Optimality condition for infinite horizon optimal control problem

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Let $X$ be a nonempty open convex subset of $\mathbb{R}^n$, $U$ be an arbitrary nonempty set in $\mathbb{R}^m$. Let us consider the functional

$$
J(u(\cdot), x_0, t_0, T) := \int_{t_0}^{T} g(x(t), u(t), t) \, dt,
$$

that may be unbounded when $T \to \infty$, subject to the dynamic constraint

$$
\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0,
$$

where $u(t) \in U$ and $x(t) \in X$ exists for all $t \geq t_0$. Such control $u(\cdot)$ and trajectory $x(\cdot)$ are called admissible. Functions $f$ and $g$ are differentiable w.r.t. their first argument, $x$, and together with those partial derivatives are defined and locally bounded, measurable in $t$ for every $(x, u) \in X \times U$, and continuous in $(x, u)$ for almost every $t \in [0, \infty)$.

In addition to the maximum principle we find a new form of necessary conditions for the two following concepts of optimality. An admissible control $\hat{u}(\cdot)$ for which the corresponding trajectory $\hat{x}(\cdot)$ exists on $[t_0, +\infty)$ is

- **overtaking optimal (OO)** if for all admissible controls $u(\cdot)$
  $$
  \limsup_{T \to \infty} (J(u(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)) \leq 0,
  $$

- **weakly overtaking optimal (WOO)** if for all admissible controls $u(\cdot)$
  $$
  \liminf_{T \to \infty} (J(u(\cdot), x_0, t_0, T) - J(\hat{u}(\cdot), x_0, t_0, T)) \leq 0.
  $$

**Proposition.** Let for all $\tau \geq t_0$

$$
\lim_{\alpha \to 0} \liminf_{T \to \infty} \left( \frac{J(\hat{u}(\cdot), \hat{x}(\tau) + \alpha \zeta, \tau, T) - J(\hat{u}(\cdot), \hat{x}(\tau), \tau, T)}{\alpha} - \langle \hat{J}_u(\tau, T), \zeta \rangle \right) \geq 0,
$$
with all perturbations of the initial conditions, \( x(\tau) = \hat{x}(\tau) + \alpha \zeta \), such that the resulting trajectories are feasible, \( x(t) \in X \) in \([\tau, \infty)\), and the following upper limit being finite

\[
\limsup_{T \to \infty} \left| \hat{J}_x(\tau, T) \right|
\]

where we denote the derivative of the functional w.r.t. initial condition as

\[
\hat{J}_x(\tau, T) := \int_{\tau}^{T} K^*(t, \tau) \frac{\partial g}{\partial x}(\hat{x}(t), \hat{u}(t), t) \, dt.
\]

If control \( \hat{u} \) is OO, then for all \( \tau \in [t_0, \infty) \) and \( u \in \hat{U} \)

\[
\limsup_{T \to \infty} \left( \mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \right) \leq 0.
\]

If control \( \hat{u} \) is WOO, then for all \( \tau \in [t_0, \infty) \) and \( u \in \hat{U} \)

\[
\liminf_{T \to \infty} \left( \mathcal{H}(\hat{x}(\tau), u, \tau, \hat{J}_x(\tau, T), 1) - \mathcal{H}(\hat{x}(\tau), \hat{u}(\tau), \tau, \hat{J}_x(\tau, T), 1) \right) \leq 0,
\]

where we use the Hamilton-Pontryagin \( \mathcal{H}(x, u, t, \psi, \lambda) = \lambda g(x, u, t) + \langle \psi, f(x, u, t) \rangle \), brackets \( \langle \cdot, \cdot \rangle \) denote scalar product of two vectors, and \( \hat{U} = \hat{U}(\hat{x}(\tau), \tau) \) is the set of control values \( u(\tau) \) of all feasible pairs \((u(\cdot), x(\cdot))\) satisfying, for some scalar \( \lambda \) and vector \( \psi_0 \) such that \((\lambda, \psi_0) \neq 0\), the maximum condition:

\[
\mathcal{H}(x(t), v, t, \psi(t), \lambda) - \mathcal{H}(x(t), u(t), t, \psi(t), \lambda) \leq 0, \quad \forall v \in U,
\]

the state equation (1) with \( x_0 = \hat{x}(\tau) \) and \( t_0 = \tau \), and the adjoint equation:

\[
\dot{\psi}(t) = \frac{\partial \mathcal{H}}{\partial x}(x(t), u(t), t, \psi(t), \lambda), \quad \psi(\tau) = \psi_0.
\]

**Corollary.** Let, in addition to the conditions of the Proposition, there exists a number \( \beta(\tau) > 0 \) such that for all \( x(\tau) \in X \) satisfying the inequality \(|x(\tau) - \hat{x}(\tau)| < \beta(\tau)\), the initial value problem (1) with \( u = \hat{u} \) and the initial condition \( x(t_0) = \hat{x}(\tau) \) at \( t_0 = \tau \) has an admissible solution solution, i.e. \( x(t) \in X \) for all \( t \geq \tau \). Then in the Proposition \( \hat{U} = U \).

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