Supporting Information for ”Horizontal pressure gradient parameterization for one-dimensional lake models”

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Introduction This Supplementary Information contains text and figures which are cited from the main text of the paper. The text mostly contains mathematical derivations which are extensive and may interrupt the smooth reading of the main text, if placed there.
Text S1. Derivation of mechanical energy conservation in a multilayer fluid

Consider a multilayer fluid governed by a set of equations (7)-(10) from the main text:

\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= -\frac{1}{\rho_i} \frac{\partial p'_i}{\partial x} + l v_i, \\
\frac{\partial v_i}{\partial t} &= -\frac{1}{\rho_i} \frac{\partial p'_i}{\partial y} - l u_i, \\
\frac{\partial h'_i}{\partial t} + H_i \left( \frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) &= 0, \\
p'_i &= g \sum_{k=1}^{N} \rho_{\min(i,k)} h'_k, \quad i = 1, N.
\end{align*}
\]

and supplemented by boundary conditions:

\[
[(u_i, v_i) \cdot n]_{\Gamma_i} = 0, \quad i = 1, N,
\]

where we assume the lateral boundaries to be the same in each layer, i.e. \( \Gamma_i = \Gamma \), confining the same domain \( A \). The kinetic energy of \( i \)-th layer is defined as:

\[
K_i = \frac{1}{2} H_i \rho_i \int_A (u_i^2 + v_i^2) dA'.
\]

Hereafter, we will use a notion \( \overline{\ldots} \) \( \equiv \int_A \ldots dA' \) for any expression \( \ldots \) integrated in horizontal. First, we can use (1),(2), (4), standard integration rules and vanishing the normal velocity component at boundary \( \Gamma_i \), to get:

\[
\frac{dK_i}{dt} = g H_i \sum_{k=1}^{N} \rho_{\min(i,k)} \left[ h'_k \left( \frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) \right].
\]

This equation may be summed up over \( i \) to get a tendency of total kinetic energy \( K = \sum_{i=1}^{N} K_i \):

\[
\frac{dK}{dt} = g \sum_{i=1}^{N} H_i \sum_{k=1}^{N} \rho_{\min(i,k)} \left[ h'_k \left( \frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} \right) \right].
\]

Then, consider a quantity:

\[
\Pi_m = g \left( \sum_{k=m}^{N} h'_k \right)^2.
\]
Involving (3), its tendency reads:

$$\frac{d\Pi_m}{dt} = -g\left\{\sum_{k=m}^{N} H_k\left(\frac{\partial u_k}{\partial x} + \frac{\partial v_k}{\partial y}\right)\right\} \cdot \left[\sum_{k=m}^{N} h'_k\right].$$  \hspace{1cm} (10)

This equation may be rewritten after some algebra as:

$$\frac{d\Pi_m}{dt} = - \sum_{k=m}^{N} \sum_{i=m}^{N} a_{ik},$$  \hspace{1cm} (11)

where we introduced

$$a_{ik} = g H_i \left[h'_k \left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right)\right].$$  \hspace{1cm} (12)

Presently, define a variable:

$$A_e \equiv \sum_{m=1}^{N} \beta_m \Pi_m,$$  \hspace{1cm} (13)

with so far arbitrary coefficients $\beta_m$, so that

$$\frac{dA_e}{dt} = - \sum_{m=1}^{N} \beta_m \sum_{k=m}^{N} \sum_{i=m}^{N} a_{ik}.$$  \hspace{1cm} (14)

Now, coefficients $\beta_m$ can be found to match a requirement $\frac{dA_e}{dt} = - \frac{dK}{dt}$. Indeed, equation (8) may be rewritten as:

$$\frac{dK}{dt} = \sum_{i=1}^{N} \sum_{k=1}^{N} \rho_{\min(i,k)} a_{ik}. $$  \hspace{1cm} (15)

One can now easily check, that the choice

$$\beta_m = \Delta \rho_m = \begin{cases} \rho_1, & m = 1 \\ \rho_m - \rho_{m-1}, & m > 1 \end{cases}$$

ensures equality $\frac{dA_e}{dt} = - \frac{dK}{dt}$ and thus the conservation of $A_e + K$ in time.

Text S2. Generalization of the 1-st mode seiche model to a case of a basin with non-uniform depth

The objective is now to construct a generalization to the model (30)-(33) of the main text with corresponding mechanical energy conservation law (36)-(37) to a case with water depth.
basin (lake) with non-uniform depth distribution. Specifically, we assume that for each constant-density layer its horizontal cross-section is a rectangle $A = [-L_{xi}/2, L_{xi}/2] \times [-L_{yi}/2, L_{yi}/2]$, with both sizes decreasing while increasing $i$ (Figure S1). We will seek a model for the 1-st horizontal seiche mode for such a basin in a form:

$$\frac{d\bar{u}_i}{dt} = -\frac{\pi g}{2L_{xi}\rho_i} \sum_{k=1}^{N} \rho_{\text{min}(i,k)} a_{ki} \Delta_x h'_k + l\bar{u}_i, \quad (16)$$

$$\frac{d\bar{v}_i}{dt} = -\frac{\pi g}{2L_{yi}\rho_i} \sum_{k=1}^{N} \rho_{\text{min}(i,k)} b_{ki} \Delta_y h'_k - l\bar{u}_i, \quad (17)$$

$$\frac{d\Delta_x h'_i}{dt} = \frac{2\pi H_i}{L_{xi}} \bar{u}_i, \quad (18)$$

$$\frac{d\Delta_y h'_i}{dt} = \frac{2\pi H_i}{L_{yi}} \bar{v}_i, \quad i = 1, N, \quad (19)$$

where equations (18)-(19) remain exact for the 1-st Fourier mode in each $i$-th layer, whereas in (16)-(17) the newly introduced coefficients $a_{ki}$ and $b_{ki}$ are to be found from additional physical constrains to a system. The physical sense of $a_{ki}$ and $b_{ki}$ is contribution of $k$-th layer thickness deviation to mean pressure gradient in the $i$-th layer, so that they may be termed as pressure gradient transfer coefficients. Obviously, for the lake with uniform depth $a_{ki} = b_{ki} = 1$, $L_{xi} = L_x$, $L_{yi} = L_y$, and the above system reduces to (30)-(33) of the main text.

The additional physical constrain to define coefficients $a_{ki}$ and $b_{ki}$ would be the presence of mechanical energy conservation law for the system (16)-(19), generalizing (36)-(37) of the main paper.

The kinetic energy of $i$-th layer is now defined as:

$$K_i = \frac{1}{2} \rho_i H_i L_{xi} L_{yi} \left( \bar{u}_i^2 + \bar{v}_i^2 \right). \quad (20)$$
The tendency of the total basin kinetic energy $K = \sum_{i=1}^{N} K_i$ thus becomes:

$$\frac{dK}{dt} = -\frac{\pi g}{2} \sum_{i=1}^{N} \left[ \left( \sum_{k=1}^{N} \rho_{\text{min}(i,k)} a_{ki} \Delta x \delta h'_k \delta u_i L_{yi} H_i \right) + \left( \sum_{k=1}^{N} \rho_{\text{min}(i,k)} b_{ki} \Delta y \delta h'_k \delta v_i L_{xi} H_i \right) \right] = \sum_{i=1}^{N} \sum_{k=1}^{N} \rho_{\text{min}(i,k)} (A_{ki} + B_{ki}), \quad (21)$$

where

$$A_{ki} = -\frac{\pi g}{2} a_{ki} \Delta x \delta h'_k \delta u_i L_{yi} H_i, \quad (22)$$

$$B_{ki} = -\frac{\pi g}{2} b_{ki} \Delta y \delta h'_k \delta v_i L_{xi} H_i. \quad (23)$$

Now turn to construction of expression for available potential energy. First seek for the definition of intermediate variable $\Pi_m$ analogous to the same-named variable in (9). In general case it should be a quadratic form of $\Delta x \delta h'_k$ and $\Delta y \delta h'_k$:

$$\Pi_m = \Pi_{mx} + \Pi_{my}, \quad (24)$$

$$\Pi_{mx} = \sum_{k=m}^{N} \sum_{i=m}^{N} C_{ki} \Delta x \delta h'_k \Delta h'_x, \quad (25)$$

$$\Pi_{my} = \sum_{k=m}^{N} \sum_{i=m}^{N} B_{ki} \Delta y \delta h'_k \Delta h'_y. \quad (26)$$

Here, $\Delta h'_x$ and $\Delta h'_y$ are not “mixed” in one quadratic expression because first modes of $h'_k$ in $x$ and $y$ do not correlate spatially. The quadratic form matrix elements $C_{ki}$ and $B_{ki}$ are then sought from a requirement analogous to (11):

$$\frac{d\Pi_{mx}}{dt} = -\sum_{i=m}^{N} \sum_{k=m}^{N} A_{ki}, \quad (27)$$

$$\frac{d\Pi_{my}}{dt} = -\sum_{i=m}^{N} \sum_{k=m}^{N} B_{ki}, \quad (28)$$

so that

$$\frac{d\Pi_m}{dt} = -\sum_{i=m}^{N} \sum_{k=m}^{N} (A_{ki} + B_{ki}). \quad (29)$$
Consider the first of these equations (the second is treated identically). Substituting (25) and using continuity (18) yields:

\[
\frac{d\Pi_{mx}}{dt} = \sum_{i=m}^{N} \sum_{k=m}^{N} \left( C_{ki} + C_{ik} \right) \left[ \frac{2\pi H_{k\bar{k}} \Delta x_i'}{L_{xk}} \right] = - \sum_{i=m}^{N} \sum_{k=m}^{N} A_{ki}.
\]

(30)

Recalling the definition of $A_{ki}$, and equating element-wise the sums presented above, we get:

\[
C_{ki} + C_{ik} = \frac{gL_{xi}L_{yi}a_{ki}}{4},
\]

(31)

\[
C_{ik} + C_{ki} = \frac{gL_{xk}L_{yk}a_{ik}}{4},
\]

(32)

and hence

\[
a_{ki}L_{xi}L_{yi} = a_{ik}L_{xk}L_{yk}.
\]

(33)

Concomitantly, taking into account that for quadratic form $C_{ik} = C_{ki}$, those coefficients are defined by:

\[
C_{ik} = \frac{g}{8} a_{ki} L_{xi} L_{yi}.
\]

(34)

Equation (33) does not provide unique solution for $a_{ki}$ but it suggests to seek it in a form

\[
a_{ki} = \left( \frac{L_{xk}L_{yk}}{L_{xi}L_{yi}} \right)^n,
\]

which leads to $n = 1/2$, so that:

\[
a_{ki} = \sqrt{\frac{L_{xk}L_{yk}}{L_{xi}L_{yi}}},
\]

(35)

\[
C_{ik} = \frac{g}{8} \sqrt{L_{xi}L_{yi}L_{xk}L_{yk}},
\]

(36)

i.e. $a_{ki}$ is a square root of ratio of layers’s areas, while $C_{ki}$ is proportional to geometric mean between these two areas. In analogous way it is found that $B_{ik} = C_{ik}$ and $b_{ki} = a_{ki}$.

Now, when $\Pi_m$ is constructed, available potential energy reads:

\[
A_e = \sum_{m=1}^{N} \Delta \rho_m \Pi_m.
\]

(37)

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It is straightforward to check that this expression with $\Pi_m$ given above reduces to definition (37) of the main paper for a basin with uniform depth.

Due to certain arbitrariness of choice of expressions for $a_{ki}$, $b_{ki}$ and hence $C_{ik}$ it should be verified if desired properties of the system (16)-(19) and physical sense of available potential energy are matched. For instance, Sylvester’s criterium allows to establish that quadratic forms $\Pi_{mx}, \Pi_{my}$ with matrix coefficients (36) are positive-semidefinite, and hence $A_e \geq 0$. Moreover, physical sense of available potential energy dictates that $A_e$ should be zero only if all $\Delta_x h_i'$ and $\Delta_y h_i'$ are zero. This property can be straightforwardly proven at least for the case $N = 2$ where equation $A_e = 0$ leads to quadratic equations for $\Delta_x h_1'/\Delta_x h_2'$ and $\Delta_y h_1'/\Delta_y h_2'$ with no real solutions, so that only trivial solution $\Delta_x h_1' = \Delta_x h_2' = \Delta_y h_1' = \Delta_y h_2' = 0$ holds.

**Text S3. Identification of constant-density layers for a multilayer fluid model in a continuously stratified basin**

Assume, a continuous vertical density distribution $\rho(z)$ is given. Hereafter, the depth of a lake is taken uniform. The task is then to define the margins of layers which can be treated as constant-density layers in a multilayer fluid model.

First consider the simplest case when two layers have to be distinguished. The frequency of the first horizontal mode in two-layer model is given by well-known formula:

$$\omega = \frac{\pi}{L} \sqrt{\frac{gH_1 H_2 \Delta \rho}{\rho_2 (H_1 + H_2)}},$$

(38)

with $H_i, i = 1, 2$ – the layers’ thicknesses at rest, $\rho_1$ and $\rho_2$ – the densities of upper and lower layers, respectively (computed as means of $\rho(z)$ over $z \in (0, H_1)$ and $z \in (H_1, H_1 + H_2)$), and $\Delta \rho = \rho_2 - \rho_1$. The amplitudes of velocity and layer thickness

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deviations of each layer are related as:

\[ \hat{h}_i' = \frac{\pi H_i \hat{u}_i}{\omega L}, \quad i = 1, 2. \]  

(39)

The condition of multilayer model validity \( \hat{h}_i' \ll H_i \) when applied to (39) and using (38) yields:

\[ \Delta \rho \gg \frac{\hat{u}_i^2 \rho_2 (H_1 + H_2)}{g H_1 H_2}. \]  

(40)

Otherwise, deviation of layer thickness may reach or exceed the mean thickness. Assessing inequality (40) needs an apriori estimate of velocity amplitude \( \hat{u}_i \) which is usually available, at least in the order of magnitude. Thus, the condition (40) is necessary for the layer interface \( z = H_1 \) to be set in a two-layer model. However, two layers are usually not sufficient to represent the vertical structure of water motions in a lake, and a case \( N = 3 \) is usually considered, reflecting classical density profile schematization (epilimnion, metalimnion and hypolimnion) and allowing for the 2-d vertical mode of velocity. Below, we consider a general case, where the number of constant-density layers is not limited and defined by features of \( \rho(z) \) function only.

Assume \( \rho \) is non-decreasing function of \( z \). This is almost always true in water bodies, as unstably stratified layers are being quickly mixed to become quasi-neutrally stratified. Let \( \Delta \rho_H = \rho(H) - \rho(0) > 0 \) be density difference between surface and bottom, where \( H \) is the maximal lake depth. Consider an integer \( N > 1 \) and define \( \rho_i = \rho(0) + i/N \Delta \rho_H, \quad i = 1, \ldots N - 1 \), and corresponding depths \( z_i : \rho(z_i) = \rho_i \). Thus, the density difference \( \Delta \rho_H \) is divided in \( N \) equal parts, and the depths where the "boundary" density values \( \rho_i \) take place are identified (the case of \( N = 3 \) is shown at Figure S2). Is the system of constant-density layers with interfaces locating at \( z_i \) meets the condition of smallness of thickness.
deviations? The answer for case $N = 2$ is defined by checking inequality (40), but the analogous solution even for $N = 3$ becomes hardly tractable in analytical way.

We propose a practical solution, where each pair of layers, contacting at a set of depths $z_i, i = 1, \ldots, N - 1$ (there are $N - 1$ pairs) is tested against (40), and the system of layers is treated as appropriate, if and only if all pairs pass the test. This means, we assume that the thickness deviations of all layers are small if in each pair the deviations are small, when this pair is considered as comprising a water body isolated from other layers. Then, different sets of layers with different $N$ can be tested, and the maximal number $N$ with the corresponding layers still conforming the above formulated condition is preferential for using in a multilayer model, as the maximal number of vertical modes can be represented.
Figure S1. The scheme of horizontal pressure gradient parameterization in 1D lake model for a basin with non-uniform depth. Numerical grid of 1D model is shown by red lines; bold black lines are margins of constant-density layers; dashed black lines represent 1-st horizontal mode deviations of layers' interfaces; blue solid line denotes a lake bottom; $L_{xi}$ are sizes of constant-density layers in $x$ ($y$ axis being perpendicular to the plot plane)
Figure S2. Interfaces of constant-density layers \((z_1, z_2)\) to be used in a multilayer model given a continuous density profile (bold red curve). Dashed red lines denote mean density values in each of three layers. See details in Text S3