?)

2. Every entry of an N×N matrix is ±1. What is the maximal possible determinant of this matrix
a. for N = 4?
b. for N = 8?

3. Find the maximal volume of a simplex which is contained in a unit cube of dimension N for
a. N = 3.
b. N = 7.
Remark. A simplex is a higher-dimensional generalization of a two-dimensional triangle and a three-dimensional tetrahedron; in other words, it is a convex hull of N+1 points which are not in one hyperplane (which happens to be a minimal number of points which might be not in one hyperplane).

4. We are given nonzero n×n matrices $A_1, A_2, \ldots, A_k$.
a. Over an infinite field, prove that it is possible to find a matrix B such that the matrix $A_1BA_2B \ldots BA_k$ is nonzero.
b. Is this true for a finite field?

5. A linear transformation $f : \mathbb{Q}^n \to \mathbb{Q}^n$ satisfies $f\left(f\left(f\left(v\right)\right)\right) = 2v$ for any vector v. Describe all the possible values of n.

התחרות הבינלאומית במתמטיקה לסטודנטים

מיון לנבחרת ישראל – שלב א'

6/1/2012

משך הבחינה: 4 שעות

1. חשב את האינטגרל (הלא מסויים)

$$\int \frac{x^2 dx}{(\cos x + x\sin x)^2}.$$

2. נתון פולינום P ממעלה 5772, עם מקדמים שלמים חיוביים. הוכח שקיים n שלם, כך שהההצגה העשרונית של המספר $P(n)$ כוללת 2011 ספרות 7 רצופות.

3. חשב את הגבול

$$\lim_{n\to\infty}\left(n\left(\frac{1}{2n+1}-\frac{1}{2n+2}+\frac{1}{2n+3}-\frac{1}{2n+4}+\cdots-\frac{1}{4n}\right)\right).$$

4. לכל m נקודות $P_1,\ldots,P_m$ במרחב $\mathbb{R}^n$, אפשר ליצור מטריצה $m\times m$, $A=(a_{ij})$, שרכיביה הם $a_{ij}=|P_iP_j|^2$. עבור n נתון, מהו הערך הגדול ביותר האפשרי של m שעבורו יש מטריצה כזו שהיא הפיכה?

5. סדרת אותיות נקראת "חופשית מריבועים" אם אין בה אף רצף החוזר על עצמו פעמיים רצופות. האם אפשר להרכיב מלה חופשית מריבועים מ־22 אותיות האלף־בית העברי, כך שאם נוסיף אות כלשהי משמאל, הרצף יפסיק להיות חופשי מריבועים? אם כן, מהו האורך המינימלי של סדרה כזו?

בהצלחה!

IMC 2012 Team Preselection Exam

Duration: 4 hours

Janurary 6, 2012

Question1 (10 points)

Calculate the indefinite integral:

$$\int \frac{x^2}{(\cos x + x\sin x)^2} dx$$

Solution:

We use the following well-known trick. Let $x = \tan y$. Then:

$$\begin{aligned} \frac{1}{\sqrt{1+x^2}}(\cos x + x\sin x) &= \frac{1}{\sqrt{1+x^2}}\cos x + \frac{x}{\sqrt{1+x^2}}\sin x \\ &= \cos y \cos x - \sin y \sin x = \cos(y-x) \end{aligned}$$

Thus:

$$\begin{aligned} &\left(\tfrac{x}{\cos x + x\sin x}\right)^2 \\ = &\left(\frac{x/\sqrt{1+x^2}}{(\cos x + x\sin x)/\sqrt{1+x^2}}\right)^2 \\ = &\left(\frac{\sin y}{\cos(y-x)}\right)^2 \\ = &\frac{1-\cos^2 y}{\cos^2(y-x)} \end{aligned}$$

Therefore:

$$\begin{aligned} \int \frac{x^2}{(\cos x + x\sin x)^2} dx &= \int \frac{1-\cos^2 y}{\cos^2(y-x)} dx = \int \frac{dx - \cos^2 y dx}{\cos^2(y-x)} \\ &= \int \frac{dx - \cos^2 y d(\tan y)}{\cos^2(y-x)} = \int \frac{dx - dy}{\cos^2(y-x)} \\ &= \int \frac{d(x-y)}{\cos^2(x-y)} = \tan(x-y) \\ &= \frac{\tan x - \tan y}{1 + \tan y \tan x} = \frac{\tan x - x}{1 + x\tan x} \end{aligned}$$

Question2 (10 points)

Let $P(n)$ be a polynomial of degree 5772, with positive integer coefficients. Prove that there is some integer n such that the decimal representation of $P(n)$ includes the digit 7 appearing 2011 times in a row.

Solution:

Note that if we take n and multiply it by a very large power of 10, then if we let $P(n) = \sum a_i n^i$, then the decimal representations of each monomial $a_i n^i$ are separated by a large number of zeroes. Hence it is sufficient to prove the question for any one monomial $P(n) = a_i n^i$. Specifically, it is sufficient to prove this for the top coefficient, $a_{5772}n^{5772}$, since $a_{5772} \neq 0$.

Now replace n with $n+1$. Then according to the same argument, it is sufficient to prove the statement for the linear coefficient, $P(n) = 5772a_{5772}n$. Let $b = 5772a_{5772}$. Then we must only consider $P(n) = bn$ with $b \neq 0$. Now we choose powers of 2 and 5 such that $2^x 5^y b = 10^z c$, where c is disjoint with 10. Then we substitute $n \mapsto 2^x 5^y n$ and since nothing changes if we divide $P(n)$ by a power of 10, we are reduced to the case $P(n) = cn$, where c and 10 are disjoint.

Then c is invertible modulu any power of 10, specifically, 10^{2011}. Therefore, we are left with $P(n) = n$, which is trivial.

Question3 (10 points)

Calculate the limit:

$$\lim_{n\to\infty}\left(n\left(\frac{1}{2n+1}-\frac{1}{2n+2}+\frac{1}{2n+3}-\frac{1}{2n+4}+\cdots-\frac{1}{4n}\right)\right)$$

Solution:

Note that the expression in the limit is equal to:

$$\begin{aligned} n\sum_{m=n}^{m=2n-1}\frac{1}{(2m+1)(2m+2)} &= n\sum_{m=0}^{m=n-1}\frac{1}{(2m+2n+1)(2m+2n+2)} \\ &= \sum_{m=0}^{m=n-1}\frac{1}{(2\frac{m}{n}+2+\frac{1}{n})(2\frac{m}{n}+2+\frac{2}{n})}\frac{1}{n} \\ &= O(\frac{1}{n})+\sum_{m=0}^{m=n-1}\frac{1}{(2\frac{m}{n}+2)(2\frac{m}{n}+2)}\frac{1}{n} \\ &= O(\frac{1}{n})+\sum_{m=0}^{m=n-1}\frac{1}{4}\frac{1}{(x_m+1)^2}(x_{m+1}-x_m) \end{aligned}$$

Where $x_m = m/n$. Thus, by the definition of the integral:

$$\ldots \quad \to \quad \int_0^1 \frac{1}{4}\frac{1}{(x+1)^2}dx = \frac{1}{8}$$

Question4 (10 points)

For any m points $P_1 \ldots P_m \in \mathbb{R}^n$, let $A = (a_{ij})$ be the matrix defined by $a_{ij} = \|P_iP_j\|^2$. For a given n, what is the largest possible value of m for which A is invertible?

Solution:

We calim that the answer is $m = n+2$. We do this by showing that the rank of A is no more than $m+2$, and then showing that this rank is achieveable.

Indeed, let $\vec{u} = \underbrace{(1,\dots,1)}_{m}$, $\vec{w} = (P_1^2, \dots, P_m^2)$ and for all $1 \le i \le n$, $\vec{v}_i = ((P_1)_i, \dots (P_m)_i)$, where $(P_k)_i$ is the i-th component of P_k. Then we will show that $\vec{u}$, $\vec{w}$ and $\vec{v}_i$ span the rows of A. Indeed, since $a_{ij} = P_i^2 + P_j^2 - 2P_iP_j$ then the k-th row of A is: $P_k^2\vec{u} + \vec{w} - 2\sum_i (P_k)_i \vec{v}_i$, which is a linear combination of the above vectors. Thus the rank of A is no more than $n+2$.

To show that this is indeed achieveable, let P_i for $1 \le i \le n+1$ be the vertices of the regular simplex in n dimensions, and P_{n+2} its center. We want to calculate A for this configuration, so we will give an explicit construction for this. Add another dimension, so that we are in $\mathbb{R}^{n+1}$. Then consider the $n+1$ unit vectors together with the vector $(\frac{1}{n+1}, \dots, \frac{1}{n+1})$. Then they are all on one hyperplane (defined by the equation $x + y + z + \cdots = 1$), and form the vertices and center of the regular simplex in that hyperplane (which has dimension n). Hence we can consider them to be vectors in $\mathbb{R}^n$.

Thus, it is easy to see that the distance between any two vertices of the simplex is $\sqrt{2}$, and that the distance to its center is $\sqrt{\frac{n}{n+1}}$. Thus A is:

$$A = \begin{pmatrix} 0 & 2 & \cdots & 2 & \frac{n}{n+1} \\ 2 & 0 & \cdots & 2 & \frac{n}{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 0 & \frac{n}{n+1} \\ \frac{n}{n+1} & \frac{n}{n+1} & \cdots & \frac{n}{n+1} & 0 \end{pmatrix}$$

And it is trivial to see that this matrix has maximal rank. Indeed, take the sum of the top $n+1$ rows, and substract an appropriate multiple of the last row, to give us the vector $(0, \dots 0, 1)$. Thus it is sufficient to show that the rows of the following matrix are independent:

$$\begin{pmatrix} 0 & 2 & \cdots & 2 \\ 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 0 \end{pmatrix}$$

Which is a well known fact.

Question5 (10 points)

A word w is called *square-free* iff it has no subword repeating twice in a row. Can you compose a square-free word using the 22 letters of the hebrew alpha-bet such that for any letter that we add on the left, the resulting word will no longer be square-free? If so, what is the minimal length of such a series?

Solution:

The answer is yes. In general, suppose that we have an alpha-bet of n letters $a_1, \dots, a_n$. Define $w_1 = a_1$, and $w_i = w_{i-1}a_iw_{i-1}$. Then by induction, it is easy to see that w_i is square-free (if there is a subword repeating twice then it cannot contain a_i, thus w_i is not square-free) and also that for all i, a_iw_i contains a square. Thus, w_n is square-free, and since it begins with w_i for all i, then the word $a_i w_n$ contains a square. Thus we have an example, of length $2^n - 1$.

Now we will prove that this is optimal. Let w be a word as in the question. Now let w_i be the shortest word such that $a_i w$ starts with $(a_i w_i)^2$ (the square word must always contain the first letter, otherwise w is not square-free). So for all i, w starts with $w_i a_i w_i$. Now suppose that w_i begins with $w_j a_j w_j$. Then we will say that $j \prec i$. Also, it is obvious that for each $i \neq j$, w_i starts with w_j or vice versa. Hence, assume that w_{i+1} starts with w_i for all i. We will show that for any $i < j$, $i \prec j$. From there, we will be done by induction. So suppose the contrary, that is, there is some i such that $i - 1 \not\prec i$.

Then suppose $w_i = w_{i-1} a_{i-1} u$, where u might be empty. Then we have $w_i a_i w_i = w_{i-1} a_{i-1} u a_i w_i$ begins with $w_{i-1} a_{i-1} w_{i-1}$. Then since the second w_{i-1} is not contained in the first w_i (otherwise $i - 1 \prec i$), we have $w_{i-1} = u a_i v$, where v might be empty. Then $w_i = w_{i-1} a_{i-1} u = u a_i v a_{i-1} u$.

But then we have that w starts with

$$
\begin{aligned}
w_i a_i w_i &= (u a_i v a_{i-1} u) a_i (u a_i v a_{i-1} u) \\
&= u a_i v a_{i-1} (u a_i)(u a_i) v a_{i-1} u \\
&= u a_i v a_{i-1} (u a_i)^2 v a_{i-1} u
\end{aligned}
$$

Therefore, w is not square-free, a constradiction!

IMC 2012 Team Selection Exam

Duration: 4 hours

June 6, 2012

Question 1 (10 points)

Prove that the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ is positive definite.

Solution:

We substract the first row from the others, and do the same with the first column. This gives:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

Repeating the process for the smaller matrix, we are done.

Question 2 (10 points)

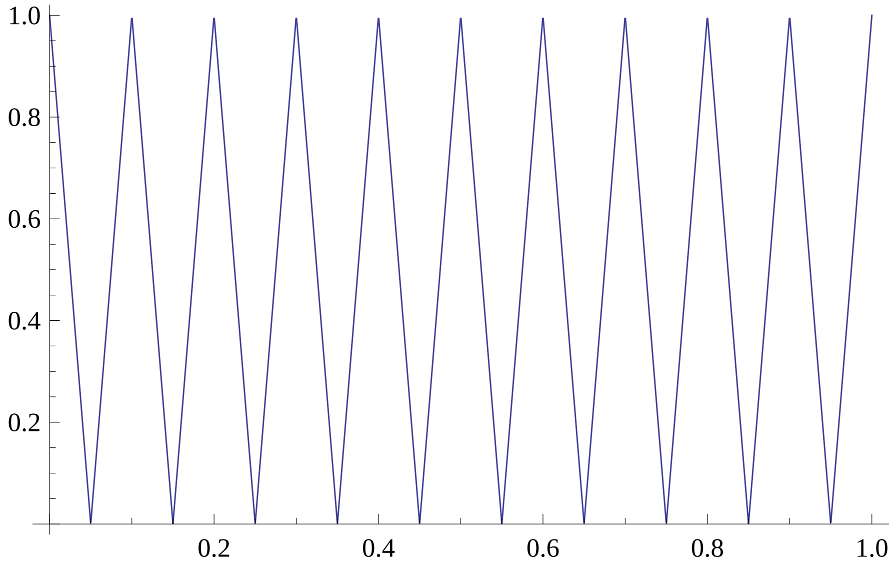
A graph of a continuous function $f : [0,1] \to [0,1]$ is a broken line consisting of 10 intervals. The graph of the function $f(f(x))$ is a broken line consisting of n intervals.

1. What is the maximal possible value of n?
2. What is the minimal possible value of n?

Solution:

1. The maximal value of n is 100. This can be easily seen by constructing a zigzag line that goes up and down many times, each time covering the entire domain:

$$f(x) = 2\left|(10x - \lfloor 10x \rfloor) - \frac{1}{2}\right|.$$



In this case, $f(x)$ covers its entire domain 10 times, and for each such time there are 10 intervals. Therefore, $f(f(x))$ is composed of a total of 100 intervals.

2. The minimal value of n is 1. Indeed, it is possible to construct such a function f which is its own inverse, meaning that $f(f(x)) = x$ has one interval. How do we construct such a function? Being its own inverse is equivalent to being symmetric around the line $y = x$. So take any 11 points symmetric with respect to that line, for example $\vec{x}_n = (\cos\frac{90^\circ n}{10}, \sin\frac{90^\circ n}{10})$, $0 \leq n \leq 10$, and the line f composed of the intervals between two such subsequent points is a funtion, inverse to itself.

Question 3 (10 points)

Let $p(x)$ be a non-constant polynomial with integer coefficients. Prove that there exists a natural number n, such that $p(n)$ has at least 5772 distinct prime divisors.

Solution:

Suppose that there was such a polynomial P. We call a prime q *good* if $P(x) \equiv 0 \pmod q$ has a solution modulo q. Now, suppose that there were 5772 good primes. By the Chinese Remainder theorem, there is a single value of x such that $P(x) = 0 \pmod q$ for each one of the 5772 good primes. In particular, $P(x)$ is divisible by at least 5772 distinct prime numbers, so we are done. As a result, there are no more than 5771 good primes.

However, note that by the definition of good primes, $P(x)$ is never divisible by a bad prime. Thus, we see that $P(x) \in \{q_1^{n_1} \cdots q_{5771}^{n_{5771}}\}$, where $q_1, \ldots, q_{5771}$ are the good primes. We will show that the set $A = \{q_1^{n_1} \cdots q_{5771}^{n_{5771}}\}$ is spread way too thinly for this to be possible.

Indeed, the number of elements in A smaller than N is significantly less than:

$$\log_{q_1} N \cdots \log_{q_{5771}} N = O(\log(N)^{5771}).$$

But, this means that the average distance between such consecutive elements is at least:

$$O\left(\frac{N}{\log(N)^{5771}}\right).$$

However, for values $x < M$, we have that $P(x) = O(M^d)$, where d is the degree of P, and the distance between consecutive values of P is $P(x+1) - P(x) < O(M^{d-1})$. But eventually, $O(M^{d-1}) < O\left(\frac{M^d}{\log(M^d)^{5771}}\right)$, so the distances grow too large for P to skip in one go, a contradiction - as we wanted to show.

Question 4 (10 points)
Let $K \subseteq \mathbb{R}^2$ be a convex shape, symmetric with respect to the origin. Suppose that $\iint_K \operatorname{dist}(\vec{x}, \partial K)\, \mathrm{d}^2\vec{x} > 2$, where $\operatorname{dist}(\vec{x}, \partial K)$ is the distance from $\vec{x}$ to the boundary of K. Prove that K contains at least 3 integer points.

Solution:

Suppose that $\iint_K \operatorname{dist}(\vec{x}, \partial K)\, \mathrm{d}^2\vec{x} > 2$. Now, if $\operatorname{dist}(\vec{0}, \partial K) > 1$, then K would contain a circle of radius more than one around the origin, and such a circle has at least 5 integer points in it, so we win. So, suppose that $\operatorname{dist}(\vec{0}, \partial K) \leq 1$.

We note two more facts: first of all, because K is symmetric, the function $f(\vec{x}) = \operatorname{dist}(\vec{x}, \partial K)$ is also symmetric: $f(-\vec{x}) = f(\vec{x})$. In addition, f is a convex function in K because K is convex (this can be very easily seen). In particular, the maximum of f in K is obtained at 0.

Now, consider the level curves of f, that is: $L_z = \{\vec{x} \in K | f(\vec{x}) = z\}$. Then $L_0 = \partial K$, and the curves are convex shapes contained in one another. In particular, if we denote by $\ell(L_z)$ the length of the curve L_z, then $\ell(L_z)$ is a monotonically decreasing function of z.

The last thing we note is that:

$$\iint_K 1 \cdot \mathrm{d}^2\vec{x} = \int_0^1 \ell(L_z)\, \mathrm{d}z,$$
$$\iint_K \operatorname{dist}(\vec{x}, \partial K)\, \mathrm{d}^2\vec{x} = \int_0^1 z\ell(L_z)\, \mathrm{d}z.$$

So, we see that:

$$\frac{\iint_K \operatorname{dist}(\vec{x}, \partial K)\, \mathrm{d}^2\vec{x}}{\iint_K 1 \cdot \mathrm{d}^2\vec{x}} = \frac{\int_0^1 z\ell(L_z)\, \mathrm{d}z}{\int_0^1 \ell(L_z)\, \mathrm{d}z}.$$

Hence, the right hand side is a weighted average of a quantity ranging from 0 to 1, with a weight which is monotonically decreasing. Thus, it is no more than $\frac{1}{2}$. As a result, we have:

$$\frac{\iint_K \operatorname{dist}(\vec{x}, \partial K)\, \mathrm{d}^2\vec{x}}{\iint_K 1 \cdot \mathrm{d}^2\vec{x}} \leq \frac{1}{2},$$

meaning that:

$$S = \iint_K 1 \cdot \mathrm{d}^2\vec{x} \geq 2 \iint_K \operatorname{dist}(\vec{x}, \partial K)\, \mathrm{d}^2\vec{x} > 4.$$

Therefore, by Minkowski's theorem, we are done.

Question 5 (10 points)
Consider words consisting of zeroes and ones. Several words of length 30 are considered

obscene. Can it happen that there exists an infinite periodic word, which contains no obscene subwords, and that all such infinite words are periodic, and their period is greater than 10^9?

Solution:

It is impossible.

Consider the following directed graph D: its vertices are all words of length 30, and two words u, v have a directed edge $u \to v$ iff there is some word w of length 29 and two letters a, b such that $u = aw$, $v = wb$. Thus, D is a directed graph such that each vertex has two ingoing and two outgoing edges. With this terminology, an infinite word is just an infinite path in this graph, and a word has an obscene subword iff it goes through a vertex corresponding to an obscene word.

We call a vertex *essentially obscene* iff all paths starting there must pass through an obscene word. The information given in the question means that this graph, with all essentially obscene words removed, is a single directed cycle of length at least 10^9. But the size of D is just 2^{30}, which is barely bigger than 10^9. So somehow, we removed almost no vertices from a graph with quite a lot of edges, and we are left with a cycle. There are many easy ways to show that this is impossible.

For example, for each essentially obscene vertex we remove, the number of edges decreases by no more than 4. Initially, there were 2^{31} edges. At the end, the graph is a cycle, so it has no more than 2^{30} edges. So at least 2^{28} vertices were removed. So there are no more than $2^{30} - 2^{28} = \frac{3}{4} \cdot 2^{30}$ vertices left. But there are at least 10^9 vertices left. Therefore, $\frac{3}{4} \cdot 2^{30} \geq 10^9$, which is plainly false.

Question 6 (10 points)

Let A, B be matrices with integer entries, such that $\det A = 1$.

1. Can we claim that $B^{-1}AB$ has integer entries?
2. Can we claim that there is a number n, such that $(B^{-1}AB)^n$ has integer entries?

Solution:

1. The answer is no. For example, if we let:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

   then we have:

$$B^{-1}AB = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}.$$

2. The answer is yes. Let us prove this. We will use the following two lemmas to reduce the question to one simple case, which we then easily solve.

Lemma 1. *If the theorem is true for all values of A and some specific value $B = B_1$, and also true for all values of A and some specific value $B = B_2$, then it is also true for all values of A and $B = B_1B_2$.*

Proof. By our assumption, there is some n_1 such that $(B_1^{-1}AB_1)^{n_1}$ has integer entries. But $(B_1^{-1}AB_1)^{n_1}$ has determinant 1 as well, so there is some n_2 such that $(B_2^{-1}(B_1^{-1}AB_1)^{n_1}B_2)^{n_2} = (B_2^{-1}B_1^{-1}AB_1B_2)^{n_1n_2}$ has integer entries. □

Lemma 2. *If $\det B = \pm 1$, then the theorem holds.*

Proof. In fact, $B^{-1} = \frac{1}{\det B} \operatorname{Adj} B$, where $\operatorname{Adj} B$ is the adjoint matrix to B which has integer entries, so B^{-1} has integer entries, meaning that the same applies to $B^{-1}AB$. □

Our proof will be by induction on $\det B$. If $\det B = \pm 1$, we have proven this above. Otherwise, there is some p such that $p|\det B$. Note that since we have seen that the statement is multiplicative on B, we can perform row operations with determinant ± 1 freely.

So, since $p|\det B$, let B_p be the reduction of B modulo p (so it is a matrix over the field $\mathbb{Z}_p$). Then, $\det B_p = 0$. Hence, there is some linear combination of the rows of B_p such that at least one of the rows (say, the first row) has coefficient 1, and the sum is equal to zero.

Going back to the original matrix B, this means that there is some linear combination of the rows of B such that at least one of the rows (say, the first row) has coefficient 1, and the sum is a vector whose entries are divisible by p. So, using row operations with determinant ± 1 (we add all other rows with the above coefficients to the first row), we bring B to a from where its first row is divisible by p. In that form,

$$B = \begin{pmatrix} p & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} B',$$

where B' has integer entries, and a smaller determinant. So by the induction hypothesis, it is sufficient to consider the case where

$$B = \begin{pmatrix} p & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Let us prove this case. We will show that there is some n such that all off-diagonal entries of A^n are divisible by p, which easily implies that

$$(B^{-1}AB)^n = B^{-1}A^nB = \begin{pmatrix} A_{11}^n & \frac{1}{p}A_{12}^n & \frac{1}{p}A_{13}^n & \dots \\ pA_{21}^n & A_{22}^n & A_{23}^n & \dots \\ pA_{31}^n & A_{32}^n & A_{33}^n & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

has integer entries.

In fact, we will show that when A is reduced modulo p to A_p, then there is some n such that $A_p^n = 1$. However, that is actually trivial - since $\det A = 1$ (this is the only place where we actually use this), then A_p is invertible modulo p. But the group $\mathrm{GL}(m, \mathbb{Z}_p)$ of $m \times m$ invertible matrices over $\mathbb{Z}_p$ has a finite order, so taking $n = \# \mathrm{GL}(m, \mathbb{Z}_p)$ is good enough. Note that this actually shows that it is sufficient to assume that $\det A$ and $\det B$ are coprime.

Good Luck!

First stage of Israeli students competition, 2012.

Duration – 4 hours.

1. Find the determinants of the following matrixes (the answer might depend on a)

a. $\begin{pmatrix} 1+a^2 & a & 0 & 0 & 0 & 0 \\ a & 1+a^2 & a & 0 & 0 & 0 \\ 0 & a & 1+a^2 & a & 0 & 0 \\ 0 & 0 & a & 1+a^2 & a & 0 \\ 0 & 0 & 0 & a & 1+a^2 & a \\ 0 & 0 & 0 & 0 & a & 1+a^2 \end{pmatrix}$ b. $\begin{pmatrix} 1+a^2+a^4 & a+a^3 & a^2 & 0 & 0 \\ a+a^3 & 1+a^2+a^4 & a+a^3 & a^2 & 0 \\ a^2 & a+a^3 & 1+a^2+a^4 & a+a^3 & a^2 \\ 0 & a^2 & a+a^3 & 1+a^2+a^4 & a+a^3 \\ 0 & 0 & a^2 & a+a^3 & 1+a^2+a^4 \end{pmatrix}$

2. A function f has the following property $\int_{-1}^{1} f(x)x^n dx = 0$ for n=0, 1, …, 5771.

Prove that f has at least 5772 roots in the interval $[-1,1]$.

3. ABC is a triangle. Consider all intervals PQ, such that P is on AB, Q is on BC, and PQ divides ABC into two parts of equal area. The union of all intervals PQ will be denoted by U. Compute $\frac{Area(U)}{Area(\mathrm{ABC})}$.

4. A sequence $\{a_n\}$ is such that $a_n > 0, \forall n$ and $a_n \underset{n\to\infty}{\to} 0$, however, $\sum_{n=1}^{\infty} a_n = \infty$.

Denote $s_n = \sum_{k=1}^{n} a_k$. Prove that $\sum_{k=1}^{\infty} \frac{a_k}{s_k^2}$ always converges.

5. Prove that there is an infinite quantity of natural numbers n such that n appears in the end of the decimal representation of 2^n (for example: 2^{36} ends with 36).

Second stage of Israeli students competition, 2012.

Duration – 4 hours.

1. Prove that the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ is positive definite.

2. A graph of a continuous function $f:[0,1]\to[0,1]$ is a broken line consisting of 10 intervals. The graph of the function $f(f(x))$ is a broken line consisting of n intervals.
(a) What is the maximal possible value of n?
(b) What is the minimal possible value of n?

3. Let $p(x)$ be a non-constant polynomial with integer coefficients. Prove that there exists a natural number n, such that $p(n)$ has at least 5772 distinct prime divisors.

4. Let $K \subseteq \mathbb{R}^2$ be a convex shape, symmetric with respect to the origin. Suppose that $\iint\limits_K dist(\vec{x},\partial K)\,d^2\vec{x} > 2$, where $dist(\vec{x},\partial K)$ is the distance from x to the boundary of K. Prove that K contains at least 3 integer points.

5. Consider words consisting of zeroes and ones. Several words of length exactly 30 are considered **obscene**. A word is called **patient** if it is of infinite length and does not contain obscene subwords. Can it happen that a patient word exists, and every patient word is periodic of period greater than 10^9?

6. Let A, B be invertible matrices with integer entries, such that $\det A = 1$.
(a) Can we claim that $B^{-1}AB$ has integer entries?
(b) Can we claim that there is a number n, such that $\left(B^{-1}AB\right)^n$ has integer entries?

Good luck!

IMC 2013 Team Preselection Exam

Duration: 4 hours

January 23, 2013

Question 1 (10 points)

Let A be an $n \times n$ matrix with real entries and non-zero determinant such that for each $v \in \mathbb{R}^n$, the vectors Av and v are orthogonal. Prove that for each $v \in \mathbb{R}^n$, the vectors A^2v and v do not form an acute angle.

Solution:

There are several approaches here. The simplest one is:

$$\begin{aligned} 0 &= \langle Av + v, A(Av + v)\rangle = \left\langle Av + v, A^2v + Av\right\rangle \\ &= \left\langle Av, A^2v\right\rangle + \langle Av, Av\rangle + \left\langle v, A^2v\right\rangle + \langle v, Av\rangle = 0 + \langle Av, Av\rangle + \left\langle v, A^2v\right\rangle + 0, \end{aligned}$$

so

$$\left\langle v, A^2v\right\rangle = -\langle Av, Av\rangle \le 0.$$

Question 2 (10 points)

Compute $\arctan^{(2012)}(0)$. (Reminder: $f^{(n)}$ means f derived n times.)

Solution:

We note that $\arctan'(x) = \frac{1}{1+x^2}$. However, it is well known that for $|x| < 1$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots - x^2011 + x^2012 - x^2013 + \ldots,$$

and the sum converges absolutely in this range. Therefore,

$$\left(\frac{1}{1+x^2}\right)^{(2012)} = 2012! - \frac{2013!}{1!}x + \frac{2014!}{2!}x^2 - \ldots.$$

As a result,

$$\arctan^{(2012)}(0) = \left(\frac{1}{1+x^2}\right)^{(2012)}\Bigg|_{x=0} = 2012!.$$

Question 3 (10 points)

An $n \times n$ table consists initially of zeroes. At each step, it is allowed to choose a 2×2 subs-quare and revert all numbers in it: all zeroes are replaced by ones, and all ones are replaced by zeroes. How many possible tables can be created this way?

Solution:

We note that the order at which we invert the sub-squares does not matter, and that inverting the same sub-square twice does nothing. So, for each of the $(n-1)^2$ different sub-squares, we need to decide whether to invert it or not, giving us $2^{(n-1)^2}$ possibilites. Let us show that all of these possibilities are distinct.

Indeed, suppose that we inverted some set S_1 of sub-squares, and then inverted some set S_2 of sub-squares, and obtained the same result, with $S_1 \neq S_2$. So, obviously this means that inverting S_1 and then inverting S_2 brings the table back to zero (because every cell is inverted either an even number of times in both of them, or an odd number of times in both of them). However, according to the above, this is the same as inverting all 2×2 sub-squares in $(S_1 \cup S_2) - (S_1 \cap S_2)$, which is not empty because $S_1 \neq S_2$.

So, we have inverted some non-empty set S of sub-squares and returned to a board of 0-s. But, let us look at the sub-squares in S whose top edge is highest (there may be more than one). Among those, look at the one whose left edge is leftmost. In that sub-square, look at the top-left corner. But that corner is contained in exactly one sub-square (because all other sub-squares are either below it or to its right). So it was inverted only once, and hence has the value 1. A contradiction!

Question 4 (10 points)

A regular tetrahedron has only integer vertices. Show that its edge length divided by $\sqrt{2}$ is an integer number.

Solution:

Let the length of the side of the tetrahedron be ℓ. Let $0, u, v, w$ be the vertices of the tetrahedron. Then $|u| = |v| = \ell$, and becuase the angle between u and v (as vectors) is $60°$, we see that

$$\frac{\ell^2}{2} = |u| \cdot |v| \cdot \cos(60°) = u \cdot v \in \mathbb{Z},$$

since $u, v \in \mathbb{Z}^2$.

However, it is well known that the volume V of the tetrahedron is equal to $V = \frac{\ell^3}{6\sqrt{2}}$. In addition, it is also known that the volume of the tetrahedron is $\frac{1}{6}$ of the the volume of the paralelipiped spanned by u, v, w:

$$V = \pm\frac{1}{6}\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \in \frac{1}{6}\mathbb{Z}.$$

Thus, we have $\frac{\ell^3}{6\sqrt{2}} \in \frac{1}{6}\mathbb{Z}$, or $\ell^3/\sqrt{2} \in \mathbb{Z}$. So,

$$\frac{\ell}{\sqrt{2}} = \frac{\ell^3/\sqrt{2}}{2 \cdot \ell^2/2} \in \mathbb{Q}.$$

But this means that $\left(\frac{\ell}{\sqrt{2}}\right)^2 = \ell^2/2 \in \mathbb{Z}$, with $\frac{\ell}{\sqrt{2}} \in \mathbb{Q}$, which is well known to imply that $\frac{\ell}{\sqrt{2}} \in \mathbb{Z}$.

Question 5 (10 points)

Let A be an infinite subset of $\mathbb{N}$. Prove that there exists a number $\alpha > 100$ such that $B = \{\lfloor \alpha^n \rfloor \mid n \in \mathbb{N}\}$ satisfies: $A \cap B$ is an infinite set.

Solution:

Let A be an infinite subset of $\mathbb{N}$. We will inductively define numbers β_j such that the set B defined by each one of them will contain more and more elements of A, and such that they converge to a limit.

We have two main observations: one is that we can always shift β_j slightly without damaging anything, and the second is that if we raise β_j to a high enough power, then that small shift can allow it to "hit" any sufficiently large element of A.

Formally, suppose that we have found a number $\beta_j > 100$, and numbers $a_1, \ldots, a_j \in A$, $n_1, \ldots, n_j \in \mathbb{N}$, such that $a_k \leq \beta_j^{n_k} < a_k + \frac{1}{2}$ for all $1 \leq k \leq j$.

Then there is some $\epsilon > 0$ such that $\beta_j^{n_k}(1+\epsilon)^{n_k} < a_k + \frac{1}{2}$ for all $1 \leq k \leq j$ (this is a finite number of right-open constraints). Choose $n_{j+1} \in \mathbb{N}$ so huge that

$$(1+\epsilon)^{n_{j+1}} > \beta_j,$$

and $n_{j+1} > n_j$. Then there is some $a_{j+1} \in A$ such that $a_{j+1} > \beta_j^{n_{j+1}}$. Increasing n_{j+1} if necessary, we may assume that

$$\beta_j^{n_{j+1}+1} \geq a_{j+1} > \beta_j^{n_{j+1}}.$$

So, choose $\beta_{j+1} = a_{j+1}^{1/n_{j+1}}$. So, $\beta_j < \beta_{j+1} \leq \beta_j^{1/n_{j+1}}\beta_j < (1+\epsilon)\beta_j$. Hence:

$$a_k \leq \beta_{j+1}^{n_k} < ((1+\epsilon)\beta_j)^{n_k} < a_k + \frac{1}{2}$$

for all $1 \leq k \leq j+1$ (note that $a_{j+1} > \beta_j^{n_{j+1}} > \beta_j^{n_j} \geq a_j$, so $a_{j+1} > a_j$).

We repeat the above inductively (finding $100 < \beta_1$ such that $\beta_1^{n_1} = a_1$ with $n_1 = 1$ and $a_1 \in A$ is obvious). So, we have an infinite sequence of numbers, $100 < \beta_1 \leq \beta_2 \leq \ldots$, $n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$, $a_1 < a_2 < a_3 < \cdots \in A$, such that

$$a_k \leq \beta_j^{n_k} < a_k + \frac{1}{2}$$

for all $1 \leq k \leq j$. In particular, $\beta_j^{n_1} < a_1 + \frac{1}{2}$ for all j, so β_j is a bounded increasing sequence. Let α be the limit. Then:

$$a_k \leq \alpha^{n_k} \leq a_k + \frac{1}{2}$$

for all k, so in particular $\lfloor \alpha^{n_k} \rfloor = a_k$ for all k.

University Students Olympiad, Stage 2

Duration - 4 hours

1. At the beginning of the game, on each cells of the lower half of the chessboard 8×8 is there is a white piece; on each cell of the upper half there is a black piece. On each move, the player is allowed to exchange two pieces on the cells that have a common side. In what minimal number of moves will he be able to move all black pieces to the lower half of the chessboard?

2. Numbers a,b,c,d are integer and each two of them are coprime. Consider two intervals in the plane: interval I whose endpoints are $(0,0)$, (a,b) and interval J whose endpoints are $(0,0),(c,d)$. Two points are called **similar**, if both coordinates of vector connecting them are integer. Let n be the number of pairs of points (P,Q) which are similar, and are internal points of I and J respectively. Compute n in terms of a,b,c,d.

3. Compute the integral $\int\limits_0^{\pi/2} \ln\left(4-\sin^2 x\right)dx$.

4. a. Compute the determinant $\left|\begin{pmatrix} a^2+ax+x^2 & a^2+ay+y^2 & a^2+az+z^2 \\ b^2+bx+x^2 & b^2+by+y^2 & b^2+bz+z^2 \\ c^2+cx+x^2 & c^2+cy+y^2 & c^2+cz+z^2 \end{pmatrix}\right|$.

b. Formulate and prove similar claim for $n\times n$ matrices instead of 3×3.

5. ABCDE is a non-planar pentagon in three-dimensional Euclidean space. All sides of ABCDE are of the same length. The angles A, B, C, D of the pentagon are right. Find the angle E (all possible values).

6. Denote by $\mathbb{R}_+$ the set of all nonnegative reals. We are given a sequence of functions $f_m:\mathbb{R}_+^m \to \mathbb{R}_+$, satisfying the following properties:

(a) Symmetry: $f_m\left(x_1,...,x_m\right)=f_m\left(x_{\sigma(1)},...,x_{\sigma(m)}\right)$ for any permutation σ.

(b) Monotonicity: $f_m\left(x,x_2,...,x_m\right)>f_m\left(y,x_2,...,x_m\right)$ if $x>y$.

(c) Homogeneous of degree 1: $f_m\left(\lambda x_1,...,\lambda x_m\right)=\lambda f_m\left(x_1,...,x_n\right)$, $\forall\lambda\in\mathbb{R}_+$.

(d) For any $k<m$:

$$f_m\left(x_1,...,x_k,x_{k+1},...,x_m\right)=f_m\left(f_k\left(x_1,...,x_k\right),...,f_k\left(x_1,...,x_k\right),x_{k+1},...,x_m\right).$$

(e) $f_2\left(0,1\right)=\frac{1}{2}$.

Prove that $f_m\left(x_1,x_2,...,x_m\right)=\frac{1}{m}\left(x_1+...+x_m\right)$ for each m.

Good luck!

University Students Olympiad, Stage 2

Duration - 4 hours

1. At the beginning of the game, on each cells of the lower half of the chessboard 8×8 is there is a white piece; on each cell of the upper half there is a black piece. On each move, the player is allowed to exchange two pieces on the cells that have a common side. In what minimal number of moves will he be able to move all black pieces to the lower half of the chessboard?

Solution. Let us invent the following "energy" function: let's number all lines from below, 1 to 8, and sum all line numbers of white pieces. The energy at the beginning is $(1+2+3+4)\cdot 8$ and it is lowest possible value.

The energy in the end is $(5+6+7+8)\cdot 8$ and it is highest possible.

The difference is $(4+4+4+4)\cdot 8 = 4\cdot 4\cdot 8 = 2^{2+2+3} = 2^7 = 128$.

By each move the energy either remains the same, or changes by 1. It is increased if the move is vertical and white goes up. Therefore the total number of moves is at least 128, and it in each move one white piece goes up it will be precise.

The strategy is, therefore, to perform only vertical moves in which white goes up and black goes down. In this case, we can promise that at each moment, there are precisely 4 pieces of each color in each column (since pieces never move to another column). So, unless in each column white pieces are the top 4, we shall be able to find a white piece below the black piece and to improve the situation by 1.

2. Numbers a,b,c,d are integer and each two of them are coprime. Consider two intervals in the plane: interval I whose endpoints are $(0,0)$, (a,b) and interval J whose endpoints are $(0,0),(c,d)$. Two points are called **similar**, if both coordinates of vector connecting them are integer. Let n be the number of pairs of points (P,Q) which are similar, and are internal points of I and J respectively. Compute n in terms of a,b,c,d.

Solution 1. Actually, n is the number of integer points in Minkowski difference I – J which are not on the boundary.

I – J is a parallelogram, its sides don't contain integer points; its vertices are integer. By Pick formula $n+\frac{4}{2}-1=S$ which is the area of the parallelogram. On the other hand, we know it is the absolute value of the determinant $|ad-bc|$. Hence $n+1=|ad-bc|$ the answer follows.

3. Compute the integral $\int_0^{\pi/2} \ln\left(4-\sin^2 x\right)dx$.

Solution. We shall use the method of parametric differentiation.

Let us consider $I(t)=\int_0^{\pi/2} \ln\left(t^2-\sin^2 x\right)dx$. We have to compute $I(2)$.

Let us compute $\frac{dI}{dt}=\frac{d}{dt}\int_0^{\pi/2} \ln\left(t^2-\sin^2 x\right)dx=\int_0^{\pi/2} \frac{2t}{t^2-\sin^2 x}dx$.

Differentiation inside the integral is allowed for $t>1$, because the function under the integral is bounded. So, if we take $\tan x = y$ then

$\sin^2 x=\frac{\tan^2 x}{1+\tan^2 x}$ and

$$dy = d\tan x = d\left(\frac{\sin x}{\cos x}\right)=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}dx=\left(1+\tan^2 x\right)dx$$

so, $\frac{dy}{1+y^2}=dx$ and

$$\frac{dI}{dt}=\int_0^{\pi/2}\frac{2t}{t^2-\sin^2 x}dx=\int_0^{\infty}\frac{2t}{t^2-\frac{y^2}{1+y^2}}\cdot\frac{dy}{1+y^2}dy=\int_0^{\infty}\frac{2t}{t^2+y^2t^2-y^2}dy=$$

$$=\int_0^{\infty}\frac{2t}{t^2+y^2t^2-y^2}dy=\int_0^{\infty}\frac{2t}{t^2+y^2\left(t^2-1\right)}dy=\frac{2}{t}\int_0^{\infty}\frac{1}{1+y^2\left(1-\frac{1}{t^2}\right)}dy=$$

$$=\frac{2}{t\sqrt{1-\frac{1}{t^2}}}\arctan\left(y\sqrt{1-\tfrac{1}{t^2}}\right)\Big|_0^{\infty}=\frac{2}{\sqrt{t^2-1}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{t^2-1}}$$

So $\frac{dI}{dt}=\frac{\pi}{\sqrt{t^2-1}}$.

Therefore $I(t)=\int \frac{\pi}{\sqrt{t^2-1}}dt$. If $t=\operatorname{ch} s$ then $dt=\operatorname{sh} s\cdot ds$ and

$\int \frac{1}{\sqrt{t^2-1}}dt=\int \frac{\operatorname{sh} s}{\operatorname{sh} s}ds=s+const$. Since $t=\frac{e^s+e^{-s}}{2}$ so $0=e^{2s}-2te^s+1$

therefore $e^s=t+\sqrt{t^2-1}$ and $s=\ln\left(t+\sqrt{t^2-1}\right)$.

So, $I(t)=\int \frac{\pi}{\sqrt{t^2-1}}dt=\pi\ln\left(t+\sqrt{t^2-1}\right)+const$.

To find the constant, let us look at large t.

$$I(t)-\pi\ln\left(t+\sqrt{t^2-1}\right)=\int_0^{\pi/2}\ln\left(t^2-\sin^2 x\right)dx-\pi\ln\left(t+\sqrt{t^2-1}\right)=$$

$$=\int_0^{\pi/2}\ln\left(t^2-\sin^2 x\right)dx-\tfrac{\pi}{2}\ln\left(t^2\right)+\pi\ln t-\pi\ln\left(t+\sqrt{t^2-1}\right)=$$

$$=\int_0^{\pi/2}\left(\ln\left(t^2-\sin^2 x\right)-\ln t^2\right)\cdot dx-\pi\ln\frac{t+\sqrt{t^2-1}}{t}=$$

$$=\int_0^{\pi/2}\ln\left(1-\frac{\sin^2 x}{t^2}\right)\cdot dx-\pi\ln\left(1+\sqrt{1-\tfrac{1}{t^2}}\right)\xrightarrow[t\to\infty]{}\int_0^{\pi/2}\ln 1\cdot dx-\pi\ln(2)=$$

$$=-\pi\ln 2$$

Since the difference is constant,

$$I(t)=\pi\ln\left(t+\sqrt{t^2-1}\right)-\pi\ln 2=\pi\ln\left(\frac{t+\sqrt{t^2-1}}{2}\right)$$

Now, we substitute $t=2$ and get the answer $I(2)=\pi\ln\left(1+\frac{\sqrt{3}}{2}\right)$.

4. a. Compute the determinant $\left|\begin{pmatrix} a^2+ax+x^2 & a^2+ay+y^2 & a^2+az+z^2 \\ b^2+bx+x^2 & b^2+by+y^2 & b^2+bz+z^2 \\ c^2+cx+x^2 & c^2+cy+y^2 & c^2+cz+z^2 \end{pmatrix}\right|$.

b. Formulate and prove similar claim for $n\times n$ matrices instead of 3×3.

Solution. Notice that:

$$\begin{pmatrix} a^2+ax+x^2 & a^2+ay+y^2 & a^2+az+z^2 \\ b^2+bx+x^2 & b^2+by+y^2 & b^2+bz+z^2 \\ c^2+cx+x^2 & c^2+cy+y^2 & c^2+cz+z^2 \end{pmatrix} =$$

$$= \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x & y & z \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x^2 & y^2 & z^2 \end{pmatrix}$$

If we multiply from the left by $\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}^{-1}$, and from the right by $\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}^{-1}$, we get the matrix

$$e_3 \cdot e_1^T + e_2 \cdot e_2^T + e_1 \cdot e_3^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, our matrix is product of 3 matrices: two Vandermonde matrices and one permutation matrix. The determinants of Vandermonde are known to be $(c-a)(c-b)(b-a)$ and $(z-y)(z-x)(y-x)$. The permutation is negative. Hence the answer:

$$-(c-a)(c-b)(b-a)(z-y)(z-x)(y-x).$$

b. The natural generalization is matrix depending on two sequences $a_1,\ldots,a_n$ and $x_1,\ldots,x_n$ whose elements are $\left(x_i^{n-1}+x_i^{n-2}a_j+\ldots+a_j^{n-1}\right)$. The proof is the same as in 3-dimensional case.

The answer is: $(-1)^{\left[\frac{n}{2}\right]}\prod_{i<k}(x_k-x_i)\prod_{j<\ell}\left(a_\ell-a_j\right)$.

If we take polynomials $\left(x_i^m+x_i^{m-1}a_j+\ldots+a_j^m\right)$, and $m<n-1$, then the determinant is zero, since the matrix is a sum of $m-1$ matrixes of rank 1 (by the same argument as above), and the rank is $m-1$ at most. If we take $m\geq n$, I don't know the answer.

5. ABCDE is a non-planar pentagon in three-dimensional Euclidean space. All sides of ABCDE are of the same length. The angles A, B, C, D of the pentagon are right. Find the angle E (all possible values).

Answer. Either 60° or $\arccos\left(\frac{5}{14}\right)$.

Solution. WLOG, length of each side is 1 (if not, apply homothety).
The intervals EB, EC are $\sqrt{2}$. Therefore, the triangle BEC is given.

One option is that ABCD is a square, and ADE is a regular triangle, such that plane ABCD is orthogonal to the plane ADE. In this case, angle E is 60^{o}, or $\frac{\pi}{3}$ radians. In this case, we can choose Cartesian coordinates in space, s. t. $A=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $B=\begin{pmatrix}0\\0\\0\end{pmatrix}$, $C=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $D=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $E=\begin{pmatrix}\frac{1}{2}\\1\\\frac{\sqrt{3}}{2}\end{pmatrix}$.

In any other case, we can still assume that coordinates of B,C,E are the same (since the triangle BCE is unique up to symmetry), and ask whether there are other possible options for A, D. On one hand, A should be in distance 1 from both B and E. This condition specifies intersection of two unit spheres with centers at B and E, or a circle in the plane of orthogonal bisector of BE. If A belongs to this circle, the condition of right angle at A is automatically satisfied, since the sides of triangle ABE are 1, 1, and $\sqrt{2}$, hence Pythagoras theorem holds.
Still, there is another condition on A: the angle ABC should be right. This condition specifies a plane through B, orthogonal to BC. Hence A should be on intersection of plane with a circle which lies in another (non-parallel) plane. This can happen only in two places A_1 which is the same as the A we guessed and A_2 which is symmetric to it with respect to the plane BCE.
Similar things can be said about the location of D: there is D_1 that we have guessed and there is D_2 symmetric with respect to BCE. It is easy to check, that at these points all conditions are satisfied. Therefore, there are only 4 possible pentagons, and they are symmetric in pairs. Pentagon A_1BCD_1E we have discussed already, it leads to 60^{o}, its symmetric image A_2BCD_2E gives the same answer. It remains to discuss A_2BCD_1E (or

A_1BCD_2E; they are symmetric so it is enough to discuss one of them).

The equation of plane through $B=\begin{pmatrix}0\\0\\0\end{pmatrix}$, $C=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $E=\begin{pmatrix}\frac{1}{2}\\1\\\frac{\sqrt{3}}{2}\end{pmatrix}$ can be written as $p(x,y,z)=\frac{1}{\sqrt{7}}\left(\sqrt{3}\cdot y-2\cdot z\right)=0$.

Its unit normal vector is $N=\begin{pmatrix}0\\\frac{\sqrt{3}}{\sqrt{7}}\\-\frac{2}{\sqrt{7}}\end{pmatrix}$. Then $p(A_1)=N\cdot A_1=-2/\sqrt{7}$.

The symmetric point $A_2=A_1-2(N\cdot A_1)N=\begin{pmatrix}0\\1\\0\end{pmatrix}-2\dfrac{\sqrt{3}}{\sqrt{7}}\begin{pmatrix}0\\\frac{\sqrt{3}}{\sqrt{7}}\\-\frac{2}{\sqrt{7}}\end{pmatrix}=\begin{pmatrix}0\\\frac{1}{7}\\\frac{4\sqrt{3}}{7}\end{pmatrix}$.

The angle A_2ED_1 is between the vector $ED_1=\begin{pmatrix}-\frac{1}{2}\\0\\\frac{\sqrt{3}}{2}\end{pmatrix}$ the vector

$EA_2=\begin{pmatrix}0\\\frac{1}{7}\\\frac{4\sqrt{3}}{7}\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\1\\\frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\-\frac{6}{7}\\\frac{\sqrt{3}}{14}\end{pmatrix}$. Both are unit vectors; hence the cosine of each angle is the scalar product: $\cos\phi=\dfrac{1}{4}+\dfrac{3}{28}=\dfrac{10}{28}=\dfrac{5}{14}$. Therefore the result is $\arccos\left(\frac{5}{14}\right)$.

6. Denote by $\mathbb{R}_+$ the set of all nonnegative reals. We are given a sequence of functions $f_m:\mathbb{R}_+^m\to\mathbb{R}_+$, satisfying the following properties:

(a) Symmetry: $f_m(x_1,\ldots,x_m)=f_m\left(x_{\sigma(1)},\ldots,x_{\sigma(m)}\right)$ for any permutation σ.

(b) Monotonicity: $f_m(x,x_2,\ldots,x_m)>f_m(y,x_2,\ldots,x_m)$ if $x>y$.

(c) Homogeneous of degree 1: $f_m(\lambda x_1,\ldots,\lambda x_m)=\lambda f_m(x_1,\ldots,x_n)$, $\forall\lambda\in\mathbb{R}_+$.

(d) For any $k<m$:

$$f_m(x_1,\ldots,x_k,x_{k+1},\ldots,x_m)=f_m\left(f_k(x_1,\ldots,x_k),\ldots,f_k(x_1,\ldots,x_k),x_{k+1},\ldots,x_m\right).$$

(e) $f_2(0,1)=\frac{1}{2}$.

Prove that $f_m(x_1, x_2, ..., x_m) = \frac{1}{m}(x_1 + ... + x_m)$.

Sketch of solution. The problem is hard, and it would take much time to reduce everything to the axioms. Therefore we shall skip the trivial steps. Denote $R_{k,m} = f_m(\underbrace{1,...,1}_{k\ times}, \underbrace{0,...,0}_{m-k\ times})$. Then it follows from axioms that

$$f_{m^s}(\underbrace{1,....,1}_{k^s\ times},0,...,0) = f_{m^s}(\underbrace{R_{k,m},....,R_{k,m}}_{m\cdot k^{s-1}\ times},0,...,0) =$$

$$= R_{k,m} \cdot f_{m^s}(\underbrace{1,....,1}_{m\cdot k^{s-1}\ times},0,...,0) = \left(R_{k,m}\right)^2 \cdot f_{m^s}(\underbrace{1,....,1}_{m^2\cdot k^{s-2}\ times},0,...,0) =$$

...

$$= \left(R_{k,m}\right)^s \cdot f_{m^s}(1,...,1) = \left(R_{k,m}\right)^s$$

In particular, $f_{2^n}(1,0,...,0) = \dfrac{1}{2^n}$ and $f_{2^n}(\underbrace{1,...,1}_{2^m},0,...0) = \dfrac{2^m}{2^n}$ by similar argument. If in $f_n(1,...,1,0,...,0)$ we shall reduce the number of ones and increase the number of zeroes (maybe changing the n) the result will be reduced; if we increase the number of ones and reduce the number of zeroes, then the result will be increased; these follow easily from monotonicity.

Therefore if $k^s > 2^p, m^s < 2^q$, then $R_{k,m}^s < 2^{p-q}$.

Similarly if $k^s < 2^{p'}, m^s > 2^{q'}$, then $R_{k,m}^s > 2^{p'-q'}$.

Let us logarithm the last two statements:

If $s\log_2 k > p,\ \ s\log_2 m < q$, then $\log R_{k,m} < \dfrac{p-q}{s}$.

If $s\log_2 k < p',\ \ s\log_2 m > q'$, then $\log R_{k,m} > \dfrac{p'-q'}{s}$.

Obviously, if $p = \lfloor s\log_2 k \rfloor, p' = \lceil s\log_2 k \rceil, q' = \lfloor s\log_2 m \rfloor, q = \lceil s\log_2 m \rceil$, then the difference between the two bounds of $\dfrac{p'-q'}{s} < \log R_{k,m} < \dfrac{p-q}{s}$ is no greater than $\dfrac{4}{s}$, and both are within distance at most $\dfrac{2}{s}$ from $\log_2 k - \log_2 m = \log_2\left(\frac{k}{m}\right)$. Therefore $R_{k,m} = \frac{k}{m}$.

First stage of Israeli students competition, 2013.

Duration: 4 hours

1. Let A be an $n \times n$ matrix with real entries and nonzero determinant, such that for each $v \in \mathbb{R}^n$, vectors Av, v are orthogonal. Prove, that for each $v \in \mathbb{R}^n$, vectors $A^2 v$, v don't form an acute angle.

2. Compute $\arctan^{(2013)}(0)$. (Reminder: $f^{(n)}$ means: f derived n times).

3. A table $n \times n$ consists initially of zeroes. At each steps, it is allowed to choose a sub-square 2×2 and revert all numbers in it: all zeroes are replaces by ones, and all ones are replaced by zeroes. How many possible tables can be created in that way?

4. A regular tetrahedron has only integer vertices. Show that its edge length divided by $\sqrt{2}$ is an integer number.

5. Let A be an infinite subset of $\mathbb{N}$. Prove that there exists a number $\alpha > 100$, such that $\mathrm{B} = \left\{ \lfloor \alpha^n \rfloor \,\middle|\, n \in \mathbb{N} \right\}$ satisfies: $\mathrm{A} \cap \mathrm{B}$ is an infinite set.

Good luck!

First stage of Israeli students competition, 2014.

Duration: 4 hours

1. A real number a is given. Find the greatest possible number of elements in the following sets:

a. $\{\sin x \mid \sin 3x = a\}$. b. $\{\sin x \mid \sin 4x = a\}$.

2. Compute $\det\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 1 & 5 & 3 \\ 3 & 5 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{pmatrix}$.

3. A grasshopper lives on a unit interval. Each second, a grasshopper chooses one of the endpoints of the unit interval (by flipping a coin) and jumps two-thirds of the way towards that endpoint (so the distance from him to one of the endpoints becomes precisely three times smaller).
130 spiders, each of the size 0.0005, can choose any positions on the interval simultaneously and stay there. Is it possible for them to catch the grasshopper?

4. For any $\varepsilon > 0$, consider the locus of points, satisfying the inequalities

$$\begin{cases} x^2 + y^2 + z^2 \le 1 \\ \left|x^2 + y^2 - z^2\right| \le \varepsilon \end{cases}$$

The volume of that locus will be denoted V_ε.

Does the $\lim\limits_{\varepsilon \to 0} \frac{V_\varepsilon}{\varepsilon}$ exist?

5. For any polynomial p with real coefficients, let

$$S(p) = \{x \in \mathbb{R} \mid p(x) \in \mathbb{Z}\}.$$

Prove that if p, q are two polynomials, such that $S(p) = S(q)$, then either $p + q$ or $p - q$ is a constant.

6. On a bookshelf, there are N tomes of the Encyclopedia in random order. Each hour, a librarian takes a tome which stands not on its place, and puts it in its place. Show that the process will stop.

Good luck!

Second Stage of Israeli Olympiad for University Students.

1. On a plane N lines in **general position** are chosen. General position means, that no two lines are parallel, no three lines are concurrent, and neither three intersection points belong to the same unchosen line.
An additional line will be called good, if it doesn't pass through any intersection points of chosen lines. Additional lines will be considered equivalent, if one can be obtained from the other by continuous motion, such that all intermediate lines are good as well. Find the number of equivalence classes of good lines.

2. Evaluate the integral $\int_0^{\infty}\left(e^{-2x}-e^{-x}\right)\frac{dx}{x}$.

3. Let $a,b,c,d,v\in\mathbb{R}^3$. Show that
$\langle[a,b],v\rangle\cdot\langle[c,d],v\rangle-\langle[a,c],v\rangle\cdot\langle[b,d],v\rangle+\langle[a,d],v\rangle\cdot\langle[b,c],v\rangle=0$.
Here $\langle,\rangle$ is scalar product and $[,]$ is vector product.

4. Let $\alpha>0$ be an irrational number, $\beta\in(0,1)$. Denote:

$$Q(m)=\min_{1\le n\le m}\{n\alpha\}$$
$$R(m)=\min_{1\le n\le m}\{\beta-n\alpha\}$$

(where n runs over integer numbers between 1 and m).
Prove that there is infinite number of values m, such that $Q(m)>R(m)$.

5. What is the maximal possible area of an ellipse, which is contained in the upper half of a unit circle?

6. Let p be an odd prime, and let $GL_2(\mathbb{F}_p)$ be the set of all invertible 2×2 matrices over the field with p elements. A partition of $GL_2(\mathbb{F}_p)$ will be called **nice** if every two matrices belonging to the same set commute. Determine the minimal number of sets in the nice partition.

Good luck!

Second Stage of Israeli Olympiad for University Students.

1. On a plane N lines in **general position** are chosen. General position means, that no two lines are parallel, no three lines are concurrent, and neither three intersection points belong to the same unchosen line.
An additional line will be called good, if it doesn't pass through any intersection points of chosen lines. Additional lines will be considered equivalent, if one can be obtained from the other by continuous motion, such that all intermediate lines are good as well. Find the number of equivalence classes of good lines.

Answer. $\dfrac{n^4+2n^3-21n^2+10n+8}{8}$

Solution. We shall use projective duality. Consider central projection of the plane *P* where all the lines lift to sphere *S*. Each line will be projected to a *big circle*, i.e. circle of maximal radius. To each big circle one can associate a point - like pole associated to equator. If we glue together pairs of opposite points on sphere we get a *projective plane*.

Concurrent lines correspond to collinear points and vice versa. Using duality, we transform the problem to following one:

n points in general position on projective plane are given. General position *means that no 3 of them are collinear (and not any two of them are collinear together with the point, corresponding to infinite line (which has no projection on the plane P)), but we don't use this condition. Consider lines joining pairs of these points. Suppose no 3 of them are concurrent apart to the given points. In how many ways one can add one extra point non-collinear with any two others, if two positions are* equivalent *if one can be obtained to another by continuous motion such that in all intermediate positions no 3 points are collinear?*

This problem can be reformulated yet again in the following way:

n points in general position on projective plane are given. General position *means that no 3 of them are collinear. Consider lines joining pairs of these points. Suppose no 3 of them are concurrent apart from the given points. On how many regions they divide projective plane?*

Let us count. We have $m=\frac{n(n-1)}{2}$ lines joining these points. If they are in general position, they will divide ordinary plane on $\frac{m(m+1)}{2}+1$ part, $2m$ of them are infinite, $\frac{m(m-3)}{2}+1$ are finite. On the projective plane opposite infinite parts are glued, and we get m infinite and $\frac{m(m-3)}{2}+1$ finite parts, $\frac{m(m-1)}{2}+1$ parts in total.

However, our lines are not generic. Via each given point $n-1$ given lines are passing. If we slightly move them to a general position we get division of plane by $n-1$ lines with $\frac{(n-1)(n-4)}{2}+1$ bounded parts which will collapse when we return these lines to their initial place. Hence in each of n points we lose $\frac{(n-1)(n-4)}{2}+1$ parts, totally $n\left(\frac{(n-1)(n-4)}{2}+1\right)$.

So the number of parts M is equal

$$M=\frac{m(m-1)}{2}+1-n\left(\frac{(n-1)(n-4)}{2}+1\right)=$$
$$=\frac{\frac{n(n-1)}{2}\cdot\left(\frac{n(n-1)}{2}-1\right)}{2}-\frac{n(n-1)(n-4)}{2}-(n-1)=$$
$$=\frac{n-1}{8}\left(n^2(n-1)-2n+4n(n-4)-8\right)=$$
$$=\frac{n-1}{8}\left(n^3-n^2-2n+4n^2-16n-8\right)=\frac{n-1}{8}\left(n^3+3n^2-18n-8\right)$$

It is easy to see that the second term is indecomposable over $\mathbb{Q}$

2. Evaluate the integral $\int\limits_0^\infty\left(e^{-2x}-e^{-x}\right)\frac{dx}{x}$.

Solution. Let's consider the more general problem,

$$I(a)=\int_0^\infty \left(e^{-ax}-e^{-x}\right)\frac{dx}{x}.$$

By differentiating we get

$$\frac{dI}{da}=\int_0^\infty -e^{-ax}dx.$$

The integral in the right hand side is absolutely convergent; therefore it can be integrated to obtain the original expression, hence differentiating inside the integral was justified.

$$\frac{dI}{da}=\int_0^\infty -e^{-ax}dx=\left.\frac{e^{-ax}}{a}\right|_0^\infty=0-\frac{1}{a}=0.$$

It is clear even to a hedgehog that $I(1)=0$. Hence

$$I(a)=\int_1^a \frac{dI}{da}\cdot da=-\int_1^a \frac{da}{a}=-\ln(a)$$

(we omit | | here, since for $a<0$ the integral is not defined anyway).

Hence $\int_0^\infty \left(e^{-2x}-e^{-x}\right)\frac{dx}{x}=I(2)=-\ln 2$.

3. Let $a,b,c,d,v\in\mathbb{R}^3$. Show that
$\langle[a,b],v\rangle\cdot\langle[c,d],v\rangle-\langle[a,c],v\rangle\cdot\langle[b,d],v\rangle+\langle[a,d],v\rangle\cdot\langle[b,c],v\rangle=0$.
Here $\langle,\rangle$ is scalar product and $[,]$ is vector product.

Solution. Denote the expression in the left-hand side by $f(a,b,c,d)$.

We claim, that f is anti-symmetric, which means that

$$f(v_1,v_2,v_3,v_4)=\operatorname{sgn}\sigma\cdot f\left(v_{\sigma(1)},v_{\sigma(2)},v_{\sigma(3)},v_{\sigma(4)}\right).$$

for each permutation σ of 4 indexes. Each anti-symmetric multi-linear expressions in 4 vectors in $\mathbb{R}^3$ is identically zero. Indeed, we could decompose each vector as a linear combination of vectors of the standard basis, and represent $f(v_1,v_2,v_3,v_4)$ as a sum of $f(e_\alpha,e_\beta,e_\gamma,e_\delta)$ with coefficients (where e_1,e_2,e_3 is the standard basis and $\alpha,\beta,\gamma,\delta$ are indexes) and in each summands at least one index is repeated among $\alpha,\beta,\gamma,\delta$. If two among the vectors substituted in the form coincide and the form is anti-symmetric, the value is zero.

It is clear that our f is multi-linear, since both vector and scalar products are multi-linear. So, it remains to show it is anti-symmetric. Each permutation can be composed of just three transpositions $(1,2)$, $(1,3)$ and $(1,4)$ (in some order, maybe some of the transpositions are used more than once or not used at all). Therefore it is enough to verify the formula for only these 3 transpositions:

$$\begin{aligned}f(b,a,c,d)=&\langle[b,a],v\rangle\cdot\langle[c,d],v\rangle-\langle[b,c],v\rangle\cdot\langle[a,d],v\rangle+\\&+\langle[b,d],v\rangle\cdot\langle[a,c],v\rangle=-\langle[b,a],v\rangle\cdot\langle[c,d],v\rangle+\\&+\langle[b,d],v\rangle\cdot\langle[a,c],v\rangle-\langle[b,c],v\rangle\cdot\langle[a,d],v\rangle=-f(a,b,c,d)\end{aligned}$$

$$\begin{aligned}f(c,b,a,d)=&\langle[c,b],v\rangle\cdot\langle[a,d],v\rangle-\langle[c,a],v\rangle\cdot\langle[b,d],v\rangle+\\&+\langle[c,d],v\rangle\cdot\langle[b,a],v\rangle=-\langle[b,c],v\rangle\cdot\langle[a,d],v\rangle+\\&+\langle[a,c],v\rangle\cdot\langle[b,d],v\rangle-\langle[c,d],v\rangle\cdot\langle[a,b],v\rangle=-f(a,b,c,d)\end{aligned}$$

$$\begin{aligned}f(d,b,c,a)=&\langle[d,b],v\rangle\cdot\langle[c,a],v\rangle-\langle[d,c],v\rangle\cdot\langle[b,a],v\rangle+\\&+\langle[d,a],v\rangle\cdot\langle[b,c],v\rangle=\langle[b,d],v\rangle\cdot\langle[a,c],v\rangle-\\&-\langle[c,d],v\rangle\cdot\langle[a,b],v\rangle-\langle[a,d],v\rangle\cdot\langle[b,c],v\rangle=-f(a,b,c,d)\end{aligned}$$

This completes the verification.

4. Let $\alpha>0$ be an irrational number, $\beta\in(0,1)$. Denote:

$$Q(m)=\min_{1\le n\le m}\{n\alpha\}$$
$$R(m)=\min_{1\le n\le m}\{\beta-n\alpha\}$$

(where *n* runs over integer numbers between 1 and *m*).
Prove that there is infinite number of values m, such that $Q(m)>R(m)$.

Solution. We have to prove that for any natural N there is $n > N$ such that $Q_n > R_n$. The case when $R_n = 0$ for some n is obvious, so we suppose that $R_n > 0$ for all n, i.e. $\beta \neq \{k\alpha\}$ $\forall k \in \mathbb{N}$. Because the set $\{n\alpha\}_{n=0}^{\infty}$ is dense, there exist $n \in \mathbb{N}$ such that $\{\beta - n\alpha\} \le \min(\beta, R_n)$. Among such n we shall choose minimal n_0. Then

$$\{\beta - n_0\alpha\} < \min(\beta, R_n) \le \{\beta - n\alpha\},\ \forall n \in [1, n_0 - 1].$$

Note that $n_0 > N$. Otherwise by the definition of R_n we would have an inequality $R_n \le \{\beta - n\alpha\}$. In that case if $R_n < Q_{n_0}$ then taking $n_0 = n$ we get just what we need.

Now we can suppose that $R_{n_0} \ge Q_{n_0} = \{m\alpha\}$ for some $m \in [1, n_0]$. Then

$$R_{m_0+m} \le \{\beta - \{n_0 + m\}\alpha\} = R_{n_0} - Q_{n_0} \qquad (1)$$

This holds because of equality

$$\{\beta - (n_0 + m)\alpha\} = \{\{\beta - n_0\alpha\} - \{m\alpha\}\} = \{R_{n_0} - Q_{n_0}\} = R_{n_0} - Q_{n_0}$$

We also have $Q_{n_0+m} = Q_{n_0} = \{m\alpha\}$.

Indeed. Suppose the contrary, i.e. $\{k\alpha\} < Q_{n_0}$ for some $k \in [n_0 + 1, n_0 + m]$. Then $0 \le n_0 + m - k < n_0$ and

$$\{\beta - (n_0 + m - k)\alpha\} < \{\beta - (n_0 + m)\alpha\} + \{k\alpha\} = R_{n_0} - Q_{n_0} + \{k\alpha\} < R_{n_0}$$

This contradicts to the choice of n_0.

Now let $n_1 = n_0 + m$.

If $R_{n_1} < Q_{n_1}$ we can put $n + n_1$ and we are done. Otherwise if $R_{n_1} \ge Q_{n_1}$ we can replace n_0 by n_1 in the equations (1) and (2) and proceed. Then put $n_2 = n_1 + m = n_0 + 2m$ etc.

Finally for some k we get $R_{n_k} < Q_{n_k} - Q_{n_0}$ because if $R_{n_k} \ge Q_{n_k}$ for any k then using (1) and (2) (n_k substituted instead of n_0) we get

$$Q_{n_0} = Q_{n_0+m} = Q_{n_0+2m} = \ldots = Q_{n_0+km} \le R_{n_0} - kQ_{n_0}$$

But for sufficiently large k we have $R_{n_0} - k \cdot Q_{n_0}$. This provides final contradiction.

5. What is the maximal possible area of an ellipse, which is contained in the upper half of a unit circle?

In the space of ellipses (center, orientation, and lengths of the axes) the set of admissible ellipses is compact; therefore a maximal ellipse exists.

First, it is obvious that the maximal ellipse should touch the boundary of the semicircle in at some points, such that their convex hull would contain the center of the ellipse. Otherwise, the ellipse can be moved to a certain direction and then homothetically expanded.

Let us prove that the maximal ellipse touches the diameter which is the boundary of the semicircle in the center. If not, we can move the ellipse a little bit towards the center, so that it will be no longer tangent to the circle but still won't contain the center. Then the ellipse can be expanded once again, so that it still won't contain the center and won't touch the circle. Of course, the ellipse can protrude outside the given semicircle, but it is still contained in some semicircle, since by convexity some line through the center of the circle doesn't intersect the ellipse. Now we can rotate the expanded ellipse back into the semicircle.

Now, there is a tangency point on the left and on the right half of the arc.

Assume the points are different; we'll denote them A and B.

Denote by ℓ the equation of the line AB and by ℓ_A, ℓ_B the equation of tangents at A and B. It is easy to see that all quadrics, passing through A, B and tangent to the circle there, are of the form $\lambda \cdot \ell_A \cdot \ell_B + \mu \cdot \ell^2$, and hence both the circle and the ellipse are symmetric with respect to the perpendicular bisector of AB.

Since the ellipse passes through the center of the circle, then AB is horizontal. Choose the natural coordinate system (the origin in the center of the circle, y upwards). Then the tangency points are $(\pm\cos t, \sin t)$.

We shall compute the area of the ellipse as a function of t.

Consider the affine transformation $T:(x,y)\mapsto(\alpha x,y)$ that will turn this ellipse into a circle. Before the transformation, the ellipse is inscribed in the triangle whose vertices are $\left(-\frac{1}{\cos t},0\right)$, $\left(\frac{1}{\cos t},0\right)$, $\left(0,\frac{1}{\sin t}\right)$, and the points of tangency are $(\pm\cos t,\sin t)$ and $(0,0)$.

After the action of T the ellipse becomes the incircle of the triangle, and the computation is easy.

But we have to find α. The tangents from $(\cos t,\sin t)$ to the ellipse should become equal, therefore the distances from $\left(\frac{\alpha}{\cos t},0\right)$ to both $(\alpha\cos t,\sin t)$ and $(0,0)$ should become equal. Hence

$$\left(\alpha\left(\cos t-\tfrac{1}{\cos t}\right)\right)^2+\sin^2 t=\frac{\alpha^2}{\cos^2 t}$$

and therefore by a standard computation,

$$\alpha^2=\frac{\sin^2 t}{1+\sin^2 t}$$

Now, the radius of the incircle can be obtained from the formula $S=pr$, where S is the area of the triangle, p is one-half its perimeter, and r is the radius of the incircle. From here, the area of the incircle is given by πr^2, and the area of the original ellipse becomes $\pi r^2/\alpha$. We compute:

$$S=\frac{1}{2}\cdot\frac{1}{\sin t}\cdot\frac{2\alpha}{\cos t}=\frac{\alpha}{\cos t\cdot\sin t}\text{ and }p=\frac{\alpha}{\cos t}+\sqrt{\frac{\alpha^2}{\cos^2 t}+\frac{1}{\sin^2 t}}.$$

This gives the area of the original ellipse is as follows:

$$f(t)=\frac{\pi S^2}{p^2\alpha}=\ldots=\pi\cdot\frac{\sin t}{\left(1+\sin^2 t\right)^{3/2}}$$

Deriving and finding the maximum gives

$$f'(t)=\pi\cdot\frac{1-\sin^2 t}{Z},\text{ where } Z>0.$$

where Z is positive. This gives $t=45^\circ$, and $f(t)=\dfrac{2\pi}{\sqrt{27}}$.

Note that we have neglected the case where the ellipse is tangent to the circle only in one point, at $(0,1)$. In that case we may thicken the ellipse sideways until it is double tangent at that point, and hence this becomes a limit case of $f(t)$ where $t\to 1$, which not the maximum of f.

6. Let p be an odd prime, and let $GL_2(\mathbb{F}_p)$ be the set of all invertible 2×2 matrices over the field with p elements. A partition of $GL_2(\mathbb{F}_p)$ will be called **nice** if every two matrices belonging to the same set commute. Determine the minimal number of sets in the nice partition.

Answer. p^2+p+1

First solution. Let $\alpha\in\mathbb{F}_p$ a non-square, and set

$$a=\begin{pmatrix}1&0\\0&2\end{pmatrix},\ b=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ c=\begin{pmatrix}0&1\\\alpha&0\end{pmatrix}.$$

Their centralizers are easy to compute:

$$A:=Z_{GL_2(\mathbb{F}_p)}(a)=\left\{\begin{pmatrix}\beta&0\\0&\gamma\end{pmatrix}\middle|\beta,\gamma\in\mathbb{F}_p^\times\right\}$$

$$B:=Z_{GL_2(\mathbb{F}_p)}(b)=\left\{\begin{pmatrix}\beta&\gamma\\0&\beta\end{pmatrix}\middle|\beta\in\mathbb{F}_p^\times,\gamma\in\mathbb{F}_p\right\}$$

$$C:=Z_{GL_2(\mathbb{F}_p)}(c)=\left\{\begin{pmatrix}\beta&\gamma\\\alpha\gamma&\beta\end{pmatrix}\middle|\beta,\gamma\in\mathbb{F}_p,\ (\beta,\gamma)\neq(0,0)\right\}$$

This can be verified by an explicit computation, but also can be deduced from linear algebra: when a linear operator L is represented as a block matrix with smallest possible blocks, and all blocks are essentially different, then centralizer consists of matrixes with the same blocks, and in each block of size k we have a polynomial of degree $k-1$ at most in a relevant block of L.

Subgroups A,B,C are **commutative** (since they consist of linear expressions in respectively a,b,c). Let

$$\mathbf{A} := \left\{ gag^{-1} \middle| g \in GL_2\left(\mathbb{F}_p\right) \right\}$$
$$\mathbf{B} := \left\{ gbg^{-1} \middle| g \in GL_2\left(\mathbb{F}_p\right) \right\}$$
$$\mathbf{C} := \left\{ gcg^{-1} \middle| g \in GL_2\left(\mathbb{F}_p\right) \right\}$$
$$\mathbf{X} := \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$$

As the centralizer of each element of $\mathbf{X}$ is commutative, the commuting relation on $\mathbf{X}$ is an equivalence relation.

We claim that every element of $GL_2\left(\mathbb{F}_p\right)$ commutes with at least one element of $\mathbf{X}$. It is known from linear algebra that every 2×2 matrix can be transformed by conjugation to one of the following three canonical forms: $\begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}, \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \beta & \gamma \\ \alpha\gamma & \beta \end{pmatrix}$.

As the centralizer of each element of $\mathbf{X}$ is commutative, this proves that the minimal size of a nice partition of $GL_2\left(\mathbb{F}_p\right)$ and of $\mathbf{X}$ is the same. Indeed, we can extend every nice partition of $\mathbf{X}$ to a nice partition of $GL_2\left(\mathbb{F}_p\right)$ by adding each element $y \in GL_2\left(\mathbb{F}_p\right) \setminus \mathbf{X}$ to a set in the partition which contains some $x \in \mathbf{X}$ which commutes with y.

It remains to determine the number of sets needed for a nice partition of $\mathbf{X}$. As the commuting relation is an equivalence on $\mathbf{X}$, this is the same as determining the number of equivalence classes in $\mathbf{X}$ modulo the commuting relation.

We claim that no element in $\mathbf{A}$ commutes with an element of $\mathbf{B}$, no element in $\mathbf{A}$ commutes with an element of $\mathbf{C}$ and no element in $\mathbf{C}$ commutes with an element of $\mathbf{B}$.

Indeed, every element which commutes with some gag^{-1} is diagonalizable over $\mathbb{F}_p$, while non-scalar matrices commuting with some gbg^{-1} are not diagonalizable and non-scalar matrices commuting with gcg^{-1} are diagonalizable over $\mathbb{F}_{p^2}$ but not over $\mathbb{F}_p$.

Therefore it's enough to count the equivalence classes in **A**, **B**, **C** separately.

Lemma 1. The number of equivalence classes in **A** is

$$\frac{(p^2-1)(p^2-p)}{2(p-1)^2}=\frac{p(p+1)}{2}.$$

Proof. The only conjugate of a in A is $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$, since A is diagonal matrices, and diagonal elements are eigenvalues which are uniquely defined up to order. By symmetry, the size of each equivalence class in **A** is 2. So $|\mathbf{A}|=|GL_2(\mathbb{F}_p)/A|=\frac{|GL_2(\mathbb{F}_p)|}{|A|}=\frac{(p^2-1)(p^2-p)}{(p-1)^2}$. The last number should be divided by 2.

Lemma 2. The number of equivalence classes in **B** is

$$\frac{(p^2-1)(p^2-p)}{p(p-1)^2}=p+1.$$

Proof. The only conjugates of b in B are $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$, where $x\in\mathbb{F}_p^\times$, since every conjugate of b has eigenvalue 1 with multiplicity 2. By symmetry, the size of each equivalence class in **B** is $p-1$. As

$$|\mathbf{B}|=|GL_2(\mathbb{F}_p)/B|=\frac{|GL_2(\mathbb{F}_p)|}{|B|}=\frac{(p^2-1)(p^2-p)}{p(p-1)}$$

To complete the proof, we divide $|\mathbf{B}|$ by the size of equivalence classes, which is $p-1$.

Lemma 3. The number of equivalence classes in **C** is

$$\frac{(p^2-1)(p^2-p)}{2(p^2-1)}=\frac{p^2-p}{2}.$$

Proof. The only conjugates of b in B are $\begin{pmatrix} 0 & 1 \\ \pm\alpha & 0 \end{pmatrix}$. This is because the eigenvalues of a matrix $\begin{pmatrix} \beta & \gamma \\ \alpha\gamma & \beta \end{pmatrix}$ are $\beta \pm \gamma\sqrt{\alpha}$ and this equals $\pm\alpha$ iff $\beta = 0$ and $\gamma = \pm 1$. By symmetry, the size of each equivalence class in $\mathbf{C}$ is 2. So $|\mathbf{C}| = \left|GL_2(\mathbb{F}_p)/C\right| = \frac{|GL_2(\mathbb{F}_p)|}{|C|} = \frac{(p^2-1)(p^2-p)}{(p^2-1)}$, which should be divided by 2.

To summarize, the minimal size of a nice partition is

$$\frac{p^2+p}{2} + (p+1) + \frac{p^2-p}{2} = p^2 + p + 1.$$

Second solution. As we have seen in the beginning of the first solutions, centralizer of non-scalar 2×2 matrix A consists of all possible (non-singular) matrices of the form $\lambda I + \alpha A$.

Notice that $gl_2(\mathbb{F}_p)/I$ is a 3-dimensional linear space, having a non-singular matrix in every line through the origin. Hence we need a separate class of partition for each point of a projective plane over $\mathbb{F}_p$, therefore we need precisely $p^2 + p + 1$ classes.

First stage of Israeli students competition, 2015.

Duration: 4 hours

אנחנו מבקשים לכתוב את הפתרונות באנגלית, אבל זה בסדר מדי פעם להשתמש בעברית
כאשר אתם מתקשים לנסח הוכחה מורכבת כלשהי. את השאלון ניתן לקחת איתכם.

1. Compute $\lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\ldots+\dfrac{1}{\sqrt{n}}}$.

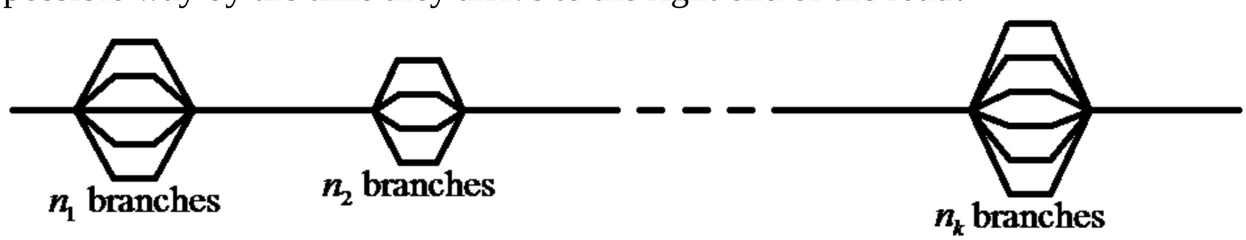
2. Compute $\det\begin{pmatrix}1&2&2&3\\2&2&3&3\\2&3&3&4\\3&3&4&4\end{pmatrix}$.

3. Let A, B, C, D be points in 3-dimensional Euclidean space not in the same plane, such that the plane ACB is orthogonal to the plane ACD, and the plane ABD is orthogonal to the plane CBD. Prove that $\dfrac{\cos(\sphericalangle ACB)}{\cos(\sphericalangle ADB)}=\dfrac{\cos(\sphericalangle CBD)}{\cos(\sphericalangle CAD)}$.

4. A finite number of polyhedrons of positive volume in 3-dimensional Euclidean space is given. Prove that one can mark a finite number of points in the same Euclidean space, so that strictly inside any two of given polyhedrons of equal volume, there will be the same number of marked points, and every given polyhedron will contain at least one point.

5. Prove that sum of digits of $2^{4^{1000001}}$ is greater than 1000000.

6. M cars move from left to right on a narrow road (they can't overtake each other, and cannot go backwards, all cars start at the left end and arrive to the right end). In k places, the road is split in parallel routes: first in n_1 parallel branches which are merged again, then in n_2 parallel branches, etc. Each branch is long enough to contain any amount of cars. For which M is it possible to reorder the cars in any possible way by the time they arrive to the right end of the road?



Good luck!

Solutions: first stage of Israeli students competition, 2015.

1. Compute $\lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}}$.

Answer. $\dfrac{1}{2}$.

First solution. By Stolz–Cesàro criterion criterion (similar to the more familiar l'Hôpital's rule but for sequences), if $\lim\limits_{n\to\infty}\dfrac{\sqrt{n}-\sqrt{n-1}}{\frac{1}{\sqrt{n}}}$ exists, then it also gives an answer to the original question. But

$$\lim_{n\to\infty}\frac{\sqrt{n}-\sqrt{n-1}}{\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{\left(\sqrt{n}+\sqrt{n-1}\right)\left(\sqrt{n}-\sqrt{n-1}\right)}{\left(\sqrt{n}+\sqrt{n-1}\right)\frac{1}{\sqrt{n}}}=\lim_{n\to\infty}\frac{1}{1+\sqrt{\frac{n-1}{n}}}=\frac{1}{1+1}=\frac{1}{2}$$

Second solution. Denote $L=\lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}}$. Then

$$\frac{1}{L}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{\frac{1}{n}}}+\frac{1}{\sqrt{\frac{2}{n}}}+\frac{1}{\sqrt{\frac{3}{n}}}+\ldots+\frac{1}{\sqrt{1}}\right)=$$

$$=\int_0^1\frac{dx}{\sqrt{x}}=2\sqrt{x}\Big|_0^1=2$$

Hence $L=\frac{1}{2}$.

2. Compute $\det\begin{pmatrix}1&2&2&3\\2&2&3&3\\2&3&3&4\\3&3&4&4\end{pmatrix}$.

First solution. We may subtract row 3 from row 4, and row 1 from row 2 without changing the determinant. We get

$$\det\begin{pmatrix}1&2&2&3\\2&2&3&3\\2&3&3&4\\3&3&4&4\end{pmatrix}=\det\begin{pmatrix}1&2&2&3\\1&0&1&0\\2&3&3&4\\1&0&1&0\end{pmatrix}$$

Now two of the rows (the second and the last) are the same, so det = 0.

Second solution. Notice that sum of two numbers in the middle is the same as the sum of two numbers at the ends of each row, so $\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$ is in the kernel, so determinant is zero.

3. Let *A*, *B*, *C*, *D* be points in 3-dimensional Euclidean space not in the same plane, such that the plane *ACB* is orthogonal to the plane *ACD*, and the plane *ABD* is orthogonal to the plane *CBD*. Prove that $\dfrac{\cos(\sphericalangle ACB)}{\cos(\sphericalangle ADB)}=\dfrac{\cos(\sphericalangle CBD)}{\cos(\sphericalangle CAD)}$.

Solution. Consider projection from *AD* to *BC* (each point *X* on *AD* is sent to the foot of perpendicular from *X* to *BC*). Length of each interval on *AD* is reduced by this projection by the same coefficient, which is $\cos\sphericalangle\left(\overrightarrow{AD},\overrightarrow{BC}\right)$.

This projection can be performed in two steps: first project point *X* on *AD* to point *Y* on *AC* (so that $XY\perp AC$, and then project point *Y* on *AC* to point *Z* on *BC*, so that $YZ\perp CD$. Indeed, planes *ACB*, *ACD* are given to be orthogonal, so *Y* is the projection of *X* to plane *ACB*, but *XY* is orthogonal to plane *ACB*, and to the line *BC*, therefore *X* and *Y* lie in the same plane, orthogonal to *BC*, hence projections of *X* and of *Y* to the line *BC* are in the same place. But now each projection is performed from a line to a line within the same plane, and each reduces distances by a coefficient which is cosine of an angle between two intersecting lines, so

$$\cos\sphericalangle\left(\overrightarrow{AD},\overrightarrow{BC}\right)=\cos\sphericalangle CAD\cdot\cos\sphericalangle ACB.$$

Now doing the same for the projection from *CD* to *AB*, using *BD* as intermediate line and the fact the plane *ABD* is orthogonal to the plane *CBD*, we conclude:

$$\cos\sphericalangle\left(\overrightarrow{CD},\overrightarrow{AB}\right)=\cos\sphericalangle ADB\cdot\cos\sphericalangle CBD.$$

Therefore $\cos\sphericalangle CAD\cdot\cos\sphericalangle ACB=\cos\sphericalangle ADB\cdot\cos\sphericalangle CBD$.

Q.E.D.

4. A finite number of polyhedrons of positive volume in 3-dimensional Euclidean space is given. Prove that one can mark a finite number of points in the same Euclidean space, so that strictly inside any two of given polyhedrons of equal volume, there will be the same number of marked points, and every given polyhedron will contain at least one point.

Solution. The number of marked points in each region (intersection of some polyhedrons and of complements to the other polyhedrons) can be denoted by a letter. Totally, there will be at most 2^N letters (some regions might be empty), where N is the number of polyhedrons, and each pair of polyhedrons of equal volume gives a linear equation. The equations are linear, and have integer coefficients, hence the solutions of the equations is linear subspace, spanned by rational vectors (which can be found by Gauss procedure). The equations have a positive solution: when to each region corresponds the volume of the region. It can be expressed as a linear combination of the rational basis vectors of the subspace, with possibly real coefficients. If the coefficients can be replaced by sufficiently close rational coefficients, we shall get a rational vector, which also has positive coordinates. Multiplying the vector by common denominator of the coordinates we get a vector with positive integer coordinates, which satisfies all the equations.

We can mark number of points in each region, according to the respective coordinates of the vector, and all conditions will be satisfied.

5. Prove that sum of digits of $2^{4^{1000001}}$ is greater than 1000000.

Solution. The last (units) digit is nonzero. It is not possible to have 3 consequent zeroes before the last digit, because the number consisting of 4 last digits has to be divisible by 16, but the last digit isn't divisible by 16. Let **u** be the number consisting of the last k digits. It is not possible to have $3k$ consequent zeroes before **u**. Otherwise **u** is divisible by $2^{4k} = 16^k > \mathbf{u} \neq 0$. So the number has at least $\log_4 d$ nonzero digits, where d is the total number of digits. The number of digits is

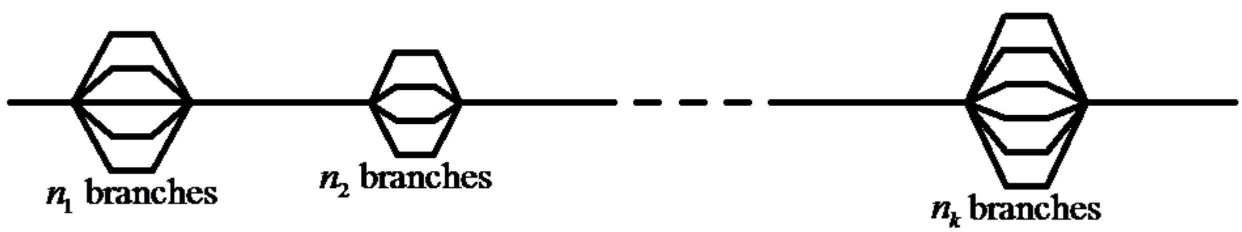
$$\log_{10}\left(2^{4^{1000001}}\right) = 4^{1000001} \cdot \log_{10} 2 \geq 4^{1000001} \cdot 0.3 > 4^{1000000}.$$

Remark. Then number 2 in this problem might be replaced by any even number which is not divisible by 5, or any odd number which is divisible by 5.

6. M cars move from left to right on a narrow road (they can't overtake each other, and cannot go backwards, all cars start at the left end and arrive to the right end). In k places, the road is split in parallel routes: first in n_1 parallel branches which are merged again, then in n_2 parallel branches, etc.

Each branch is long enough to contain any amount of cars.

For which M is it possible to reorder the cars in any possible way by the time they arrive to the right end of the road?



Answer. When $M \leq n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

Solution. The trajectory of each car is completely determined by the choice of branches, which has $N = n_1 \cdot n_2 \cdot \ldots \cdot n_k$ possibilities. If $M > N$, then by pigeonhole principle two of the cars have the same trajectory, hence they will be in the same order in the end as in the beginning.

Now we shall show that for $M \le N$, any rearrangement is possible. We shall give each car a k-digit number. The last (least signifcant) digit of a number may be anything between 1 and n_1, the second least significant digit may be anything between 1 and n_2, and so on, the leading digit may be anything between 1 and n_k. If $n_i \ge 10$ we shall invent new digits.

The total number of possible numbers is N, so if $M \le N$ we can assign each car a different number; we shall do it in such a way, that a car with a smaller number will be a car that should arrive earlier.
For each possible order of arrivals, such numbering is possible.

The driving will be according to the following three rules:

(a) All cars should arrive to the i'th split road and take its place on one of n_i parallel branches, before any car is allowed to continue to the next split road.

(b) The first to live i'th split road are the cars on the first parallel branch, the second are the cars of second parallel branch, the third are the cars of third parallel branch and so on.

(c) The decision for each car of which branch to take on the i'th split road is based on i'th least significant digit of its number (1 means first branch, 2 means second branch, and so on).

It is easy to see, that the less is the most significant digit of a number, the sooner the car will arrive, and given that first several digits of numbers of some two cars are the same, the car will arrive sooner if its next digit is less. Which means that the cars will arrive in order according to their numbers. Since numbers can be given in any possible order, the cars can arrive in any possible order.

Second stage of Israeli students competition, 2015.

Duration: 4 hours

1. Find such $x>0$, for which $\int\limits_0^x \frac{dt}{t^{1+\ln t}} = \int\limits_x^\infty \frac{dt}{t^{1+\ln t}}$.

2. N people must travel from one end of the road to another. The length of the road is L. They have K bicycles (K < N). The velocity of walking man is v_1, and the velocity of a bicycle is v_2 (obviously, $v_1 < v_2$). How much time is required?

3. A unit cube in 4-dimensional Euclidean space contains a 3-dimensional Euclidean ball of radius R. What is the greatest possible value of R?

4. The sequence $\{a_n\}$ is defined by recurrent formula $a_{n+1} = a_n + \sqrt{1+a_n^2}$, and $a_1 = 1$. Compute $\lim\limits_{n\to\infty} \frac{2^n}{a_n}$.

5. Polynomials $P(x)$ and $Q(x)$ of odd degree are such that for each integer x there is an integer y such that $P(x)=Q(y)$. Prove that there exists a polynomial R, such that $P(x)=Q(R(x))$ for each x.

6. For given 2×2 matrices A,B there is only finite number n of 2×2 matrices X such that $X^2 + AX + B = 0$. Find the maximal possible value of n. (All matrices in this questions have complex entries.)

Good luck!

Solutions: second stage of Israeli students competition, 2015.

1. Find such $x>0$, for which $\int\limits_0^x \frac{dt}{t^{1+\ln t}} = \int\limits_x^\infty \frac{dt}{t^{1+\ln t}}$.

Answer. $x=1$.

First solution. We shall start with general remarks on convergence. When $t\to 0$ we have $\ln t < -2$, so $\frac{1}{t^{1+\ln t}} < t$, so integral is well-defined at 0. As $t\to\infty$, $\ln t > 2$, so $\frac{1}{t^{\ln t}} < \frac{1}{t^2}$, so the integral converges at ∞.

Notice, that the integrated function is positive, so as x is increasing, the left hand side increasing, and right hand side is decreasing. So there can be only one answer. Since the integral is well-defined at both ends, the LHS is sufficiently small when x is close to zero, and RHS is sufficiently small when x is large, so by continuity an answer exists.

Consider a substitution $s=\frac{1}{t}$. Then $\ln t = -\ln s$, and $dt = -\frac{ds}{s^2}$, but we can skip the minus sign if we revert the endpoints of the integral (which is a logical thing to do, since the substitution reverses the order. So we get

$$\int\limits_{1/x}^\infty s^{1-\ln s}\frac{ds}{s^2} = \int\limits_0^{1/x} s^{1-\ln s}\frac{ds}{s^2}$$

$$\int\limits_{1/x}^\infty s^{1-\ln s}\frac{ds}{s^2} = \int\limits_0^{1/x} s^{1-\ln s}\frac{ds}{s^2}$$

$$\int\limits_{1/x}^\infty s^{1-\ln s}\frac{ds}{s^2} = \int\limits_0^{1/x} s^{1-\ln s}\frac{ds}{s^2}$$

$$\int\limits_{1/x}^\infty \frac{ds}{s^{1+\ln s}} = \int\limits_0^{1/x} \frac{ds}{s^{1+\ln s}}$$

So, if x is an answer then $\frac{1}{x}$ is also an answer. But the answer is unique, so $x=\frac{1}{x}$, hence $x=1$.

Second solution. We will apply the substitution $\ln t = y$, which means $t=e^y$. Then $dt = \frac{dy}{y}$, and the new condition is $\int\limits_{-\infty}^{\ln x} \frac{dy}{\left(e^y\right)^y} = \int\limits_{\ln y}^\infty \frac{dx}{\left(e^y\right)^y}$.

$$\int_{-\infty}^{\ln x} e^{-y^2}dy = \int_{\ln x}^{\infty} e^{-y^2}dy$$

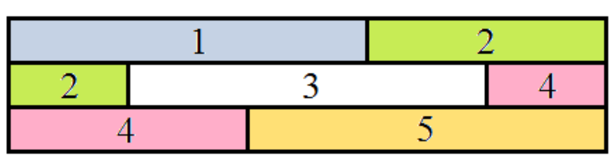
The integral $\int e^{-y^2}dy$ is famous (especially in probability theory) , but it is not an elementary function. The function e^{-y^2} is even, positive and quickly decreasing, so it is obvious that the only point which cuts the integral in half is 0.
Hence $\ln x = 0$, and $x = 1$.

2. N people must travel from one end of the road to another. The length of the road is L. They have K bicycles ($K < N$). The velocity of walking man is v_1, and the velocity of a bicycle is v_2 (obviously, $v_1 < v_2$). How much time is required?

Answer. $\frac{L}{N}\left(\frac{K}{v_2} + \frac{N-K}{v_1}\right)$

Solution. We shall introduce natural coordinates on the road: the first end is zero, and the target end is L. The total displacement of the bicycles is KL at most, and that happens only if all bicycles make all the way from 0 to the target (if someone fancies riding a bicycle in the opposite direction for some reason, it is regarded as negative displacement). So, one of the people who was the least advanced by the bicycles, made $\frac{KL}{N}$ at most by the bicycle, and the rest of the way, $L - \frac{KL}{N}$ at least, by foot, so he spent no less than $\frac{KL}{N}\Big/v_2 + \left(L - \frac{KL}{N}\right)\Big/v_1 = \frac{L}{N}\left(\frac{K}{v_2} + \frac{N-K}{v_1}\right)$.

The hard part is to prove that this number can be achieved. It is easy to guess from the first part of the proof, that in order to transport all people in this amount of time, all of them must constantly move forward, and do precisely $\frac{K}{N}$ of the way by bicycle. We shall build a table of height K and length L, and we shall pack it with blocks of height 1 and length $\frac{KL}{N}$.



In the picture there is an example for $K = 3, N = 5$. The first block is in the first line but it doesn't take the whole line; each block starts in the same place, where the previous block stops, but if it is too long, then part of the block for which there is not enough place in the current line, is chopped away and moved to the beginning of the next line. So, all blocks have the same length, even if some are divided.

This table we've built is a schedule of bicycle usage (a term schedule usually means time-table, but in our case it is distance-table). Blocks correspond to people, lines of the table correspond to bicycles; the horizontal direction to the locations on the road. So, on this table we see which bicycle on which part of the road can be used by which person.

If someone arrives to a spot, where he (according to our schedule) has to take a bicycle, the previous owner of the bicycle enjoyed more bicycle-time than he, so he is more advanced along the road, so he has already left him a bicycle in precisely this spot. So this schedule can be implemented. Hence each person can do precisely $\frac{K}{N}$ of the way by bicycle, and then they all can arrive in the time we computed.

3. A unit cube in 4-dimensional Euclidean space contains a 3-dimensional Euclidean ball of radius R. What is the greatest possible value of R?

Answer. $\frac{1}{\sqrt{3}}$.

Solution. Coordinates in $\mathbb{R}^4$ will be denoted x_1, x_2, x_3, x_4, and we can assume the unit cube is $\left[-\frac{1}{2}, \frac{1}{2}\right]^4$. Consider a hyperplane $a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 + a_4 \cdot x_4 = s$.

Without loss of generality, we can assume that $\sum_{i=1}^{4} a_i^2 = 1$. In short, we can describe the hyperplane by the equation $\langle n, x \rangle = s$, where $n = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$.

Consider also a vector $v = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ b_4 \end{pmatrix}$, such that $\langle v_4, n \rangle = 0.$

It means in coordinates that $0 = a_1^2 + a_2^2 + a_3^2 + a_4 b_4 = 1 - a_4^2 + a_4 b_4 = 0$, so $b_4 = \frac{a_4^2 - 1}{a_4} = a_4 - \frac{1}{a_4}$.

Therefore $|v|^2 = a_1^2 + a_2^2 + a_3^2 + b_4^2 = 1 - a_4^2 + a_4^2 - 2 + \frac{1}{a_4^2} = \frac{1}{a_4^2} - 1.$

We can find a three vectors v_1, v_2, v_3 which are of unit length, and orthogonal to each other and to n, such that $v_1 = \frac{v}{|v|}$. Since v_2, v_3 are orthogonal to both v and n, they are orthogonal both to $\frac{v+n}{2} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{pmatrix}$ and to $\frac{v-n}{2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ t \end{pmatrix}$, (where $t = -\frac{1}{a_4}$) so they have zero last coordinate. Any vector parallel to the hyperplane can be expressed as $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, and the length of the vector is $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ but its projection to the x_4-axis is α_1 times the last coordinate of v_1. The diameter of the 3-dimensional ball of radius R in the hyperplane, which has the longest projection on the x_4-axis, is parallel to the vector $2Rv_1 = 2R\frac{v}{|v|}$, and its last coordinate is

$$2R\frac{b_4}{|v|} = {2R\left(a_4 - \frac{1}{a_4}\right)}\Bigg/{\sqrt{\frac{1}{a_4^2} - 1}} = {2R\left(a_4^2 - 1\right)}\Big/{\sqrt{1 - a_4^2}} = -2R\sqrt{1 - a_4^2}$$

To have the ball inside the unit cube, we should have $2R\sqrt{1 - a_4^2} \le 1$, therefore $\sqrt{1 - a_4^2} \le \frac{1}{2R}$, hence $1 - \frac{1}{4R^2} \le a_4^2$.

But similar argument holds for each coordinate, hence $1-\frac{1}{4R^2}\le a_i^2$, so $4-\frac{1}{R^2}\le a_1^2+a_2^2+a_3^2+a_4^2=1$, hence $3\le\frac{1}{R^2}$, so $R\le\frac{1}{\sqrt{3}}$.

Just to be sure, let us verify, that a 3-dimensional ball of radius $\frac{1}{\sqrt{3}}$ can be inserted into the 4-dimensional unit cube. It is easy to guess the hyperplane; that is the case when all inequalities we wrote turn to the equalities. So, take the hyperplane $x_1+x_2+x_3+x_4=0$, and in it take a ball of radius $\frac{1}{\sqrt{3}}$. We have to verify that

$$\begin{cases} x_1+x_2+x_3+x_4=0 \\ x_1^2+x_2^2+x_3^2+x_4^2\le\frac{1}{3} \end{cases}$$

implies $-\frac{1}{2}\le x_i\le\frac{1}{2}$ for each i, but by symmetry it is enough to verify for $i=4$ by symmetry, also, it is possible to revert sign of all x_i simultaneously, and so it is enough to show that $x_4\le\frac{1}{4}$.

Obviously $(x_1-x_2)^2+(x_1-x_3)^2+(x_2-x_3)^2\ge 0$, hence

$$2\left(x_1^2+x_2^2+x_3^2\right)\ge 2\left(x_1x_2+x_1x_3+x_2x_3\right)$$

$$3\left(x_1^2+x_2^2+x_3^2\right)\ge x_1^2+x_2^2+x_3^2+2\left(x_1x_2+x_1x_3+x_2x_3\right)=\left(x_1+x_2+x_3\right)^2$$

$$x_1^2+x_2^2+x_3^2\ge\frac{1}{3}\cdot\left(x_1+x_2+x_3\right)^2=\frac{1}{3}\cdot\left(-x_4\right)^2=\frac{x_4^2}{3}$$

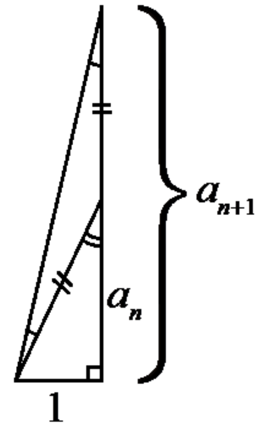
$$\frac{1}{3}\ge x_1^2+x_2^2+x_3^2+x_4^2\ge\frac{4x_4^2}{3}$$

$$\frac{1}{4}\ge x_4^2.$$

Q.E.D.

Remark. We could argue that in our example all inequalities in the first discussion turn into equalities, and skip some algebra, but it is good to verify an argument in independent way and so to make sure that we didn't have an arithmetic mistake.

4. The sequence $\{a_n\}$ is defined by recurrent formula $a_{n+1} = a_n + \sqrt{1+a_n^2}$, and $a_1 = 1$. Compute $\lim\limits_{n\to\infty} \frac{2^n}{a_n}$.



Answer. $\frac{2}{\pi}$.

First solution. The formula becomes clear, if you look at its geometric meaning. Construct a right-angled triangle, the short sides of which are 1 and a_n, and the long side is, by Pythagoras theorem, $\sqrt{1+a_n^2}$. The iteration of the process is prolonging the a_n side by the same length as the hypotenuse. This means, we append an isosceles triangle to our right-angle triangle. So, by an almost obvious angle computation, the angle opposite to the side of length 1 becomes half of what it was with each step. Since we start with $45^\circ = \frac{\pi}{4}$ at step1, the angle at step n is $\frac{\pi}{2^{n+1}}$. So

$$\lim_{n\to\infty} \frac{2^n}{a_n} = \lim_{n\to\infty} 2^n \tan\frac{\pi}{2^{n+1}} = \lim_{n\to\infty} \frac{2}{\pi} \cdot \frac{\tan\frac{\pi}{2^{n+1}}}{\frac{\pi}{2^{n+1}}} = \frac{2}{\pi}.$$

Since $\frac{\tan x}{x} \underset{x\to 0}{\to} 1$

Second solution. Denote $\alpha_n = \arctan a_n$, then $\tan\alpha_n = a_n$. Then

$$\tan(\alpha_{n+1}) = \tan\alpha_n + \sqrt{1+\tan^2\alpha_n} = \tan\alpha_n + \frac{1}{\cos\alpha_n} = \frac{\sin\alpha_n + 1}{\cos\alpha_n} = \frac{\cos\left(\frac{\pi}{2} - \alpha_n\right) + 1}{\sin\left(\frac{\pi}{2} - \alpha_n\right)} =$$

$$= \frac{2\cos^2\left(\frac{\pi}{4} - \frac{1}{2}\alpha_n\right)}{2\cos\left(\frac{\pi}{4} - \frac{1}{2}\alpha_n\right)\sin\left(\frac{\pi}{4} - \frac{1}{2}\alpha_n\right)} = \frac{\cos\left(\frac{\pi}{4} - \frac{1}{2}\alpha_n\right)}{\sin\left(\frac{\pi}{4} - \frac{1}{2}\alpha_n\right)} = \tan\left(\frac{\pi}{4} + \frac{1}{2}\alpha_n\right)$$

Therefore $\alpha_{n+1} = \frac{\pi}{4} + \frac{1}{2}\alpha_n$.

It is easier to consider take $\alpha_n = \frac{\pi}{2} - \beta_n$.

Then $\frac{\pi}{2} - \beta_{n+1} = \frac{\pi}{4} + \frac{\pi}{4} - \frac{1}{2}\beta_n$.

As $\beta_{n+1} = \frac{1}{2}\beta_n$.

Since $\alpha_1 = \arctan a_1 = \arctan 1 = \frac{\pi}{4}$, and $\beta_1 = \frac{\pi}{4}$ as well, hence $\beta_n = \dfrac{\pi}{2^{n+1}}$.

So $a_n = \tan \alpha_n = \cot \beta_n = \cot \dfrac{\pi}{2^{n+1}}$. We finish as in the first solution.

5. Polynomials $P(x)$ and $Q(x)$ of odd degree are such that for each integer x there is integer y such that $P(x) = Q(y)$. Prove that there exists a polynomial R, such that $P(x) = Q(R(x))$ for each x.

Remark. The condition of having odd degree is artificial. It makes problem technically much simpler and more suitable for a competition with limited time, but ideologically the same. We shall further comment regarding how to remove this restriction.

Solution. For $|x|$ large enough, both polynomial are monotone. We may assume WLOG that both P and Q have positive leading coefficient; indeed, if P has negative leading coefficient, we can replace P and Q by $-P$ and $-Q$, and if Q has negative leading coefficient, we can replace $Q(x)$ by $Q(-x)$.
It is enough to prove the equality for large x, since it is equality of polynomials. For large x, both P and Q are monotonically increasing, therefore the function $F = Q^{-1} \circ P$ in well-defined. It is an algebraic function, which receives integer values at integer points. By algebraic function we mean a function, Satisfying an equation of the form $a_n(x)F^n + a_{n-1}(x)F^{n-1} + ... + a_0(x) = 0$, where $a_i(x)$ are polynomials.
To separate the ideology of solution from technical details, we shall formulate several lemmas.

We shall say that $U(x) \prec V(x)$ if for sufficiently large x, $\left|\dfrac{U(x)}{V(x)}\right| < const$.

Lemma 1. $F(x) \prec Cx^{m/\ell}$, where $m = \deg P$, $\ell = \deg Q$.

Lemma 2. $F^{(s)}(x) \prec x^{\frac{m}{\ell} - s}$.

Define discrete derivative: $\Delta f(x)=f(x+1)-f(x)$. Discrete derivative can be applied several times, to obtain $\Delta^2 F=f(x+2)-2f(x+1)+f(x)$, and so on; we will get formulas with alternating signs and binomial coefficients.

Lemma 3. For every n, there exists universal constant c_n, such that for every function f which has n continuous derivatives, and for each real x, it is possible to choose $y\in[x,x+n]$ such that $\Delta^{(n)}f(x)=c_n f^{(n)}(y)$.

Lemma 4. If for some n, $\Delta^{(n)}f(x)=0$ for all integer x which is large enough, then f is a polynomial at sufficiently large integers (of degree less than n).

Using these lemmas, we can solve the problem easily. Indeed, choosing $s>\frac{m}{\ell}+1$, we will have that $F^{(s)}(x)\underset{x\to\infty}{\to}0$. Therefore, $\Delta^{(s)}F(x)\underset{x\to\infty}{\to}0$, by lemma 3. However, F and hence $\Delta^{(s)}F$ are integer for sufficiently large integers, hence it is zero for sufficiently large integers, so by lemma 4, F is a polynomial for sufficiently large integers. Hence there is a polynomial R such that $P(x)=Q(R(x))$ at infinite number of points, hence it is true at all points (since nonzero polynomial cannot have infinite number of roots).

Now it remains to prove the lemmas, but first we shall hint about what problems can appear when we remove the condition of odd degree, and how to treat them. The point is, that when we define $F=Q^{-1}\circ P$ for sufficiently large x, sometimes we shall use values of Q at points far from zero, but not necessarily from the same side. It is better to define F_1 and F_2, one them will be integer for any sufficiently large integer x, but one tends to $+\infty$ and another to $-\infty$. This defines a way to paint sufficiently large integers in two colors. By Van der Waerden theorem, it is possible to choose an arbitrarily long monochromatic arithmetic sequences, and then the argument can be concluded in a similar way. We shall not explain the details of the general case here.

So, to formally complete the proof, we need to prove lemmas 1-4.

Proof of lemma 1. We shall write $f \sim g$, if there exist positive number c, C su ch that $c < \left|\frac{f}{g}\right| < C$ for sufficiently large x.

Then $x^m \sim P(x) = Q\big(F(x)\big) \sim \big(F(x)\big)^{\ell}$, hence $x^{m/\ell} \sim F(x)$.

Proof or lemma 2. The proof is by induction on s. The case $s = 0$ is lemma 1.

For the inductive step, differentiate s times the relation $Q(F(x)) = P(x)$.

We get a slightly terrifying expression of the form

$$\sum A_{k,t_1,\ldots,t_u} Q^{(k)}\big(F(x)\big) \cdot F^{(t_1)}(x) \cdot F^{(t_2)}(x) \cdot \ldots \cdot F^{(t_k)}(x) = P^{(s)}(x)$$

where A_* universal constants. In each summand, $t_1 + t_2 + \ldots + t_k = s$.

The only term that contains $F^{(s)}$ is $Q'\big(F(x)\big) \cdot F^{(s)}(x)$.

Notice that by induction $F^{(t)} \prec x^{\frac{m}{k}-t}$ for all $t < s$.

Hence $F^{(t_1)}(x) \cdot F^{(t_2)}(x) \cdot \ldots \cdot F^{(t_k)}(x) \prec x^{k\frac{m}{\ell}-t_1-t_2-\ldots-t_k} = x^{k\cdot\frac{m}{\ell}-s}$.

$Q^{(k)}$ is a polynomial of degree $\ell - k$, so $Q^{(k)}\big(F(x)\big) \sim F^{\ell-k} \sim x^{\frac{m}{\ell}(\ell-k)}$.

Hence each term

$$A_{k,t_1,\ldots,t_u} Q^{(k)}\big(F(x)\big) \cdot F^{(t_1)}(x) \cdot F^{(t_2)}(x) \cdot \ldots \cdot F^{(t_k)}(x) \prec x^{k\cdot\frac{m}{\ell}-s} \cdot x^{\frac{m}{\ell}(\ell-k)} = x^{m-s}.$$

Also, $P^{(s)} \sim x^{m-s}$.

If in the identity we move all terms except $Q'\big(F(x)\big) \cdot F^{(s)}(x)$ to the right hand side, we get

$$Q'\big(F(x)\big) \cdot F^{(s)}(x) \prec x^{m-s}.$$

But Q' is a polynomial of degree $s-1$, so $Q'(F) \sim F^{\ell-1} \sim x^{(\ell-1)\frac{m}{\ell}}$

$$x^{(\ell-1)\frac{m}{\ell}} \cdot F^{(s)}(x) \prec x^{m-s}$$

So $x^{(\ell-1)\frac{m}{\ell}} \cdot F^{(s)}(x) \prec x^{m-s-m+\frac{m}{\ell}} = x^{\frac{m}{\ell}-s}$. QED.

Proof of lemma 3. It is possible to choose such polynomial $p(x)$ of degree at most n, which has precisely the same values as f at points

$$x, x+1, ..., x+n.$$

The function $f-p$ has $n+1$ root in $[x, x+n]$, so by iteration of Rolle theorem, $(f-p)^{(n)}$ has at least one root y in $[x, x+n]$. Then $p^{(n)}(y) = f^{(n)}(y)$.

Then $p^{(n)}$ is a constant. It easy to see that Δ of a polynomial is a polynomial of degree 1 less. So $\Delta^{(n)}p$ is a constants, which depends linearly on the coefficient of x^n, as well as $p^{(n)}$. Hence $p^{(n)}(y) = f^{(n)}(y)$ is a universal constant times $\Delta^{(n)}p(x) = \Delta^{(n)}f(x)$.

Exercise to the reader. Compute this universal constant as a function of n ☺.

Proof of lemma 4. One shows inductively, that is Δp is polynomial of degree n for natural x, then p is a polynomial of degree $n+1$ for natural x. If you survived so far, you probably prefer to prove it yourself.

6. For given 2×2 matrices A, B there is only finite number n of 2×2 matrices X such that $X^2 + AX + B = 0$. Find the maximal possible value of n. (All matrices in this questions have complex entries.)

Solution. We shall denote $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$.

Let v be an eigenvector of X, $Xv = \lambda v$.

Then $X^2v + AXv + Bv = 0$, hence $\lambda^2 v + \lambda Av + Bv = 0$, so v is in the kernel of linear operator $P(\lambda) = \lambda^2 + \lambda A + B$. Therefore

$$\lambda^2 + \lambda A + B = \begin{pmatrix} \lambda^2 + a_1\lambda + b_1 & a_2\lambda + b_2 \\ a_3\lambda + b_3 & \lambda^2 + a_4\lambda + b_4 \end{pmatrix}$$

should be a degenerate matrix. Hence λ should be a root of the polynomial

$$p(\lambda) = \det(P(\lambda)) = \det(\lambda^2 + \lambda A + B) = \det\begin{pmatrix} \lambda^2 + a_1\lambda + b_1 & a_2\lambda + b_2 \\ a_3\lambda + b_3 & \lambda^2 + a_4\lambda + b_4 \end{pmatrix} =$$

$$= (\lambda^2 + a_1\lambda + b_1)(\lambda^2 + a_4\lambda + b_4) - (a_2\lambda + b_2)(a_3\lambda + b_3)$$

In the solution, we shall use a derivative of this polynomial, so we shall compute it now.

$$p'(\lambda) =$$

$$= (2\lambda + a_1)(\lambda^2 + a_4\lambda + b_4) - a_2(a_3\lambda + b_3) + (\lambda^2 + a_1\lambda + b_1)(2\lambda + a_4) - a_3(a_2\lambda + b_2)$$

We can open the brackets, but we won't.

If $P(\lambda_0)$ is a zero matrix, it means all entries of $P(\lambda)$ are divisible by $\lambda - \lambda_0$, so the $p(\lambda)$ is divisible by $(\lambda - \lambda_0)^2$. So, in this case λ_0 has to be a multiple root of $p(\lambda)$. Hence, if λ is a root of multiplicity 1 of p, then $P(\lambda)$ is not a zero matrix, hence the kernel of $P(\lambda)$ is one-dimensional.

Also, the eigenvector v of X has to be in the kernel of $\lambda^2 + \lambda A + B$. So if λ is a root of multiplicity 1 of p, an λ is an eigenvalue of X, then the direction of eigenvector v is defined uniquely.

(1) Assume that $p(\lambda)$ has no multiple roots. There are two cases: X can have distinct eigenvalues, or multiple eigenvalues. If X has distinct eigenvalues, they can be chosen in $\binom{4}{2}$ ways among the roots of $p(\lambda)$. Once we have chosen eigenvalues of X, the directions of eigenvectors are defined uniquely, and if eigenvectors are chosen, then X is defined uniquely in its eigenbasis. Hence each choice of two distinct eigenvalues of X among the roots of $p(\lambda)$ defines a unique matrix X, so there are 6 such matrices.

Now assume, that X has only one eigenvalue λ (of algebraic multiplicity 2). It also has to be a root of p. In this case $X=\lambda+N$, where N is a nilpotent matrix: $N^2=0$. Then

$$0=X^2+AX+B=(\lambda+N)^2+A(\lambda+N)+B=P(\lambda)+(2\lambda+A)N$$

$P(\lambda)$ is not a zero matrix, but it is degenerate, so it has one-dimensional kernel, which will be denoted K.

N also has a nontrivial kernel, and $(2\lambda+A)N$ has at least the same kernel, so N has to have the same kernel K. Being nilpotent, N specifies a mapping to its kernel, which is uniquely defined by specifying for a given vector outside K its image in K. So N is defined up to scaling, $N=sN_0$, where N_0 is a specific nilpotent matrix and s is a number. The condition that we have to satisfy is

$$0=P(\lambda)+(2\lambda+A)N=P(\lambda)+(2\lambda+A)sN_0$$

Is a linear condition in s. It either has an infinite number of solutions, or at most one solution. Let us multiply the last equation by the adjoint matrix of $2\lambda+A$.

$$0=\operatorname{adj}(2\lambda+A)\cdot P(\lambda)+\det(2\lambda+A)N$$

The second summand has trace zero. So the first also should have trace zero.

It is a necessary condition for existence of N. In coordinates

$$P(\lambda)=\begin{pmatrix}\lambda^2+a_1\lambda+b_1 & a_2\lambda+b_2\\ a_3\lambda+b_3 & \lambda^2+a_4\lambda+b_4\end{pmatrix},\quad 2\lambda+A=\begin{pmatrix}2\lambda+a_1 & a_2\\ a_3 & 2\lambda+a_4\end{pmatrix},$$

$$\operatorname{adj}(2\lambda+A)\cdot P(\lambda)=\begin{pmatrix}2\lambda+a_4 & -a_2\\ -a_3 & 2\lambda+a_1\end{pmatrix}\begin{pmatrix}\lambda^2+a_1\lambda+b_1 & a_2\lambda+b_2\\ a_3\lambda+b_3 & \lambda^2+a_4\lambda+b_4\end{pmatrix}=$$

$$=\begin{pmatrix}(2\lambda+a_4)(\lambda^2+a_1\lambda+b_1)-a_2(a_3\lambda+b_3) & *\\ * & (2\lambda+a_1)(\lambda^2+a_4\lambda+b_4)-a_3(a_2\lambda+b_2)\end{pmatrix}$$

We have computed only diagonal elements, since we aim to compute trace. So

$$0=\operatorname{tr}\left(\operatorname{adj}(2\lambda+A)\cdot P(\lambda)\right)=(2\lambda+a_4)(\lambda^2+a_1\lambda+b_1)-a_2(a_3\lambda+b_3)+$$
$$+(2\lambda+a_1)(\lambda^2+a_4\lambda+b_4)-a_3(a_2\lambda+b_2)=p'(\lambda)$$

But in our case, when p has no roots with multiplicity greater than one, there are no common roots for p and p', so there are no such solutions.

So, in this case, we have only 6 possible values for X.

(2) Assume that p has a root of multiplicity greater than 1. In this case, p has at most 3 distinct roots. We shall distinguish two cases, depending on whether there exists λ such that $P(\lambda)$ is a zero matrix.

(2.0) If there exists λ_0, such that $P(\lambda_0)$ is a zero matrix. Assume there is also $\lambda_1 \neq \lambda_0$ such that $p(\lambda_1)=0$. There is a nonzero vector v_1, such that $P(\lambda_1)\cdot v_1=0$. For any vector v_0 we have $P(\lambda_0)\cdot v_0=0$. If we choose arbitrary v_0, which is not multiple of v_1, then there is a unique matrix X such that $Xv_0=\lambda_0 v_0$ and $Xv_1=\lambda_1 v_1$, and since there is infinite number of ways to choose the direction of v_0, the are infinite number of ways to construct such X. In all cases, $P(X)$ is zero on v_0 and v_1, and hence it is a zero matrix. So there are infinite number of solutions, which is forbidden.
Now assume that all four roots of p are equal to the same value λ_0. In this case all eigenvalues of X are λ_0. Then $X=\lambda_0+N$, where N is nilpotent. So

$$P(X)=(\lambda_0+N)^2+A(\lambda_0+N)+B=P(\lambda_0)+(2\lambda_0+A)N=(2\lambda_0+A)N=0.$$

If we have at least one option for N which is not a zero matrix, then for each number μ, also μN works. Hence there is either infinite number of solutions (which is forbidden) or there is just one solution in this case.

(2.1) Now we assume that p has a multiple root, but $P(\lambda)$ is never a zero matrix. Than we have at most 3 different roots, so we can choose two distinct roots $\lambda_1 \neq \lambda_2$ in at most 3 ways. Matrices $P(\lambda_1)$ and $P(\lambda_2)$ are degenerate but non-zero, so a non-zero vectors $v_1 \in \ker P(\lambda_1)$, $v_2 \in \ker P(\lambda_2)$ are defined uniquely up to scaling, Therefore a matrix X such that $Xv_i=\lambda_i v_i$ for $i=1,2$ is unique. So there are only 3 solutions with distinct eigenvalues.
Assume that X has just one eigenvalue (of algebraic multiplicity 2). This eigenvalue λ can be chosen in 3 possible ways. Assume we have chosen λ, then $X=\lambda+N$, where N is a nilpotent matrix, and therefore

$$0=P(X)=P(\lambda+N)=P(\lambda)+(2\lambda+A)N.$$

Now $P(\lambda)$ is a degenerate but nonzero matrix, so as in part (1), N has the same kernel as $P(\lambda)$, and hence $N = sN_0$, so where N_0 is a specific nilpotent matrix and s is a number. The equation is linear in s, so it has either infinite number of solutions (which is forbidden), or at most one solution.

So, in this case (with multiplicities) we get at most 3 solutions with distinct eigenvalues, and at most 3 solutions with double eigenvalues, so in total at most 6 solutions.

Remark. In this case, the estimate can be improved, but there's no need to.

An example for having precisely 6 solutions: $P(X) = X^2 + \begin{pmatrix} 0 & 10 \\ 1 & 0 \end{pmatrix} X + \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

$$p(\lambda) = \det\begin{pmatrix} \lambda^2+1 & 10\lambda \\ \lambda & \lambda^2+4 \end{pmatrix} = (\lambda^2+1)(\lambda^2+4) - 10\lambda^2 = \lambda^4 + 5\lambda^2 + 4 - 10\lambda^2 =$$

$$= \lambda^4 - 5\lambda^2 + 4 = (\lambda^2-1)(\lambda^2-4) = (\lambda-1)(\lambda+1)(\lambda-2)(\lambda+2)$$

So for arbitrary choice of $\lambda_1 \neq \lambda_2$ from the set $\{-2,-1,1,2\}$, we can find non-zero vectors $v_1 \in \ker P(\lambda_1)$ and $v_2 \in \ker P(\lambda_2)$. It is easy to see that v_1, v_2 are linearly independent. Indeed, if there is a vector v_0 in $\ker P(\lambda_1) \cap \ker P(\lambda_2)$ then

$$0 = \left(P(\lambda_1) - P(\lambda_2)\right)v_0 = \left(\lambda_1^2 - \lambda_2^2 + (\lambda_1 - \lambda_2)\begin{pmatrix} 0 & 10 \\ 1 & 0 \end{pmatrix}\right)v =$$

$$= (\lambda_1 - \lambda_2)\left(\lambda_1 + \lambda_2 + \begin{pmatrix} 0 & 10 \\ 1 & 0 \end{pmatrix}\right)v = (\lambda_1 - \lambda_2)\begin{pmatrix} \lambda_1+\lambda_2 & 10 \\ 1 & \lambda_1+\lambda_2 \end{pmatrix}v$$

So either $\lambda_1 = \lambda_2$, or $v = 0$, or $\det\begin{pmatrix} \lambda_1+\lambda_2 & 10 \\ 1 & \lambda_1+\lambda_2 \end{pmatrix} = (\lambda_1+\lambda_2)^2 - 10 = 0$.

The last possibility doesn't exist, since summing numbers from $\{-2,-1,1,2\}$ won't produce $\pm\sqrt{10}$. Therefore, we can construct a unique matrix satisfying $Xv_i = \lambda_i v_i$ for $i = 1, 2$, so in this case we have precisely 6 solutions (and we don't have more, because of the discussion of case (1).

First stage of Israeli students competition, 2016.

Please try to write your solutions in English.

Duration: 4 hours

1. Compute $\det\begin{pmatrix} 5 & 1 & & & & & \\ 1 & & 1 & & & & \\ & 1 & 7 & 1 & & & \\ & & 1 & & 1 & & \\ & & & 1 & 7 & 1 & \\ & & & & 1 & & 1 \\ & & & & & 1 & 6 \end{pmatrix}$. The empty places are zeroes.

2. Let $f_1:\mathbb{R}\to\mathbb{R}$ be a continuous function. Define a sequence of functions: $f_n(x)=\int\limits_0^x f_{n-1}(t)\,dt$.

Compute $\lim\limits_{n\to\infty} f_n(1000000)$.

3. Does there exist a regular polygon, such that the set of its vertices V has a subset $S\subset V$, which satisfies the both following conditions:

(a) $|S|\ge\dfrac{99}{100}\cdot|V|$,

(b) the union of any 10 rotations of S doesn't cover V ?

4. Let $P(x)=2x^2-1$, and let $Q(x)=P\big(P\big(P(x)\big)\big)$. Let $R(x)$ be a polynomial of degree 8 such that $R(0)=1$, and $-1\le R(x)\le 1$ for each $x\in[-1,1]$. Prove that $Q(x)\le R(x)$ for each $x\in\left[-\frac{1}{10},\frac{1}{10}\right]$.

5. Let ABC be a triangle a Euclidean plane, X a point in the same plane, and M the centroid (נקודת מפגש התיכונים) of the triangle. Show that

$$\frac{\mathrm{AX}^3}{\mathrm{AB}\cdot\mathrm{AC}}+\frac{\mathrm{BX}^3}{\mathrm{BA}\cdot\mathrm{BC}}+\frac{\mathrm{CX}^3}{\mathrm{CA}\cdot\mathrm{CB}}\ge 3\cdot\mathrm{MX}.$$

6. An open unit disc (in a Euclidean plane) is covered by open equilateral triangles (משולשים משוכללים) which are contained in the disc, but may overlap (in other words, each point inside the disc is inside one of the triangles, and each point inside one of the triangles is inside the disc). Is it possible for the sum of sides of all triangles to be finite?

Good luck!

(אפשר לקחת שאלון איתכם בסוף המבחן)

First stage of Israeli students competition, 2016.

1. Compute $\det\begin{pmatrix} 5 & 1 & & & & & \\ 1 & & 1 & & & & \\ & 1 & 7 & 1 & & & \\ & & 1 & & 1 & & \\ & & & 1 & 7 & 1 & \\ & & & & 1 & & 1 \\ & & & & & 1 & 6 \end{pmatrix}$.

Answer. -25.

Solution. The determinant can be computed as a sum of products over all permutations. If "5" in the upper left corner participates in a permutation which gives a nonzero product, then we can't take any other element in first row or column, but we must take something both from second row and second column, which means that the cells $(2,3)$ and $(3,2)$ participate (we number rows and columns starting with 1), but then we can't take more numbers from second and third row or column, which leaves only cell $(4,5)$ in fourth row and only $(5,4)$ in the fourth column. The remaining two cells must be just in the two last rows and two last columns, so they are $(6,7)$ and $(7,6)$. The numbers in all these cells are 1, but the permutation is odd (three transpositions), so minus sign should be added, so the only contribution of permutation containing upper-left corner is -5.

Similarly, the contribution of the only permutation containing lower-right corner is -6, and it comes from different permutation.

Consider now permutations containing one of the "7" numbers, for instance in the cell $(3,3)$. It disallows 4 other nonzero cells, so in each of the lines 2, 4 and also rows 2,4 a unique choice remains: cells $(1,2)$, $(2,1)$, $(4,5)$, $(5,4)$ must all be taken, and then from two last rows and two last columns we are forced to take the cells $(6,7)$ and $(7,6)$. So, the contribution is -7 and it comes from just one permutation, not using any other diagonal elements.

For similar reason, there's just one contribution of -7 from a permutation containing the cell $(5,5)$ and none of the other diagonal elements.

Now consider permutations not using diagonal elements, which is

$$\det\begin{pmatrix} & 1 & & & & & \\ 1 & & 1 & & & & \\ & 1 & & 1 & & & \\ & & 1 & & 1 & & \\ & & & 1 & & 1 & \\ & & & & 1 & & 1 \\ & & & & & 1 & \end{pmatrix}$$

This is zero, which is easy to see from chessboard coloring. The cell (i, j) of a matrix is called "black" if $i + j$ is even, and "white" if $i + j$ is odd.

Lemma. Any 7×7 matrix, all nonzero entries of which are white, has zero determinant.

Using this lemma we can complete the computation: permutations containing diagonal (black) cells contribute $-(5+7+7+6) = -25$, and others contribute zero. It still remains to prove the lemma.

Proof of the lemma. Each white cell either belongs to one of the three even rows, or to one of the three even columns. Therefore, the matrix of such type is sum of two matrices of rank at most 3. Therefore, the rank of the matrix is at most 6. Hence the matrix is degenerate.

2. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define a sequence of functions:

$f_n(x) = \int_0^x f_{n-1}(x)\,dt$. Compute $\lim_{n\to\infty} f_n(1000000)$.

Answer. 0.

Solution. We can choose a positive number M such that $|f(x)| \le M$ for each $x \in [0, 1000000]$ (since f is continuous). Then

$$|f_2(x)| \le \int_0^x |f_1(t)|\,dt \le Mx \text{ for each } x \in [0, 1000000],$$

$$\left|f_3(x)\right|\le\int_0^x M\cdot x\cdot dt\le M\frac{x^2}{2} \text{ for each } x\in[0,1000000],$$

$$\left|f_4(x)\right|\le\int_0^x M\cdot\frac{x^2}{2}\cdot dt\le M\frac{x^3}{3!} \text{ for each } x\in[0,1000000],$$

and so on, and therefore

$$\left|f_{n+1}(x)\right|\le M\cdot\frac{x^n}{n!} \text{ for each } x\in[0,1000000],$$

Take $n\ge 2000000$, and denote $C=M\cdot\frac{1000000^{1000000}}{1000000!}$.

Then $\left|f_n(100000)\right|\le C\cdot\left(\frac{1}{2}\right)^{n-1000000}$. Hence $f_n(1000000)\underset{n\to\infty}{\rightarrow}0$.

3. Let V be the set of all vertices of a regular polygon. Might it be possible to find a subset $S\subset V$ so that the following two conditions are both satisfied:

(a) $|S|\ge\frac{99}{100}\cdot|V|$,

(b) union of any 10 rotations of S doesn't cover V ?

Answer. Yes, but a large number of vertices might be required.

First solution. Let us take a 10000^{10}-gon. Its vertices will be numbered in natural cyclic order, not with decimal numeration, but in base 10000 (meaning we use 10000 different digits instead of ten), by numbers of length 10.

The set S will consist of all numbers, not having at 0 or 1 as one of its digits.

Then $\frac{|S|}{|V|}=\left(1-\frac{2}{10000}\right)^{10}\ge\frac{99}{100}$, for instance by Bernoulli inequality:

$$\left(1-\frac{2}{10000}\right)^{10}\ge 1-10\cdot\frac{2}{10000}=1-\frac{2}{1000}\ge 1-\frac{1}{100}.$$

A rotation of S is like adding a number to all elements. If digit at position k of the added number is a, then none of the numbers in the rotated set will have $a+1(\mathrm{mod}10000)$ as a digit at position k. Therefore, if $S_1,S_2,\ldots,S_{10}$ are ten rotations of S, we can take some digits $d_1,\ldots,d_{10}$ such that neither element of S_i has d_i at i'th position, so the number $\overline{d_1d_2\ldots d_{10}}$ is not in $S_1\cup S_2\cup\ldots\cup S_n$.

Solution. Consider n-gon. Let set S be chosen randomly. There are n^{10} ways to consider n different rotation of S. The probability of having a hole at some specific place is $0.01^{10}=10^{-20}$. The expectation of number of holes in the union of ten given rotations of S is $n\cdot 10^{-20}$.

4. Let $P(x)=2x^2-1$, and let $Q(x)=P\left(P\left(P(x)\right)\right)$. Let $R(x)$ be a polynomial of degree 8 such that $R(0)=1$, and $-1\le R(x)\le 1$ for each $x\in[-1,1]$. Prove that $Q(x)\le R(x)$ for each $x\in\left[-\frac{1}{10},\frac{1}{10}\right]$.

Solution. When $x\in\left[0,\frac{1}{10}\right]$, then $P(x)\in[-1,-0.98]$, and

$$P\left(P(x)\right)\in\left[2\left(1-\tfrac{2}{100}\right)^2-1,1\right]=\left[1-\tfrac{8}{100}+\tfrac{8}{10000},1\right]\subset\left[1-\tfrac{1}{10},1\right]=[0.9,1]$$

Therefore $Q(x)=P\left(P\left(P(x)\right)\right)\in\left[2\cdot 0.9^2-1,1\right]=[2\cdot 0.81-1,1]\subset[0.6,1]$.

So, $Q(x)$ is positive on $\left[0,\frac{1}{10}\right]$, and even $\left[-\frac{1}{10},\frac{1}{10}\right]$ since it is even.

While x goes from -1 to 1, $P(x)$ goes monotonically from 1 to -1 and back again monotonically from -1 to 1. So, $P\left(P(x)\right)$ goes from 1 to -1 then to 1 then to -1 and back to 1 (and monotonic on 4 subintervals). Similarly, $Q(x)=P\left(P\left(P(x)\right)\right)$ on $[-1,1]$ travels back and forth 8 times, starting from 1 and going monotonically once to -1 and back to 1, four times.

Polynomial $R(x)$ on interval $[-1,1]$ produces values in $[-1,1]$, so the graph of R intersects each of 8 monotonic segments in the graph of Q. It may happen that the graph of R meets two of the segments at their common endpoint. In these case, both R and Q are tangent to the same horizontal line at that common point. Either way, $R-Q$ has 8 roots counting with multiplicities (which is maximal allowed number, since the degree of polynomial is at most 8. Another way to say it that within the square $[-1,1]^2$, graphs of R and Q intersect at least 8 times, if tangency is counted as double intersection.

We are required to prove that $Q(x)\le R(x)$ for each $x\in\left[-\frac{1}{10},\frac{1}{10}\right]$. Assume the opposite: $Q(x_0)>R(x_0)$, where $x_0\in[-1,1]$. $x_0\ne 0$, since $R(0)=1$.

WLOG, $x_0 > 0$ (otherwise we could substitute $-x$ instead of x). The graph of R has to intersect the graph of Q at four monotonic segments to the right of x_0, at three leftmost monotonic segments, and it also is tangent to the graph at point $(0,1)$, so the number of intersections counted with multiplicities is at least 9. Therefore $R-Q$ has 9 roots counting with multiplicities, but it is a polynomial of degree 8, so it is identically zero. But $Q(x_0) > R(x_0)$, which is a contradiction.

5. Let ABC be a triangle a Euclidean plane, X a point in the same plane, and M the centroid (נקודת מפגש התיכונים) of the triangle. Show that

$$\frac{\mathrm{AX}^3}{\mathrm{AB}\cdot\mathrm{AC}}+\frac{\mathrm{BX}^3}{\mathrm{BA}\cdot\mathrm{BC}}+\frac{\mathrm{CX}^3}{\mathrm{CA}\cdot\mathrm{CB}} \geq 3\cdot\mathrm{MX}.$$

Solution. Let a,b,c be complex numbers representing points A, B, C in the complex plane, where coordinated are chosen so that X is 0, the origin.
The following identity is easily verified:

$$a^3(b-c)+b^3(c-a)+c^3(a-b)=(a+b+c)(a-b)(a-c)(b-c)$$

We shall leave the computation to the reader. By triangle inequality

$$\left|a^3(b-c)\right|+\left|b^3(c-a)\right|+\left|c^3(a-b)\right| = \left|(a+b+c)(a-b)(a-c)(b-c)\right|$$

$$\frac{|a|^3}{|a-b|\cdot|a-c|}+\frac{|b|^3}{|b-a|\cdot|b-c|}+\frac{|c|^3}{|c-a|\cdot|c-b|}=|a+b+c|$$

This is precisely what we were required to prove, if you recall that $\dfrac{a+b+c}{3}$ represents M.

6. An open unit disc (in Euclidean plane) is covered by open regular triangles which are contained in the disc, but may overlap (in other words, each point inside the disc is inside one of the triangles, and each inside one of the triangles is inside the disc). Is it possible for the sum of sides of all triangles to be finite?

Answer. No.
Solution. There are two cases: the set of triangles might be countable or uncountable.

If the set is uncountable the sum of sides is infinite anyway. Indeed, let S_k be the set of triangles whose side is greater than $\frac{1}{k}$ but not greater than $\frac{1}{k-1}$ (where k is a natural number). If at least one among the sets S_k is infinite, then sum of the length even in S_k is infinite. If all are finite, their union is countable.

So, it enough to deal with the countable (or finite) case. There is uncountable set of directions, so we can rotate the picture in such a way, that neither of the triangles has horizontal side.

Each horizontal chord of the circle is covered by triangles, each triangle cuts an interval on such horizontal chords. Only countable number of endpoints of horizontal chords might be corners of triangle, so all horizontal chords except countable number intersect infinite number of triangles. Therefore almost all horizontal chords intersect infinite amount of triangles.

Integral of number of intersections of horizontal chords with triangles is precisely the total length of all projections of the sides of all triangles to the y-axis. We see that that integral is infinite, so the sum of all sides is infinite.

Second stage of Israeli students competition, 2016.
Please try to write your solutions in English.
Duration: 4 hours

1. Let $f:[0,1]\to[e,+\infty)$ be a monotonically increasing function. Prove that there exist $x,y\in[0,1]$ such that $f(y)\le 2f(x)$ and $y-x\ge\dfrac{1}{10(\ln f(x))^2}$.

2. Prove that for every irrational number $\alpha\in(0,1)$, there exists a non-decreasing sequence $\{a_n\}_{n=0}^{\infty}$ of positive integers, such that $\alpha=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{a_n a_{n+1}}$.

3. Numbers $a_1,\dots,a_{16}$ are written in this order around a circle. At each move, all numbers are simultaneously replaced: a_i is replaced by $|a_i-a_{i-1}|$ (we take $a_0=a_{16}$ since the order is circular). Is it true that after finite number of moves, all numbers around the circle will become zeroes, assuming that
(a) the numbers are rational?
(b) the numbers are real?

4. Prove that for each $a,b,c,d\in\mathbb{R}$, the following inequality holds:

$$\sqrt{a^2+b^2+\tfrac{2}{3}ab}\cdot\sqrt{c^2+d^2+\tfrac{2}{3}cd}+\sqrt{b^2+c^2+\tfrac{2}{3}bc}\cdot\sqrt{a^2+d^2+\tfrac{2}{3}ad}\ge\sqrt{a^2+c^2+\tfrac{2}{3}ac}\cdot\sqrt{b^2+d^2+\tfrac{2}{3}bd}.$$

5. Show that the polynomial $x^n+x^{n-1}+x^{n-2}+\dots+x^2+x-1$ is irreducible over $\mathbb{Q}$.

6. There are N boys and N girls at the school. Each girl is acquainted with precisely K boys and each boy is acquainted with precisely K girls (the number K is the same for all boys and girls). For any two girls there are precisely C boys they are both acquainted with. Prove that for any two boys there are precisely C girls they are both acquainted with.

7. The set $\mathbb{L}=\mathbb{R}^2\times[0,\tfrac{1}{100}]\subset\mathbb{R}^3$ is colored in 4 colors. Prove that there exist two points in this set which are of the same color, and the distance between them is 1.

Good luck!

(אפשר לקחת שאלון איתכם בסוף המבחן)

Second stage of Israeli students competition, 2016.

Please try to write your solutions in English.

Duration: 4 hours

1. Let $f:[0,1]\to[e,+\infty)$ be a monotonically increasing function. Prove that there exist $x,y\in[0,1]$ such that $f(y)\le 2f(x)$ and $y-x\ge\frac{1}{10(\ln f(x))^2}$.

Solution. Assume the contrary. Then for $y=x+\frac{1}{10\cdot(\ln f(x))^2}$ (if $y\le 1$) we get $f(y)\ge 2f(x)$. Or, if we define $g:[0,1]\to[1,+\infty)$, $g(x):=\ln(f(x))$, then for $y=x+\frac{1}{10\cdot(g(x))^2}$ we get $g(y)\ge g(x)+\ln 2$.

Define a sequence: $x_0=0$, $x_{n+1}=x_n+\frac{1}{10\cdot(g(x_n))^2}$. Then, as long as the sequence is defined, $g(x_n)\ge 1+n\cdot\ln 2$, and hence

$$x_n=\sum_{k=0}^{n-1}\frac{1}{10\cdot(g(x_k))^2}\le\sum_{k=0}^{n-1}\frac{1}{10\cdot(1+k\cdot\ln 2)^2}\le\sum_{k=0}^{n-1}\frac{1}{(3+3k\cdot\ln 2)^2}\le\sum_{k=0}^{n-1}\frac{1}{(3+k\cdot\ln 8)^2}\le$$
$$\le\sum_{k=0}^{n-1}\frac{1}{(3+k)^2}<\sum_{k=0}^{n-1}\frac{1}{(2+k)(3+k)}=\sum_{k=0}^{n-1}\left(\frac{1}{(2+k)}-\frac{1}{(3+k)}\right)=\frac{1}{2}-\frac{1}{2+n}<\frac{1}{2}$$

as long as x_n is defined. But that means x_n is always well defined, since if $x_n<1$ then $g(x_n)$ is also defined, and x_{n+1} is well defined, and we have even proved that $x_n<\frac{1}{2}$. To summarize: we proved existence of x_n such that $f(x_n)\ge 2^n f(0)\ge e\cdot 2^n$ and therefore $f(x_n)\to+\infty$, but $f(x_n)\le f\left(\frac{1}{2}\right)$ which is a contradiction.

2. Prove that for every irrational number $\alpha\in(0,1)$, there exists a non-decreasing sequence $\{a_n\}_{n=0}^{\infty}$ of positive integers, such that $\alpha=\sum_{n=0}^{\infty}\frac{(-1)^n}{a_n a_{n+1}}$.

Proof. Recall that continued fraction produces for each irrational number α an infinite sequence of rational approximations $\frac{p_n}{q_n}$, which is rapidly converging to α, such that the first element is $\frac{p_0}{q_0}=\frac{1}{q_0}$, and $\frac{p_k}{q_k}-\frac{p_{k-1}}{q_{k-1}}=\frac{(-1)^k}{q_k q_{k-1}}$, the sequence $\{q_i\}$ is a strictly increasing sequence of positive integers and therefore

$$\alpha=\lim_{n\to\infty}\frac{p_n}{q_n}=\lim_{n\to\infty}\left(\frac{p_0}{q_0}+\sum_{k=0}^{n-1}\left(\frac{p_{k+1}}{q_{k+1}}-\frac{p_k}{q_k}\right)\right)=\lim_{n\to\infty}\left(\frac{1}{q_0}+\sum_{k=1}^{n}\frac{(-1)^k}{q_{k-1}q_k}\right)=$$
$$=\frac{1}{1\cdot q_0}-\frac{1}{q_0\cdot q_1}+\frac{1}{q_1\cdot q_2}-\frac{1}{q_2\cdot q_3}+\frac{1}{q_3\cdot q_4}-\frac{1}{q_4\cdot q_5}+\dots$$

Q.E.D.

3. Numbers $a_1,\dots,a_{16}$ are written in this order around a circle. At each move, all numbers are simultaneously replaced: a_i is replaced by $|a_i-a_{i-1}|$ (we take $a_0=a_{16}$ since the order is circular). Is it true that after finite number of moves, all numbers around the circle will become zeroes, assuming that
(a) the numbers are rational?
(b) the numbers are real?

Answers. (a) yes, (b) no.

Solution. (a) We can multiply all fractions numbers by their common denominator, and play similar (but equivalent) game with integer numbers. Starting with the second move, all numbers are non-negative; so from now on we shall assume all numbers around the circle are nonnegative.

Let $M=\max a_i$. Then also $|a_i-a_{i+1}|\le M$ (for non-negative numbers). So $a_i<M$ throughout the game. If there is a common divisor d for all a_i, we can divide all numbers by d and consider an equivalent game with smaller M. We shall prove that **after 16 moves all the numbers in the circle become even**. If that is true,

then after 16 moves we can replace M by $\frac{M}{2}$. Applying this idea several times, we see that after $16\left(1+\left[\log_2 M\right]\right)$ all numbers are zeroes.

So, to prove that all numbers become even after 16 moves, we consider the process mod 2. Then the rules are more simple: a_i is replaced by $a_i + a_{i+1}$. Iterating this process, we get that after k moves the number at place i is $\sum_j \binom{k}{j} a_{i-j}$.

Recall that $\binom{16}{j} = 0 \pmod{2}$ for $0 < j < 16$ (if you didn't know it, prove it by using 4 times the identity $(1+x)^2 = 1+x^2 \pmod{2}$). Therefore after precisely 16 moves, we get that at place i we get a number $\binom{16}{0}a_i + \binom{16}{16}a_i = a_i + a_i = 0 \pmod{16}$, which completes the proof.

Remark. Here it is important that 16 is a power of 2. Otherwise it is wrong.

(b) Let us write around the circle the geometric sequence: $1, x, x^2, x^3, ..., x^{15}$ in this order (we shall choose $x > 1$ later). Then after one move we get

$$x-1, x^2-x, ..., x^{15}-x^{14}, x^{15}-1$$

or in other words

$$z, zx, zx^2, ..., zx^{14}, x^{15}-1$$

where $z = x-1$ is a positive number. If also $x^{15}-1 = z\left(x^{16}-x\right)$, then all the numbers during this move were simply multiplied by z. So, from now on with each move all the numbers will be multiplied by z, and will remain nonzero forever. So, it remains to choose $x > 1$, such that $x^{15}-1 = (x-1)\left(x^{16}-1\right)$. Which means $x^{14}+x^{13}+x^{12}+...+1 = x^{16}-1$, or that x is a root of the polynomial

$$x^{14}+x^{13}+x^{12}+...+2-x^{16}.$$

The polynomial is positive for $x=1$ and negative for $x=2$ so there has to be a root somewhere in between.

4. Prove that for each $a,b,c,d\in\mathbb{R}$, the following inequality holds:

$$\sqrt{a^2+b^2+\tfrac{2}{3}ab}\cdot\sqrt{c^2+d^2+\tfrac{2}{3}cd}+\sqrt{b^2+c^2+\tfrac{2}{3}bc}\cdot\sqrt{a^2+d^2+\tfrac{2}{3}ad}$$
$$\geq\sqrt{a^2+c^2+\tfrac{2}{3}ac}\cdot\sqrt{b^2+d^2+\tfrac{2}{3}bd}.$$

Solution. Consider a regular tetrahedron in $\mathbb{R}^3$, inscribed in a sphere of unit radius, with the center at the origin. The vertices of the tetrahedron are represented by vectors v_1,v_2,v_3,v_4. Then $v_1+v_2+v_3+v_4=0$, and hence

$$0=\left\langle\sum_i v_i,\sum_j v_j\right\rangle=\sum_{i=j}\left\langle v_i,v_j\right\rangle+\sum_{i\neq j}\left\langle v_i,v_j\right\rangle=4+12\left\langle v_1,v_2\right\rangle,$$

so cosine of the angle between each two different vectors is $-\frac{1}{3}$.

Consider 4 points in $\mathbb{R}^3$: $A=av_1, B=bv_2, C=cv_3, D=dv_4$. Ptolemy inequality states that for each 4 points, $AB\cdot CD+BC\cdot AD\geq AC\cdot BD$. In our case all the lengths might be expressed with cosine theorem, for example $AB^2=a^2+b^2+\frac{2}{3}ab$. Substitute all such expressions in Ptolemy inequality, Q. E. D.

5. Show that the polynomial $x^n+x^{n-1}+x^{n-2}+...+x^2+x-1$ is irreducible over $\mathbb{Q}$.

Proof. After reversing the coefficients we reduce to showing that $h(x)=x^n\cdot f\left(\frac{1}{x}\right)$ is irreducible. By the lemma of Gauss it is sufficient to show irreducibility over $\mathbb{Z}$. Any factorization of h over $\mathbb{Z}$ determines a partition of the roots of h into two sets such that the product of each set is a nonzero integer (the constant term of the corresponding factor), and hence has norm ≥ 1. Thus, each of these sets must contain an element of norm ≥ 1. Therefore, if we show that h has at most one root of norm ≥ 1 then no factorization can exist and we are done. For that it is sufficient to show that f has at most one root of norm ≤ 1. Observe that $f(x)(x-1)=x^{n+1}-2x+1$ so every root of f is a fixed point of the complex mapping $T(z)=\dfrac{z^{n+1}+1}{2}$.

Suppose that f has two distinct roots z_1,z_2 of norm ≤ 1. If for some $i\in\{1,2\}$ we have $|z_i|=1$ then

$$1 = |z_i| = \left|\frac{z_i^{n+1}+1}{2}\right| \le \frac{\left|z_i^{n+1}\right|+1}{2} \le \frac{1+1}{2} = 1$$

so we have equality in the triangle inequality which means that $z_i^{n+1} \in \mathbb{R}$ which means that $z_i = T(z_i) \in \mathbb{R}$ but $f(\pm 1) \neq 0$. So z_1, z_2 are two distinct fixed points of the map T which maps the open unit disk to itself. By the equality case of the Schwarz-Pick theorem (or as a consequence of Schwarz lemma) T is a Moebius transformation which is a contradiction (unless $n=1$, but in this case irreducibility is obvious).

The proof is complete, but for those who don't remember it we shall remind the proof of Schwarz-Pick. We want to show that if there is a complex analytic mapping from the open unit disc to itself with two fixed internal points, then it is identity. By conjugation with some Moebius transformation, we may assume that one of the fixed points is 0, and another will be denoted by z_0.

So, we have a complex analytic function in the open unit disc $\Delta = \{z \mid |z| < 1\}$, satisfying $f(0) = 0$, $|f(z_0)| = |z_0|$ for some $z_0 \in \Delta$, and $f(\Delta) \subset \Delta$. Then is possible to define a complex analytic function $g(x) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$.

It is easy to see that on a circle $|z| = r$ for each $r < 1$ we get $|g| \le \frac{1}{r}$. Hence by maximum principle in the disc $|z| \le r$, we get $|g| \le \frac{1}{r}$. When r tends to 1 from below, we conclude that $|g(z)| \le 1$ for all $z \in \Delta$. But $|g(z_0)| = 1$. Hence by maximum principle (non-constant holomorphic function can not have internal maximum) the function g is constant. From this the lemma follows.

6. There are N boys and N girls at the school. Each girl is acquainted with precisely K boys and each boy is acquainted with precisely K girls (the number K is the same for all boys and girls). For any two girls there are precisely C boys they are both acquainted with. Prove that for any two boys there are precisely C girls they are both acquainted with.

First solution. We shall number the boys from 1 to N, and number the girls from 1 to N. We shall construct and $N \times N$ matrix A as follows: the number at row i column j is 1 if the i'th girl i is acquainted with the j'th boy and zero otherwise.

As usual, let $e_1, e_2, ..., e_n$ be the vectors of the standard basis. Then vectors Ae_i represent (by zeroes and ones) the friends of the i'th boy, and the vectors $A^T e_j$ represent the friends of the j'th girl. All the sentences in the formulation of the problem can be translated to the language of Linear Algebra (which is shorter):

It is given that $\left\langle A^T e_i, A^T e_j \right\rangle$ is K when $i = j$, and C when $i \neq j$. It is also given that $\left\langle Ae_i, Ae_j \right\rangle$ is K when $i = j$, and we must prove it is C when $i \neq j$.

Or even shorter: *the matrix AA^T is given: the entries on the main diagonal are all equal K, the other entries are all equal C. Show that $A^T A$ is the same.*

Notice that the vector $u = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector of all matrices we've mentioned: indeed, $Au = Ku$, and $A^T u = Ku$, so it is also eigenvector of $A^T A$ and of AA^T.

Remark. From here it is easy to see the relation between K and C: on one hand, $AA^T u = K \cdot Au = K^2 u$, on the other hand, by direct computation $AA^T u = \left(K + (n-1)C\right)u$, so $K^2 - K = (n-1)C$. This formula might be also obtained by elementary counting in two ways of triples "boy and two girls he knows" (see the third solution). Anyway, we won't use it in this solution.

Let's discuss eigenvalues and eigenvectors of matrix AA^T.

Let us start with matrix U, all entries of which are 1. This matrix is a matrix of rank 1, so 0 is eigenvalue of U of multiplicity $n-1$. Another eigenvalue is n, of multiplicity 1, and the corresponding eigenvector is u. Since matrix is symmetric the eigenvectors of 0 (a.k.a. the kernel) is $n-1$-dimensional space, consisting of all vectors orthogonal to u.

The matrix $AA^T = C\cdot U + (K-C)\cdot I$ has the same set of eigenvectors, but eigenvalues are different: for u the eigenvalue is K^2, and for its orthogonal complement the eigenvalue is $K-C$.

Recall that for any real square matrix, AA^T is similar to A^TA. Indeed, by SVD decomposition, there are two orthogonal matrices L,R and a diagonal matrix D such that $A=LDR$. Then $AA^T = LDRR^TDL^T = LD^2L^{-1}$, so it is similar to D^2 and also to $A^TA = R^TD^TL^TLDR = R^{-1}D^2R$.

Let us summarize what we know about A^TA by now. It is similar to the A^TA which is known precisely. It is also a symmetric matrix, so it has eigenspaces of maximal allowed dimensions for all eigenvalues, and different eigenspaces are orthogonal to each other. There are two distinct eigenvalues, same as for AA^T: one of multiplicity 1, another of multiplicity $n-1$. The eigenspace for the first eigenvalue is spanned by u, so the eigenspace of the other eigenvalue is its orthogonal complement. So eigenvalues and eigenspaces of A^TA are the same as for AA^T, and hence the matrices are the same.

Second solution. As before, we reformulate the problem with linear algebra. We shall use the same notations as in the previous solution. $AU = KU$, $A^TU = KU$, where U is matrix of ones.

If $K=C$, every two girls share all their acquaintances, so each boy knows either none of them or all of them. In this case, the problem is easy. We shall assume that from now on that $C<K$.

$$\frac{A^T}{K-C}\cdot\left(A-\frac{C}{K}U\right)=\frac{1}{K-C}\cdot\left(A^TA-\tfrac{C}{K}A^TU\right)=$$
$$=\frac{1}{K-C}\cdot\left((K-C)I+\cancel{CU}-\cancel{\tfrac{C}{K}KU}\right)=I$$

So A^T is inverse of $A-\frac{C}{K}U$, but $A^TU=KU=\left(AU\right)^T=U^TA^T$, so A^T commutes with U, so A^T with A. Therefore $A^TA=AA^T$.

Third solution. Let us compute the number M of triples

two boys and a girl who knows both.

On one hand, each girl participates in $\binom{K}{2}$ such triples, since among her friends there are $\binom{K}{2}$ pairs of boys, so $M=N\cdot\binom{K}{2}$. On the other hand, if the average number of girls that boys i and j both know in common is $C_{i,j}$, then .

Therefore $$N\cdot\binom{K}{2}=\sum_{i<j}C_{i,j}\ .$$

Similarly, if we compute in two ways number of triples

two girls and a boy who knows both,

we get that $$N\cdot\binom{K}{2}=\binom{N}{2}\cdot C$$

Now let's compute the number of quadruples:

two boys and two girls who know both.

On one hand, any two girls participate in precisely $\binom{C}{2}$ such groups (since we have to choose 2 boys among their common friends. On the other hand, boys i and j participate in precisely $\binom{C_{i,j}}{2}$ such groups. So we get another identity

$$\binom{N}{2}\binom{C}{2}=\sum_{i<j}\binom{C_{i,j}}{2}$$

We have found two identities on the numbers $C_{i,j}$. Both identities are satisfied if $C_{i,j} = C$ for all i, j. But is there any other way to satisfy both identities?

Recall that $\binom{x}{2}$ is a strictly convex quadratic function. Therefore, by Jensen's inequality, given $\sum_{i<j} C_{i,j}$, the value of $\sum_{i<j} \binom{C_{i,j}}{2}$ will be minimal if and only if all $C_{i,j}$ are equal to each other. Therefore, $C_{i,j} = C \ \ \forall i, j$ is a unique solution for both equalities. So for any two boys there are precisely C girls who know both.

7. The set $\mathbb{L} = \mathbb{R}^2 \times \left[0, \frac{1}{100}\right]$ (which is a subset of $\mathbb{R}^3$) is colored in 4 colors. Prove that there exist two points in $\mathbb{L}$ which are of the same color, and the distance between them is 1.

Solution. Consider a circle of radius $R > \frac{1}{2}$ in $\mathbb{R}^2$. Construct a sequence of points $A_1, A_2, \ldots$ on the circle such that A_i does not coincide with A_{i+2}, and the length of $A_i A_{i+1}$ is 1. If for some n, the point A_{2n} coincides with A_1, then R will be called ***forbidden radius***. It is easy to see that when n is large enough, then small changing of R causes big changing of the location of A_{2n}. Therefore, the set of forbidden radii is a dense subset of $\left(\frac{1}{2}, \infty\right)$.

It is easy to see, that whenever we get a circle of forbidden radius and its point are colored in less than 3 colors, then there have to be two points of the same color at distance 1.

Now take the point $Q = \left(0, 0, \frac{1}{1000}\right)$. Let $\mathbb{W}_Q$ be the set of all points $W = (x, y, z)$ satisfying the following three conditions:

- $x^2 + y^2 < \dfrac{1}{10^9}$
- $\dfrac{1}{200} < z < \dfrac{1}{100}$
- $QW = 2\sqrt{1 - R^2}$, where R is a forbidden radius.

Consider the circle $\mathbf{C}=\{X \mid XQ=1=XW\}$. Since the direction of the interval QW is almost vertical, the plane of the circle is almost horizontal, so $\mathbf{C}\subset\mathbb{L}$. It is easy to see by Pythagoras theorem that $\mathbf{C}$ is of radius R.
Assume there are no points at unit distance of the same color. So the circle $\mathbf{C}$ has points of at least 3 colors. But the colors of points Q and W are different from all colors of $\mathbf{C}$. So Q and W are of the same color (since there are just 4 colors in total), for any $W\in\mathbb{W}_Q$.

Now take a huge natural number n, and take vectors $u=\begin{pmatrix}\frac{1}{2n}\\ 0\\ z\end{pmatrix}$, $u=\begin{pmatrix}\frac{1}{2n}\\ 0\\ -z\end{pmatrix}$, such that $\left(\frac{1}{2n}\right)^2+z^2=R^2$, $z>0$. Take the sequence of points $Q_0=Q$, and then inductively $W_i=Q_i+u$, $Q_{i+1}=W_i+v$. All this points have to be of the same color. So Q_0 and Q_n are of the same color, but the distance between them is 1.