Jacobs University Bremen
School of Engineering and Science

MatBoj
June 21, 2008
Alexei Belov, Michael Stoll

MAT-BOJ

Математический Бой: The Mathematical Battle

Problems

МБ **1.** Two points A and B are chosen on the circle ω. C is the midpoint of one of the arcs AB or BA. D is an arbitrary point on the segment $[A, B]$. The circle ω_1 is tangent to the segment $[B, D]$ (in the point B_1) and also to the line segment $[C, D]$ and to the circle ω (in different points). The circle ω_2 is tangent to the continuation of $[A, B]$ beyond the point B (in the point B_2), to the circle ω (in the point K) and also to the continuation of $[C, D]$ beyond the point D. Prove that $\angle B_1KB_2 = \pi/2$.

МБ **2.** Let $(u_n)_{n\geq 1}$ be an increasing sequence of real numbers such that $u_1 = 1$, $u_2 = 2$ and $u_{mn} = u_m u_n$ for all $m, n \geq 1$. Prove that $u_k = k$ for all $k \in \mathbb{N}$.

МБ **3.** The diagonals of a convex quadrilateral $ABCD$ intersect in the point O. The points K, L, M, N are the orthogonal projections of O to the edges $[A, B]$, $[B, C]$, $[C, D]$, $[D, A]$ and lie inside the corresponding sides of the quadrilateral $ABCD$. Prove that $2S_{KLMN} \leq S_{ABCD}$, where S_Q denotes the area of the quadrilateral Q.

МБ **4.** The sequence (a_n) is given recursively: $a_1 = 1$, a_n is the smallest natural number k distinct from $a_1, \ldots, a_{n-1}$ such that $a_1 + \ldots + a_{n-1} + k$ is divisible by n. Prove that the mapping $n \to a_n$ is a bijection $\mathbb{N} \to \mathbb{N}$.

МБ **5.** Let x, y, z be nonnegative real numbers with $x + y + z = 1$. Prove that

$$\sqrt{1 - 3xy} + \sqrt{1 - 3xz} + \sqrt{1 - 3yz} \geq \sqrt{6}\,.$$

МБ **6.** A necklace consists of R red and B blue beads. We say that it is *good*, if for any $1 \leq k < R+B$ and any two substrings of length k the number of red beads in the substrings differs by at most 1. Prove that for all R and B, a good necklace exists and is unique up to rotation.

МБ **7.** Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a polynomial of total degree n, with a global minimum at $(0, 0)$, and $F(0, 0) = 0$. Does there exist a constant $\varepsilon > 0$ such that for all $-1 \leq x, y \leq 1$ the following inequality holds: $F(x, y) > \varepsilon(|x| + |y|)^n$?

МБ **8.** Two players are playing the following game on an infinite strip of cells. The first player marks two cells with an X, and the second player marks one cell with an O in each move. After 10^{12} moves the game is over. The first player wins if there are 100 consecutive cells marked with an X, else the second player wins. Is there a winning strategy for the first player?

МБ **9.** Is it possible to divide the plane into squares of pairwise different sizes such that only finitely many of these squares meet any bounded part of plane?

МБ **10.** A square matrix is called *doubly stochastic* if all its entries are nonnegative and the sum of the entries in each column and in each row is 1. Prove that any doubly stochastic matrix is a linear combination of permutation matrices, with nonegative coefficients. (Matrix is called *permutation* if it has just one unit entry in each row and in each column and all other entries are zeroes.)

МБ **11.** For which n can we draw the complete graph on n vertices in the plane in such a way that each arc has at most one inner point in common with another arc, in which case exactly two arcs are meeting there transversely (i.e, crossing each other), and no 3 arcs has common point except it is a vertex of all of them?

МБ **12.** A natural number k is considered *good*, if for each n the number $1^k + 2^k + \ldots + n^k$ is divisible by $1 + 2 + \ldots + n$. Describe the set of all good numbers.

Jacobs University Bremen
School of Engineering and Science

IMO Training 2009
MatBoj
Sat, June 20, 2009

MAT-BOJ

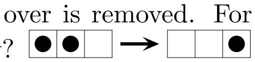
Математический Бой: The Mathematical Battle

Problems

МБ **1.** Let T be a regular tetrahedron. Find a piecewise linear closed curve of minimal length that has a point in common with every face of T.

МБ **2.** Can a polynomial with rational coefficients have $\sqrt{2}$ as its minimal value (on $\mathbb{R}$)?

МБ **3.** An infinitely wise but shortsighted cockroach is trying to find the truth (which is a point of the Euclidean plane). If its distance from the truth is at most 1, it will reach it with its next step. After each step (of unit length) it is told whether it got closer to the truth or not. In the beginning it knows that it is at (integral) distance $N > 0$ from the truth. Prove that it can reach the truth in at most $N + 10\log_2(N)$ steps.

МБ **4.** On an infinite and otherwise empty chessboard there is a rectangular array of $m \times n$ pieces. We play the *Solitaire* game: a move consists in a piece jumping over an adjacent piece into the cell beyond, which has to be empty; the piece that was jumped over is removed. For which pairs (m, n) is it possible to remove all pieces but one in this way? 

МБ **5.** Prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{3k+1}$ is never an integer, for any $n \geq 1$.

МБ **6.** Show that in a group of 50 people there are two that have an even number of common friends (possibly zero), assuming that friendship is a symmetric relation (and nobody is considered their own friend).

МБ **7.** Prove that there is no real-valued function on the open interval $]-1, 1[$ that has only a finite number of discontinuities and such that its graph is invariant under rotation by a right angle around the origin.

МБ **8.** Let C be a convex polygon and P a point inside it. Let N denote the number of vertices such that the line segment connecting P to the vertex divides the angle of C at this vertex into two acute angles. Denote by n the number of sides of C such that the foot of perpendicular from P to that side is strictly inside that side. Show that $N = n$.

МБ **9.** A grasshopper, starting at the origin, performs an infinite sequence of jumps on the real line. The length of the nth jump is n^{2009}. For each jump, it can choose the direction (left or right). Show that the grasshopper can visit all integers.

МБ **10.** Consider a shape consisting of a finite number of unit square cells. We try to cover a board of $m \times n$ unit square cells by equivalent (i.e., translated, rotated and/or reflected) copies of that shape, so that each cell of the board is covered equally often.
Prove that this is impossible if and only if we can write a real number in each cell of the board, in such a way that the sum of all those numbers is strictly negative, while a sum that can be covered by the given shape is strictly positive (wherever we place it on the board).

МБ **11.** Let $L_1, \ldots, L_4$ be four lines and $P_1, \ldots, P_4$ four points in the plane, such that $P_i \in L_j$ if and only if $i = j$. Assume that for each subset of three lines, there exists a conic section that is tangent to these lines at the given points. Show that there is a conic section that is tangent to all four lines at the given points.

МБ **12.** In a table there are n columns and m rows, $n > m$. Some cells are marked by a star, and in each column there is at least one star. Show that there is a star such that there are fewer stars in its column than in its row.

Jacobs University Bremen
School of Engineering and Science

MatBoj
19 June 2010

MAT-BOJ

Математический Бой: The Mathematical Battle

Problems

МБ **1.** Is it possible to cut two disks into a finite number of parts that can be rearranged (rotated and translated) so as to fill a single disk? The cuts have to be made along line segments and circle arcs.

МБ **2.** A function c on the set of natural numbers is defined as follows: $c(n) = 0$ if the binary representation of n has an even number of ones, and $c(n) = 1$ otherwise. We fix a positive integer k. Let $l(N)$ be the number of integers $0 \leq n \leq N$ such that $c(k+n) \neq c(k)$. Prove that $\lim_{n\to\infty} l(N)/N$ exists and belongs to $[1/3, 2/3]$.

МБ **3.** Every pair of vertices of a graph G can be connected by an edge path of length $n-1$, and the shortest length of a cycle in G is at least $2n-1$. Prove that all vertices of G have the same degree.

МБ **4.** Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$f(x+y) - f(x-y) = 2yf'(x) \quad$ for all $x, y \in \mathbb{R}$.

МБ **5.** Find all $a \in \mathbb{R}$ such that there exist non-negative $x_1, \ldots, x_n \in \mathbb{R}$ satisfying

$$\sum_{i=1}^{n} kx_k = a\,, \qquad \sum_{i=1}^{n} k^3 x_k = a^2\,, \qquad \sum_{i=1}^{n} k^5 x_k = a^3\,.$$

МБ **6.** Two players play a game on the infinite chess-board. the first player plays with 3 white pieces called *sheep*, and the second player plays with 3 black pieces, called *wolves*. They move in turn. In his move each player can move only one piece to an adjacent cell (having a common side with its previous cell). Sheep can be moved only horizontally. If a wolf and a sheep happen to be in the same cell, the wolf eats the sheep. Is it always possible for the wolves to catch at least one sheep?

МБ **7.** An $N \times M$ table of real numbers is given. The sum in every sub-square of 3×3 cells is positive, and the sum in any sub-square of 5×5 cells is negative. For which pairs (N, M) is this possible?

МБ **8.** Consider the surface of a cone with cone angle α (the total angle at the vertex). Find the maximal number of intersection points of a geodesic line on this surface that does not pass through the vertex of the cone (a line is called *geodesic* if it projects locally onto a straight line segment when unrolling the cone onto a flat piece of paper).

МБ **9.** The diagonals of a convex pentagon M divide it into ten triangles and another convex pentagon M'. Let $\Delta(M)$ be the difference between the sum of the areas of the five triangles adjacent to the sides of M and the area of M'.

Prove that $\Delta(M) > \Delta(M')$.

МБ **10.** Does there exist an integer $0 < x < 4 \cdot 99!$ such that $x(x+1)$ is divisible by $100!$?

МБ **11.** One needs to guess an integer number in $[1, 2010]$. Only the following tyoe of question is allowed: "is this number less than n?". For an answer "Yes" one has to pay 1 Euro, for an answer "No", 3 Euros. Find the minimal amount of Euros that is sufficient to find the number in all cases.

МБ **12.** Consider the map $f\colon x \mapsto 2x \pmod 1$ on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Let $I \subset \mathbb{S}^1$ be a closed interval, let $\mathcal{I}_0 := \{I\}$ and let $\mathcal{I}_1$ be the set containing the two intervals J such that $f(J) = I$. For $n \geq 1$, let $\mathcal{I}'_{n+1}$ be the set of all intervals $J \subset \mathbb{S}^1$ so that $f(J) \in \mathcal{I}_n$ and let $\mathcal{I}_{n+1}$ be the set $\mathcal{I}'_{n+1}$ from which one (arbitrary) interval is removed. Finally, let $A_n := \mathbb{S}^1 \setminus \bigcup_{k=0}^{n} \bigcup_{J\in\mathcal{I}_k} J$.

Let M_n be the minimal number of intervals of length 2^{-n} needed to cover A_n. Show that there is an $0 < \alpha < 2$ so that $M_n < \alpha^n$ for all sufficiently large n.

Jacobs University Bremen
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MatBoj
18 June, 2011

MAT-BOJ

Математический Бой: The Mathematical Battle

Problems

МБ **1.** A set S of unit circles is given on the plane such that each pair of circles intersect in two points. Prove that one can put four nails in the plane so that each circle encloses at least one nail.

МБ **2.** Find the 11111-th decimal digit of $\sqrt[10]{0.999\dots999}$, where there are 1233 digits '9'.

МБ **3.** Two players are playing the following game on the infinite plane. A move of Blue is to color one point on the plane blue, a move of Red is to color 2011 points red. It is not allowed to change the color of an already colored point. Can Blue always make sure that at some point there is a regular hexagon with blue vertices?

МБ **4.** Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(f(x))) = \cos(x)$?

МБ **5.** Somebody draws a regular triangle and a square on the plane. Prove that at least one of the 12 distances between a vertex of the triangle and a vertex of the square is irrational.

МБ **6.** A particle is moving on the plane for one second, starting in the origin O and ending in O. The absolute value of its acceleration is always ≤ 1.

МБ **7.** There is a finite set L of lamps and a set B of buttons. Each button $b \in B$ is connected to some lamps in L; when b is pressed, each connected lamp is switched on if it was off or switched off if it was on. For any subset $L' \subseteq L$ there exists a button connected to an odd number of lamps from L'. Initially some lamps are switched off, some are switched on. Prove that all lamps can be switched off.

МБ **8.** Let a_0 and b be positive integers and define a sequence (a_n) by $a_{n+1} = b^{a_n}$. Prove that the sequence of remainders of a_n modulo 2011 is eventually constant.

МБ **9.** What is the smallest number n such that the points of the plane with rational coordinates can be colored with n colors such that no two points at a distance of 1 get the same color?

МБ **10.** An infinite sequence of digits from 1 to 9 is given. Prove that either the sequence contains 10 non-overlapping 1000-digit numbers in decreasing order, or else it has a (non-empty) subsequence consisting of 100 repetitions of some finite sequence. (Each of the 1000-digit numbers and the subsequence have to be formed by consecutive digits of the original sequence.)

МБ **11.** Does there exist a rational number x such that

$$-2x^6 + 4x^5 - x^4 - 5x^3 + 12x^2 - 6x - 4$$

is the square of a rational number?

МБ **12.** On some island there lives a special species of chameleon. These animals can show 4 044 121 different colors. If a chameleon is tickled, it changes its color, and the new color only depends on the color it had before. A chameleon can also be tackled, with similar effects. By a suitable sequence of tickling and tackling, we can get a chameleon to show all its possible different colors, but this is not possible by only tickling or only tackling it. If a certain sequence S of ticklings and tacklings, applied to some color, reproduces that color, the same is true of every color, and the sequence S is called *unnecessary.* Show that the following is an unneccessary sequence: tickle 1005 times, tackle 1006 times, tickle 1006 times, tackle 1005 times.

Jacobs University Bremen
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IMO Training 2012
MatBoj
23 June, 2012

MAT-BOJ

Математический Бой: The Mathematical Battle

Problems

МБ **1.** Let $P_1, \ldots, P_{2012}$ be 2012 points in the 2011-dimensional unit cube $[0,1]^{2011}$ and let S be their convex hull. What is the largest possible volume of S?

МБ **2.** Let $A \subset \mathbb{R}$ be the set consisting of all numbers of the form

$$a = a_1\sqrt{b_1} + a_2\sqrt{b_2} + \ldots + a_n\sqrt{b_n}$$

with integers $a_1, a_2, \ldots, a_n$ and positive integers $b_1, b_2, \ldots, b_n$.
Find all solutions $(x_1, x_2, \ldots, x_{2012}) \in A^{2012}$ of the equation $x_1^2 + x_2^2 + \ldots + x_{2012}^2 = 1$.

МБ **3.** Is it possible to find uncountably many infinite sets of natural numbers such that any two of these sets have only finitely many common elements?

МБ **4.** Let $x_1, x_2, \ldots, x_{2012}$ be distinct elements of a finite set X. We choose a random permutation σ of X. Find the probability that $x_1, x_2, \ldots, x_{2012}$ belong to the same cycle of σ.

МБ **5.** Does there exist an infinite sequence of symbols a, b, c such that the sequence does not contain ss as a subsequence for any (non-empty) finite sequence s?

МБ **6.** We fix n lines in general position in the plane (so that no three lines pass through the same point, lines are not parallel, and if three intersection points are collinear, they are on one of the given lines). A further line is *good* if it does not pass through a point of intersection of the given lines, and two good lines are *equivalent* if one can be continuously moved to the position of the other with only good lines occurring in the process. Find the number of equivalence classes of good lines!

МБ **7.** Let $F : [0,1] \to [0,1]$ be a continuous function. Assume that there is some $x_0 \in [0,1]$ such that $F(x_0) \neq x_0$, but $F(F(F(x_0))) = x_0$. Show that there is some $y_0 \in [0,1]$ such that $F(y_0) \neq y_0$, but $F(F(F(F(y_0)))) = y_0$.

МБ **8.** Let $p > 3$ be a prime number. Let S be the set of all natural numbers less than p^2 that are coprime with p. Write

$$\sum_{n \in S} \frac{1}{n} = \frac{k}{l}$$

with integers k and l. Prove that k is divisible by p^2.

МБ **9.** Let T be a tetrahedron. Show that one can find two planes α and β such that the ratio of the areas of the orthogonal projections of T to α and β is at least $\sqrt{2}$.

МБ **10.** Let ABC be a triangle with $\angle CBA = 80°$ and $|AB| = |BC|$. Let O be an inner point of the triangle such that $\angle OAC = 10°$ and $\angle ACO = 30°$. Find $\angle OBA$.

МБ **11.** Let $S = \{\pm 1\}^n$ be the set of sequences of length n with entries ± 1. If $x = (x_1, \ldots, x_n) \in S$ and $y = (y_1, \ldots, y_n) \in S$, then we write $x * y = (x_1y_1, \ldots, x_ny_n)$. If $T \subseteq S$ and $z \in S$, then we set $T * z = \{t * z \mid t \in T\}$. Prove that for any subset $Z \subseteq S$ of cardinality k one can find a sequence $s \in S$ such that $Z \cap (Z * s)$ has at most $k^2 2^{-n}$ elements.

МБ **12.** Let f be a continuous function such that for all $a, b > 0$ we have $\lim_{n\to\infty} f(an + b) = 0$, where the limit is over natural numbers $n \in \mathbb{N}$. Does it follow that $\lim_{x\to+\infty} f(x) = 0$?

Further problems:

1. Let T be a tetrahedron of unit volume. Let us choose one point on each edge of T. Consider the set M of all barycenters for all possible such sets of six points. Calculate the volume of M.

2. A natural number k is considered *good*, if for each n the number $1^k + 2^k + \cdots + n^k$ is divisible by $1 + 2 + \cdots + n$. Describe the set of all good numbers.

3. The street map of IMO City is a square grid with n vertical and n horizontal streets. A tramway line runs from the south-west corner to the north-east corner. Prove that a pedestrian walking from the north-west corner to the south-east corner crosses the tramway line in an odd number of points.

Jacobs University Bremen
School of Engineering and Science
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IMO Training 2013
MatBoj
22 June, 2013

MAT-BOJ

Математический Бой: The Mathematical Battle

Problems

МБ **1.** Let a, b, c, d be integers, coprime in pairs. Consider two line segments in the plane $\mathbb{R}^2$: I with endpoints $(0,0)$, (a,b) and J with endpoints $(0,0)$, (c,d). Two points (x,y) and $(x',y') \in \mathbb{R}^2$ are said to be *similar* if $x'-x$ and $y'-y$ are both integers. Let n be the number of pairs of points (P,Q) which are similar and such that P is an internal point of I and Q is an internal point of J. Compute n in terms of a, b, c, d.

МБ **2.** Let C be a convex polygon and P a point in its interior. Let N be the number of vertices V such that the line joining P to V divides the internal angle of C at V into two acute angles. Denote by n the number of sides of C such that the foot of the perpendicular from P to that side is strictly inside that side. Prove that $N = n$.

МБ **3.** Denote by $\mathbb{R}_+$ the set of nonnegative real numbers. We are given a sequence of functions $f_m \colon \mathbb{R}_+^m \to \mathbb{R}_+$, satisfying the following properties:

(a) Symmetry: $f_m(x_1, \ldots, x_m) = f_m(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ for any permutation σ.

(b) Monotonicity: $f_m(x, x_2, \ldots, x_m) > f_m(y, x_2, \ldots, x_m)$ if $x > y$.

(c) Homogeneity of degree 1: $f_m(\lambda x_1, \ldots, \lambda x_m) = \lambda f_m(x_1, \ldots, x_m)$ for all $\lambda \in \mathbb{R}_+$.

(d) For any $k < m$:

$$f_m(x_1, \ldots, x_m) = f_m\big(f_k(x_1, \ldots, x_k), \ldots, f_k(x_1, \ldots, x_k), x_{k+1}, \ldots, x_m\big).$$

(e) $f_2(0,1) = \frac{1}{2}$.

Prove that $f_m(x_1, \ldots, x_m) = \frac{1}{m}(x_1 + \ldots + x_m)$.

МБ **4.** The sequence (x_n) is defined by the initial value $x_0 \in [0,1]$ and the recursive formula $x_{n+1} = \dfrac{1 - \sqrt{1 - x_n}}{2}$. Find $\lim_{n\to\infty} 4^n x_n$.

МБ **5.** Consider a sequence of digits $\ldots\ldots 625$, infinite to the left, such that for every $n \geq 1$, the last n digits form an n-digit number x_n (possibly with some leading zeros) such that x_n^2 ends with x_n. Prove that the sequence is not eventually periodic.

МБ **6.** For non-negative $a, b, c \in \mathbb{R}$ prove that

$$\sqrt{a^2+b^2+c^2} + 2\sqrt{ab+bc+ca} \geq \sqrt{a^2+2bc} + \sqrt{b^2+2ca} + \sqrt{c^2+2ab}.$$

МБ **7.** Let A be an infinite set of natural numbers. Prove that there exists a real number $z > 2013$ such that $A \cap \{[z^n] : n \in \mathbb{N}\}$ is infinite.

МБ **8.** For $x \in \mathbb{R}$ let $s(x)$ denote the distance from x to the nearest integer. For $q \in \mathbb{R}$, $|q| < 1$, set

$$f_q(x) = \sum_{n=0}^{\infty} q^n s(2^n x)\,.$$

For which values of q does there exist a polynomial p such that $f_q(x) = p(x)$ for all $0 \leq x \leq 1$?

МБ **9.** Find a closed form for u_n, where $u_0 = 1$, $u_1 = 3$, $u_2 = 135$ and

$$u_{n+3} = \frac{u_{n+2}^2(36u_{n+2}u_{n+1}^4u_n^3 + 21u_{n+2}^2u_n^6 - 35u_{n+1}^8)}{18u_{n+1}^3u_n^6}\,.$$

МБ **10.** Three bankers sit around a table. The bankers have together n coins, whose values are $1, 2, \ldots, n$. A legal move consists in passing the most valuable coin a banker has in its possession to his right neighbor; the value of this coin must be larger than the value of any coin the right neighbor might already possess. What is the smallest number of legal moves necessary to transfer all the coins from one banker to his left neighbor?

МБ **11.** Consider a coin moving on a strip of six squares, numbered from 0 to 5. For each move, a fair die that carries the numbers $1, 1, 2, 2, 3, 3$ is rolled; the coin has to move the number of squares shown by the die either to the left or to the right. The game ends when the coin reaches square 0. At which of the squares 1 to 5 should you place the coin to achieve the smallest expected number of moves, and how large is this expected number?

МБ **12.** Can the number $27\,000$ be written as a sum of two squares and the ninth power of an integer?

Jacobs University Bremen
IMO Training Weekend 2014
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MatBoj
Sunday, June 8, 2014

MAT-BOJ

Математический Бой: The Mathematical Battle

Problems

МБ **1.** Let $a_1, \ldots, a_n$ be real numbers such that $a_j < a_{j+1} < a_j + 1$ for all $1 \le j < n$, and let $k < n$. Is it always possible to select indices $1 = j_0 < j_1 < \ldots < j_{k-1} < j_k = n$ such that

$$\max\{a_{j_{m+1}} - a_{j_m} : 0 \le m < k\} - \min\{a_{j_{m+1}} - a_{j_m} : 0 \le m < k\} < 1\,?$$

МБ **2.** Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x-y) = 2f(x)\cos y \qquad \text{for all } x, y \in \mathbb{R}.$$

МБ **3.** Let P_1, P_2, P_3, P_4 be four points in the plane, not all on a line. When is it true that the barycenter B of the four points minimizes the sum of the distances to P_1, P_2, P_3, P_4?

МБ **4.** Describe all maps $f\colon \mathbb{Q} \to \mathbb{Q}$ such that

$$|x-y| = |x-z| \implies |f(x) - f(y)| = |f(x) - f(z)| \qquad \text{for all } x, y, z \text{ in } \mathbb{Q}.$$

МБ **5.** Let $f\colon [0,1] \to \mathbb{R}$ be such that $f(tx + (1-t)y) > tf(x) + (1-t)f(y)$ for all $0 \le x < y \le 1$ and all $0 < t < 1$. Show that there is a constant $C > 0$ such that for all $n \ge 1$, the number of $(x,y) \in \frac{1}{n}\mathbb{Z} \times \frac{1}{n}\mathbb{Z}$ with $0 \le x \le 1$ and $y = f(x)$ is bounded by $Cn^{2/3}$.

МБ **6.** Define a sequence of integers by $a_0 = 0$, $a_{n+1} = a_n^{2014} + 2014$. Show that there is a sequence $(b_n)_{n \ge 1}$ of integers, coprime in pairs, such that for all $n > 0$, a_n is the product of the b_d, where d runs through the positive divisors of n.

МБ **7.** A game is played on an 8×8 chessboard. At the beginning of the game, each cell of the lower half of the chessboard contains a white piece and each cell of the upper half contains a black piece. A move consists in exchanging two pieces on cells that share a common side. What is the minimal number of moves necessary to move all black pieces to the lower half of the chessboard?

МБ **8.** We play chess on an $n \times n \times n$-cube. A *rook* threatens all fields in the three rows parallel to the coordinate axes the rook is placed in (i.e., all fields that share at least two of the coordinates with the rook's field). What is the minimal number of rooks needed to threaten every field of the $n \times n \times n$-cube?

МБ **9.** We define $a_n \in \{1, \ldots, 9\}$ to be the leading decimal digit of 2^n, for $n \ge 0$. Fix $k \ge 1$. Show that there is some m such that every subsequence of (a_n) consisting of m consecutive terms contains $(a_0, a_1, \ldots, a_k)$ (again as a subsequence of consecutive terms).

МБ **10.** Determine the maximal number of distinct points P_j in the plane, no three on a line, such that each line seqment P_iP_j (for $i < j$) intersects at most one other line seqment.

МБ **11.** The quadrilateral $ABCD$ is inscribed in a circle with center O. Its diagonals intersect at the point K. A circle with center on the segment OK intersect the side AB at points A_1, B_1 and the side CD at points C_1, D_1, such that the points A_1, K, C_1 are collinear and $|A_1K| \ne |KC_1|$. Prove that the points B_1, K, D_1 are also collinear.

МБ **12.** By a 'convex body', we mean a convex polyhedron with interior points. Is it possible to place 2014 convex bodies in 3-dimensional space such that any two of them have a common point, but no common inner point, and no three of them have a common point?

МБ **13.** (Reserve:) Does every group of order 2014 occur as the group of (orientation-preserving) symmetries of a figure in 3-dimensional space?

Jacobs University Bremen
IMO Training Weekend 2015
Alexei Belov, Michael Stoll

MatBoj
Saturday, June 27, 2015

MAT-BOJ

Математический Бой: The Mathematical Battle

Problems

МБ **1.** The quadrilateral $ABCD$ is inscribed into a circle with center O. Prove that the centers of the incribed circles of the triangles OAB, OBC, OCD, ODA lie on a circle if and only if $ABCD$ has an inscribed circle.

МБ **2.** Determine the set of all prime numbers p such that the size of the set

$$M_p = \{(a, b, c, d) \mid a, b, c, d \in \mathbb{Z}/p\mathbb{Z}, ad \neq bc\}$$

has at most three distinct prime divisors.

МБ **3.** A policeman is chasing a gangster in a city which is built like a cross formed by two segments of length 2 that intersect at right angles in the middle. The policeman can see the gangster only if he bumps into him, and they both can only move along the cross. The speed of the policeman is 10 times that of the gangster. Is there some $T > 0$ such that the policeman will be able to catch the gangster in time at most T?

МБ **4.** In a group of n people, subgroups with common interests (football, dancing, philately, ...) are formed. The number of distinct subgroups equals 2^{n-1}, and any three (not necessarily distinct) subgroups have a common member. Prove that in fact *all* subgroups have a common member.

МБ **5.** Let $Q = [0, 1]^4 \subset \mathbb{R}^4$ be the four-dimensional unit cube and let $L \subset \mathbb{R}^4$ be a two-dimensional affine plane. What is the maximal area of the intersection $Q \cap L$?

МБ **6.** In Syldavia, there are coins of value 1/2, 1/3, 1/4 and so on. There is a law that forbids every Syldavian to have more than one coin of any particular value in their possession at midnight. One day, it is decided that the higher-valued coins should be removed from circulation, and so it is made illegal to possess coins of value $\geq 1/N$ on day N after the law is enacted. To help the citizens comply with the new law, the Syldavian central bank generously offers to pay a coin of value $1/(n + 1)$ and a coin of value $1/m$ where $m = n(n + 1)/2$ in exchange for a coin of value $1/n$ (for any $n \geq 2$). Can all Syldavians keep avoiding violating the laws from now to eternity?

МБ **7.** Let a, b, c, a', b', c' be positive real numbers such that

$$a' + b' + c' \geq a + b + c, \quad a'b' + b'c' + c'a' \geq ab + bc + ca, \quad a'b'c' = abc\,.$$

Show that $(\log a')^2 + (\log b')^2 + (\log c')^2 \geq (\log a)^2 + (\log b)^2 + (\log c)^2$.

МБ **8.** Let $n \equiv 2 \bmod 6$ be a positive integer and set $m = (n^2 - n + 1)/3$. Show that m^2 divides $n^n - (n - 1)^{n-1}$.

МБ **9.** For $n \geq 1$, the nth Chebyshev polynomial $T_n(x) \in \mathbb{Q}[x]$ is characterized by the property that $T_n(\cos t) = \cos nt$ for all $t \in \mathbb{R}$.
Determine the set of positive integers n for which $T_n(x)$ is irreducible in $\mathbb{Q}[x]$.

МБ **10.** In Borduria, the postage you have to pay for sending a rectangular parcel with the Bordurian Mail is proportional to the sum of its height, width and depth. Can it ever be possible to save money when sending a parcel by putting it into another rectangular parcel?

МБ **11.** Determine the 2 500th digit to the right of the decimal point of the number $(\sqrt{101} + 10)^{2015}$.

МБ **12.** Let $f(x) = x^3 + ax^2 + bx + c \in \mathbb{Q}[x]$, with roots $\alpha, \beta, \gamma \in \mathbb{C}$. Assume that there are integers m, n, k with $(m, n) \neq (0, 0)$ such that $m\alpha + n\beta = k$. Show that one of α, β, γ is rational.

R 1. Let S_1, S_2 and S_3 be spheres that are tangent to one another externally. Suppose that there is a plane Π tangent to the spheres S_1, S_2 and S_3 at the points A, B, and C, respectively. Consider the sphere tangent to S_1, S_2, S_3 externally and tangent to the plane Π at some point D. Prove that the projections of D onto the lines AB, BC, and CA are the vertices of an equilateral triangle.

R 2. Let $0 \leq k, l \leq n$ be integers such that $n \leq k + l$. Show that

$$\frac{(2n)!\,(k+l)!\,k!\,l!}{n!\,(n-k)!\,(n-l)!\,(2k)!\,(2l)!\,(k+l-n)!} \in \mathbb{Z}\,.$$

Jacobs University Bremen
IMO Training 2016
Alexei Belov, Michael Stoll

MatBoj
Thursday, June 16, 2016

MAT-BOJ
Математический Бой: The Mathematical Battle

Problems

МБ **1.** (Alexei)
n girls and n boys attend a certain school. Some of the boys and girls are acquainted. Each girl is acquainted with precisely k boys and each boy is acquainted with precisely k girls. For any two girls there are precisely m boys they are both acquainted with. Prove that for any two boys there are precisely m girls they are both acquainted with.

Solution.
Let $A = (a_{ij})$ be the $n \times n$ matrix with entry 1 if boy i is acquainted with girl j and 0 otherwise. The assumptions imply that $A^\top A = (k-m)I + mee^\top$, where e is the all-ones column vector. We have to show that $AA^\top = (k-m)I + mee^\top$ as well. The determinant of $A^\top A$ is $(k+(n-1)m)(k-m)^{n-1}$ (since the matrix is circulant, it is easy to find the eigenvalues) and so is $\neq 0$ unless $k = m$. Since $Ae = ke = A^\top e$ (again from the assumptions), we have $k^2 e = A^\top A e = (k-m)e + mne$, and so $k = m$ implies $k = n$, which is the trivial case that all boys and girls know one another. So assume now $k < n$. Then A is invertible, and

$$\begin{aligned} AA^\top &= A(A^\top A)A^{-1} = A((k-m)I + mee^\top)A^{-1} \\ &= (k-m)I + m(Ae)(A^{-\top}e)^\top = (k-m)I + mkek^{-1}e^\top = (k-m)I + mee^\top \end{aligned}$$

as desired. (Note that $A^\top e = ke$ implies $A^{-\top}e = k^{-1}e$.)

МБ **2.** (Alexei)
At one end of a road of length ℓ, there are n people with $k < n$ bicycles. A bicycle can carry only one person. A walking person has speed v_1, and a cycling person has speed $v_2 > v_1$. What is the minimum time required to get everybody to the other end of the road?

Solution.
Since we can scale ℓ, v_1 and v_2 by a common factor without changing the result, we can assume that $\ell = n$.

The total distance traveled by the bicycles is (at most) kn, so there must be at least one person traveling at most distance k on a bicycle. So for this person, the required time is at least $k/v_2 + (n-k)/v_1$. On the other hand, this lower bound can be achieved by the following scheme. Split the n people into k plus $n-k$; say there are k women and $n-k$ men. Number the women from 1 to k and the men from 1 to $n-k$. Then woman number j takes a bicycle up to point j on the road, leaves it there, walks to point $n-k+j$ and takes a bicycle from there to the end of the road (if $j < k$). Man number j first walks to point j, then takes the waiting bicycle to point $k+j$, then leaves the bicycle and walks the remaining distance. Then everybody cycles distance k and walks distance $n-k$. We only have to check that there is always a bicycle available when one is required, which is not hard to do.

МБ **3.** Let $P(x) \in \mathbb{Z}[x]$ be a polynomial such that $P(n)$ is a square for all $n \in \mathbb{Z}$. Is $P(x)$ necessarily the square of a polynomial in $\mathbb{Z}[x]$?

Solution.
The answer is **Yes**.

We can clearly assume that P is not the zero polynomial. Observe that we can replace $\mathbb{Z}$ by $\mathbb{Q}$: If $P \in \mathbb{Z}[x]$ satisfies $P = Q^2$ with $Q \in \mathbb{Q}[x]$, then Q must have integral coefficients (because $\mathbb{Z}$ is integrally closed). Pick $a \in \mathbb{Z}_{>0}$ such that $P(x)$ and $P(x+a)$ have no common factors. The polynomial $P(x)P(x+a)$ has even degree and its leading coefficient is a square, so we can write $P(x)P(x+a) = Q(x)^2 + R(x)$ with polynomials $Q, R \in \mathbb{Q}[x]$ such that $\deg Q = \deg P$, Q has positive leading coefficient and $\deg R < \deg P$ (by successively completing the square). Let d be a common denominator of the coefficients of Q. Then we have

$$d^2 P(x)P(x+a) = (dQ(x))^2 + d^2 R(x)$$

and dQ and d^2R have coefficients in $\mathbb{Z}$. Then for $b \in \mathbb{Z}$ large enough, $|d^2R(b)| < 2dQ(b) - 1$. For any such b, we have that $(dQ(b))^2 + d^2R(b) = d^2P(b)P(b+a)$ is a square in $\mathbb{Z}$ by the assumption on P. Since $(dQ(b) \pm 1)^2 = (dQ(b))^2 \pm 2dQ(b) + 1$ and $|d^2R(b)| < 2dQ(b) - 1$, this is only possible when $R(b) = 0$. So the polynomial R vanishes at infinitely many points; therefore $R = 0$ and $P(x)P(x+a) = Q(x)^2$ (and $d = 1$). Since $P(x)$ and $P(x+a)$ are coprime, this implies that we can write $Q(x) = cQ_1(x)Q_2(x)$ such that $P(x) = cQ_1(x)^2$ and $P(x+a) = cQ_2(x)^2$. Setting x equal to some integer that is not a root of P then shows that $c = c_1^2$ is a square, hence $P(x) = (c_1Q_1(x))^2$ as desired.

МБ **4.** (Henri Cohen)
A and B play the following game: A chooses a real number a_0, then B chooses a real number a_1. For $n \geq 2$, they compute $a_n = |a_{n-1}| - a_{n-2}$. B wins if $a_{1\,000\,001} = \pi$, otherwise A wins. Who can enforce a win?

Solution.
The key observation is that the sequence (a_n) is always periodic with period 9. This can be shown by a suitable case distinction. So we have $a_{1\,000\,001} = a_2 = |a_1| - a_0 \geq -a_0$. This implies that A can prevent B from winning by taking $a_0 < -\pi$, so A can enforce a win.

МБ **5.** (Reserve geometry problem from 2015)
Let S_1, S_2 and S_3 be spheres that are tangent to one another externally. Suppose that there is a plane Π tangent to the spheres S_1, S_2 and S_3 at the points A, B, and C, respectively. Consider the sphere tangent to S_1, S_2, S_3 externally and tangent to the plane Π at some point D. Prove that the projections of D onto the lines AB, BC, and CA are the vertices of an equilateral triangle.

Solution.
This is a 'brute-force' solution. We can assume that D is the origin of the plane Π and that the radius of the fourth sphere is 1. We use the Euclidean inner product on Π. Let a, b, c be the radii of the first three spheres. An easy argument with the quadrilateral formed by A, B and the midpoints of the two spheres involved shows that $(B-A)^2 = 2ab$ (and similarly for the other points). An analogous argument shows $A^2 = 2a$ etc. This implies that $A \cdot B = a + b - ab$ etc. Let X, Y, Z be the projections of D onto the lines AB, BC, CA. We can express X as a convex combination of A and B by solving $X \cdot (B - A) = 0$, and similarly for Y and Z. We finally find that $(Y-X)^2 - (Z-Y)^2$ is a rational expression in a, b, c whose denominator divides $(abc)^2$ and whose numerator is a multiple of the Gram determinant

$$\begin{vmatrix} A^2 & A \cdot B & A \cdot C \\ B \cdot A & B^2 & B \cdot C \\ C \cdot A & C \cdot B & C^2 \end{vmatrix} = \begin{vmatrix} 2a & a+b-ab & a+c-ac \\ a+b-ab & 2b & b+c-bc \\ a+c-ac & b+c-bc & 2c \end{vmatrix},$$

which vanishes, since A, B, C span a plane. So $|X - Y| = |Y - Z| = |Z - X|$ (the latter by symmetry), which is exactly the claim.

МБ **6.** Let a, b, c be positive integers. Show that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ can never be an odd integer.

Solution.
See http://mathoverflow.net/questions/227713/ .

МБ **7.** For $m \in \mathbb{Z}_{>0}$ define $N(m) = \{n \in \mathbb{Z}_{>0} : n \mid m^n - 1\}$. For which m is $N(m)$ an infinite set?

Solution.
The answer is that $N(m)$ is infinite exactly when $m \neq 2$.

First assume that $m > 2$. Note that $m - 1 \in N(m)$. I claim that all powers of $m - 1$ are in $N(m)$. This is proved by induction: knowing that $(m-1)^k$ divides $m^{(m-1)^k} - 1$, we have to show that $(m-1)^{k+1}$ divides $m^{(m-1)^{k+1}} - 1$. By assumption, $m^{(m-1)^k} = 1 + A(m-1)^k$ for some $A \in \mathbb{Z}$. Raising this equality to the $(m-1)$st power gives

$$m^{(m-1)^{k+1}} = \bigl(1 + A(m-1)^k\bigr)^{m-1} = 1 + A(m-1)^{k+1} + \sum_{j=2}^{m-1} \binom{m-1}{j} A^j (m-1)^{jk}.$$

Since $jk \geq k+1$ for $k \geq 1$ and $j \geq 2$, this is $\equiv 1 \bmod (m-1)^{k+1}$. Since $m \geq 3$, we have $m-1 \geq 2$, and so the powers of $m-1$ give infinitely many distinct elements of $N(m)$.

It is clear that $N(1) = \mathbb{Z}_{>0}$. So it remains to show that $N(2)$ is finite. In fact, we have $N(2) = \{1\}$. To see this, assume the contrary and let $1 < n \in N(2)$ be minimal. We have $2^n \equiv 1 \bmod n$, but also (by Euler's theorem) $2^{\varphi(n)} \equiv 1 \bmod n$, where φ is the Euler phi function. This implies $2^d \equiv 1 \bmod n$, where $d = \gcd(n, \varphi(n))$. Since d divides n, this also implies $2^d \equiv 1 \bmod d$, so $d \in N(2)$. By our choice of n, we must have $d = 1$. But then $2 = 2^d \equiv 1 \bmod n$, so $n = 1$, a contradiction.

МБ **8.** (Hendrik W. Lenstra, Jr.; via Henri Cohen)
Let P be a convex polygon with vertices $A_1, A_2, \ldots, A_n$ in cyclic order, such that all sides have equal lengths and the internal angles at $A_2, A_3, \ldots, A_{n-1}$ are rational multiples of π. Show that the remaining two angles are also rational multiples of π.

Solution.
We identify the plane with the complex numbers. Without loss of generality, $A_1 = 0$ and $A_2 = 1$. Then A_n is a sum of roots of unity such that $|A_n| = 1$. Let n be a multiple of all the orders of the roots of unity involved and let $K = \mathbb{Q}(\zeta)$ be the nth cyclotomic field, where ζ is a primitive nth root of unity, so that $A_n \in K$. The automorphisms of K are of the form σ_k with $\sigma_k(\zeta) = \zeta^k$ and $k \in (\mathbb{Z}/n\mathbb{Z})^\times$. In particular, σ_{-1} is complex conjugation (and the automorphism group is abelian). $1 = |A_n|^2 = \sigma_1(A_n)\sigma_{-1}(A_n)$ implies that

$$1 = \sigma_k(1) = \sigma_k(A_n)\sigma_{-k}(A_n) = |\sigma_k(A_n)|^2$$

for all k. So A_n is an algebraic integer all of whose conjugates have absolute value 1, therefore A_n must be (zero or) a root of unity. So the angle at A_1 is a rational multiple of π, which then implies that the last angle must also be a rational multiple of π.

МБ **9.** (Otfried Cheong)
Let $\mathcal{C} = \{x + [0,a]^3 : x \in \mathbb{R}^3, a > 0\}$ be the set of all axis-parallel cubes in $\mathbb{R}^3$. The union S of finitely many cubes from $\mathcal{C}$ is a (usually non-convex) polyhedron; we write $v(S)$ for the number of vertices of S. Let, for $n \geq 1$,

$$V(n) = \max\{v(C_1 \cup \ldots \cup C_n) : C_1, \ldots, C_n \in \mathcal{C}\}\,.$$

Find a real number k such that $\alpha n^k \leq V(n) \leq \beta n^k$ for all $n \geq 1$ with suitable $0 < \alpha < \beta$.

Solution.
The answer is $k = 2$.

The first step is to solve the corresponding problem in the plane, with $k = 1$. The lower bound is clear (use n pairwise disjoint squares, for example). For the upper bound, we observe that a vertex of the union is either a vertex of one of the original squares (and there are at most $4n$ of them), or else it is the point of intersection of sides of two distinct squares. We associate such a vertex to the smaller of the two squares involved (if they have the same size, we pick one arbitrarily). Now each side of a given square can give rise to at most two such vertices from intersections with squares that have at least the same size (since such an intersecting square will cover the side from the intersection point to one of its ends), which gives a linear bound (e.g., $8n$) for the number of vertices of this kind.

Now for the upper bound in $\mathbb{R}^3$. Any vertex of the union will be contained in a side of one of the cubes, in which case it will be a vertex of the projection of the union to the plane that contains the side. By the above, the number of such vertices is at most $12n$ (say), so the total number is at most $6n \cdot 12n = 72n^2$ (which is clearly not the best possible bound).

To get a lower bound of the correct size, we assume $n = 2m$ is even. We take m unit cubes with lower left bottom vertices at $(0, j\varepsilon, j\varepsilon)$ for $j = 0, 1, \ldots, m-1$, where $0 < \varepsilon < 2/n^2$. The cross-section perpendicular to the x-axis of their union has as part of its boundary a broken line from $(0,1)$ to $((m-1)\varepsilon, 1 + (m-1)\varepsilon)$ with $2m-1$ vertices. Now we arrange m further cubes of size $m\varepsilon < 1/(2m)$ with lower left bottom vertices at $(\varepsilon + k/m, -\varepsilon/2, 1 - \varepsilon/2)$ for $k = 0, 1, \ldots, m-1$. Then two sides of each of the small cubes will each form $2m-1$ vertices in the corresponding cross-section, which gives a lower bound of $2m(2m-1) = n^2 - n$ vertices in total.

МБ **10.** (Alexei)

The points of the set $L = \mathbb{R}^2 \times \left[0, \frac{1}{2016}\right]$ are colored in four colors. Prove that there exist two points at distance 1 in L that have the same color.

Solution.

We have the following lemma:

Lemma. *Fix $\varepsilon > 0$. Then there is $0 < \delta < \varepsilon$ with the following property. If the points of the cylinder $D \times [-\delta, \delta]$ (where D is the closed unit disk) are colored in four colors, then there are two points at distance 1 with the same color, or the points $(0, \delta)$ and $(0, -\delta)$ have the same color.*

Proof. We have

$$2 \arcsin \frac{1}{2\sqrt{1-\delta^2}} = \frac{\pi}{3} + \varphi(\delta)$$

with a continuous function φ such that $\varphi(\delta) \searrow 0$ as $\delta \searrow 0$. So we can choose $0 < \delta < \varepsilon$ such that $\varphi(\delta) = \frac{\pi}{3(6n-1)}$ for some $n \geq 1$. Consider D as a subset of $\mathbb{C}$ and let

$$S = \{\sqrt{1-\delta^2}\omega^m : m = 0, 1, \ldots, 6n-2\} \subset D\,,$$

where

$$\omega = e^{(\pi/3+\varphi(\delta))i} = e^{(2n\pi i)/(6n-1)}$$

is an $(6n-1)$st root of unity. Note that we have chosen δ precisely in such a way that $\sqrt{1-\delta^2}|\omega - 1| = 1$. This implies that two successive points (in the cyclic order) in S have distance 1. Now let Γ be the unit distance graph on the vertex set $(S \times \{0\}) \cup \{(0, \delta), (0, -\delta)\}$. Then Γ consists of a cycle of odd length $6n-1$ formed by the points in $S \times \{0\}$, together with two further vertices that are each connected to all points in the cycle. Either some pair of adjacent vertices of Γ has the same color, or Γ is properly 4-colored. In the latter case, assume that $(0, \delta)$ is red (say). Then the points in the cycle must be colored by the three other colors, and since the cycle has odd length, all three colors must be used. But this forces $(0, -\delta)$ to be also red. $\square$

Now assume the claim is false. Then in the lemma, we are always in the second case. Picking $\varepsilon > 0$ sufficiently small, the lemma implies that there is $\delta > 0$ such that for any two points $P, Q \in L$ at distance 2δ such that P and Q are not too close to the boundary of L and such that their projections to $\mathbb{R}^2$ are sufficiently close must have the same color. It is then easy to construct a sequence $P_1, P_2, \ldots, P_n$ of points such that all pairs of successive points satisfy the conditions above and $d(P_1, P_n) = 1$, which gives the desired contradiction.

Jacobs University Bremen
IMO Training 2016
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Математический Бой: The Mathematical Battle

Problems

МБ **1.** n girls and n boys attend a certain school. Some of the boys and girls are acquainted. Each girl is acquainted with precisely k boys and each boy is acquainted with precisely k girls. For any two girls there are precisely m boys they are both acquainted with. Prove that for any two boys there are precisely m girls they are both acquainted with.

МБ **2.** At one end of a road of length ℓ, there are n people with $k < n$ bicycles. A bicycle can carry only one person. A walking person has speed v_1, and a cycling person has speed $v_2 > v_1$. What is the minimum time required to get everybody to the other end of the road?

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Find a real number k such that $\alpha n^k \leq V(n) \leq \beta n^k$ for all $n \geq 1$ with suitable $0 < \alpha < \beta$.

МБ **10.** The points of the set $L = \mathbb{R}^2 \times \left[0, \frac{1}{2016}\right]$ are colored in four colors. Prove that there exist two points at distance 1 in L that have the same color.

R 1. For which $n > m > 0$ is the polynomial $X^n + X^m + 2$ irreducible in $\mathbb{Q}[X]$?

R 2. Let $0 \le k, l \le n$ be integers such that $n \le k + l$. Show that

$$\frac{(2n)!\,(k+l)!\,k!\,l!}{n!\,(n-k)!\,(n-l)!\,(2k)!\,(2l)!\,(k+l-n)!} \in \mathbb{Z}\,.$$

R 3. Determine the set of all prime numbers p such that the size of the set

$$M_p = \big\{(a,b,c,d) \;\big|\; a,b,c,d \in \mathbb{Z}/p\mathbb{Z}, ad \neq bc\big\}$$

has at most three distinct prime divisors.

R 4. Let R be an integral domain (i.e., R is a commutative ring with 1 such that $1 \neq 0$ and for all $a, b \in R$, $ab = 0$ implies $a = 0$ or $b = 0$). Let $a, b \in R$. We say that a *divides* b if there is some $c \in R$ such that $b = ac$. Now assume that a^2 divides b^2. Does it necessarily follow that a divides b?

R 5. Let $p \equiv 1 \bmod 3$ be a prime number. Show that

$$\sum_{\substack{0<k<p \\ k\equiv 1 \bmod 3}} \frac{1}{k} \equiv \frac{(-3)^{(p-1)/2} - 1}{p} \bmod p$$

in the sense that the difference of the two sides, when written as a fraction in lowest terms, has numerator divisible by p.

R 6. (Recycled from 2005)
Let ABC be a triangle. Consider the family of all triangles that have the same circumcircle and inscribed circle as ABC. Show that the barycenters of these triangles all lie on a circle.

R 7. (Recycled from 2005)
Let P be a convex n-gon ($n \ge 4$) that is subdivided into finitely many triangles. Prove that two of the triangles have a side in common.

R 8. (Recycled from 2005)
Each point of 3-dimensional space is colored in one of 5 colors, all of which occur. Prove that there exists a plane that is colored in at least 4 colors.

R 9. A very primitive computer can perform only two operations: add 1 to the natural number in its only memory cell or double it. A program for this computer consists of a finite sequence of these operations. When the program starts, the memory cell contains the number zero. How many different natural numbers can you compute on this machine with a program that consists of exactly n steps?