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First-Degree Entailment and its Relatives

Abstract. We consider a family of logical systems for representing entailment relations of various kinds. This family has its root in the logic of first-degree entailment formulated as a binary consequence system, i.e. a proof system dealing with the expressions of the form $\varphi \vdash \psi$, where both φ and ψ are single formulas. We generalize this approach by constructing consequence systems that allow manipulating with sets of formulas, either to the right or left (or both) of the turnstile. In this way, it is possible to capture proof-theoretically not only the entailment relation of the standard four-valued Belnap's logic, but also its dual version, as well as some of their interesting extensions. The proof systems we propose are, in a sense, of a hybrid Hilbert–Gentzen nature. We examine some important properties of these systems and establish their completeness with respect to the corresponding entailment relations.

Keywords: First-degree entailment, Belnap's logic, Four-valued logic, Consequence system.

1. Preliminaries: First-Degree Entailment, a Consequence System and Four Truth Values

The notion of *first-degree entailment* was first put into circulation by Belnap in a short abstract of his talk at the twenty-fourth annual meeting of the Association for Symbolic Logic held on Monday, December 28, 1959 at Columbia University in New York [4]. It was defined there as an expression of the form $\varphi \to \psi$, where both φ and ψ are formulas containing only $\wedge$, $\vee$, $\sim$ (and maybe other truth-functional connectives defined by these). For a justification of first-degree entailments Belnap developed a machinery of "tautological entailments" conceived as a tool of their "validation". Tautological entailments are essentially expressions of the form $\varphi_1 \vee \ldots \vee \varphi_m \to \psi_1 \wedge \ldots \wedge \psi_n$ (or reducible to them by special replacement rules), where every $\varphi_i \to \psi_j$ is in its turn of the form $\chi_1 \wedge \ldots \wedge \chi_m \to \xi_1 \vee \ldots \vee \xi_n$, where $\chi_1, \ldots, \chi_m, \xi_1, \ldots, \xi_n$ are all atoms (i.e. propositional variables or the

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negates thereof), and with some atom χ_i being the same as some atom ξ_j. Belnap remarks that tautological entailmenthood is effectively decidable, and observes strong equality between the set of first-degree theorems of the system **E** (of entailment) and the set of tautological entailments, see also [5].

In [2, Section 15.2] this set was formalized by a proof system operating with the first-degree entailments as primitive expressions. This system was called there $\mathbf{E}_{\mathrm{fde}}$ to emphasize that it presents, in fact, the first-degree entailment fragment of the calculus **E**. Dunn in [12] uses the label $\mathbf{R}_{\mathrm{fde}}$, because the first-degree entailment fragments of systems **R** and **E** are the same. Taking into account that, on an object language level, the semantic relation of entailment is often represented by the consequence sign ($\vdash$), we reproduce here this formalism as a "binary consequence system", the expressions of which are all of the form $\varphi \vdash \psi$, to be read as "φ has ψ as a consequence" (see, e.g., [13, p. 302]). We will refer to this system as **FDE**, which is most common nowadays. It consists of initial consequences taken as axioms, and also rules for transforming one consequences into the others:

System **FDE**:

$a1_{\mathrm{fde}}.\ \varphi \wedge \psi \vdash \varphi \qquad a2_{\mathrm{fde}}.\ \varphi \wedge \psi \vdash \psi$

$a3_{\mathrm{fde}}.\ \varphi \vdash \varphi \vee \psi \qquad a4_{\mathrm{fde}}.\ \psi \vdash \varphi \vee \psi$

$a5_{\mathrm{fde}}.\ \varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi$

$a6_{\mathrm{fde}}.\ \varphi \vdash {\sim}{\sim}\varphi \qquad a7_{\mathrm{fde}}.\ {\sim}{\sim}\varphi \vdash \varphi$

$r1_{\mathrm{fde}}.\ \varphi \vdash \psi;\ \psi \vdash \chi\,/\,\varphi \vdash \chi$

$r2_{\mathrm{fde}}.\ \varphi \vdash \psi;\ \varphi \vdash \chi\,/\,\varphi \vdash \psi \wedge \chi \qquad r3_{\mathrm{fde}}.\ \varphi \vdash \chi;\ \psi \vdash \chi\,/\,\varphi \vee \psi \vdash \chi$

$r4_{\mathrm{fde}}.\ \varphi \vdash \psi\,/\,{\sim}\psi \vdash {\sim}\varphi.$

Note, that four De Morgan laws (${\sim}\varphi \wedge {\sim}\psi \vdash {\sim}(\varphi \vee \psi)$, ${\sim}(\varphi \wedge \psi) \vdash {\sim}\varphi \vee {\sim}\psi$, ${\sim}\varphi \vee {\sim}\psi \vdash {\sim}(\varphi \wedge \psi)$, ${\sim}(\varphi \vee \psi) \vdash {\sim}\varphi \wedge {\sim}\psi$) are derivable in **FDE**. Alternatively, these laws can be taken as initial postulates, whereby $r4_{\mathrm{fde}}$ can be excluded from the list of initial rules, remaining admissible (see [14, pp. 14–15]).

Dunn in [10] initiated a highly innovative research program for semantic justification of the first-degree entailments, culminating in his paper [11]. The main point of the program consists in allowing underdetermined and overdetermined valuations that can in certain situations falsify logical laws or verify contradictions (see [26]). One way to achieve this is to treat valuation as a function from the set of sentences of a language to the powerset of classical truth values $\{t, f\}$. A truth-value function, so defined, produces exactly four possible assignments that can be ascribed to propositions: $\varnothing, \{f\}, \{t\}, \{f, t\}$.

Belnap in his seminal papers [6] and [7] (reproduced in [3] as Section 81) famously explicated these assignments as new *truth values*: $N = \varnothing$, $F = \{f\}$, $T = \{t\}$ and $B = \{f, t\}$, thus obtaining a "useful four-valued logic" for a "computer-based reasoning". Truth values for compound formulas containing $\wedge, \vee$ and $\sim$ are determined by the following matrices:

$\sim$	
T	F
B	B
N	N
F	T

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	B	N	F
T	T	T	T	T
B	T	B	T	B
N	T	T	N	N
F	T	B	N	F

If a sentence has the truth value T, it is said to be *exactly true*; if it has one of the values T or B, it can be viewed as *at least true*, and analogously for falsehood. By defining entailment as a relation between sentences, one may rely on a basic understanding that valid inference always preserves truth as well as non-falsity—from a premise to the conclusion. Belnap implements this understanding in such a way that if the premise is *at least* true, so is the conclusion, and if the conclusion is *at least* false, so is the premise (cf. [3, p. 519]).

To grasp this idea formally, let a valuation v be a map from propositional variables to the four truth values, and let it be extended to compound formulas in accordance with the above matrices. Then we have:

Definition 1.1. $\varphi \models_{\text{fde}} \psi =_{df} \forall v : t \in v(\varphi) \Rightarrow t \in v(\psi)$.

Note, that this definition in fact presupposes the treatment of both T and B as designated truth values. Moreover, it also ensures the non-falsity preservation mentioned above:

Lemma 1.2. *$\varphi \models_{\text{fde}} \psi \Leftrightarrow \forall v : f \notin v(\varphi) \Rightarrow f \notin v(\psi)$.*

Proof. See Proposition 4 in [14]. ∎

Let $\varphi \vdash_{\text{fde}} \psi$ means that $\varphi \vdash \psi$ is derivable in **FDE**. Then we have the following fundamental theorem establishing soundness and completeness of **FDE** with respect to Definition 1.1:

Theorem 1.3. *$\varphi \models_{\text{fde}} \psi \Leftrightarrow \varphi \vdash_{\text{fde}} \psi$.*

Proof. See Theorem 7 in [14]. ∎

2. Proofs from the Assumptions: Belnap's Logic

A distinctive feature of the first-degree entailment as introduced originally by Belnap, and as determined both by the system **FDE** and Definition 1.1, is that it is construed as a relation between *single sentences*. In this way, we deal with a *pure relation of entailment* explicated as a set of pairs of sentences. Correspondingly, **FDE** is regarded as a *binary* consequence system involving only expressions of the formula–formula type, i.e., of the form $\varphi \vdash \psi$.

However, a standard logical practice is to allow inferring from the *sets* of premises. Entailment is to be defined, then, as a relation between a nonempty (maybe infinite) set of sentences (Γ) and a single sentence (ψ). It will then be a relation of the set–formula type.

Take for example Priest's treatment of first-degree entailment in [23]. On p. 144 of this comprehensive textbook we find the following definition (notation adjusted):

Definition 2.1. $\Gamma \vDash_{\mathrm{bl}} \psi =_{df} \forall v : (\forall \varphi \in \Gamma : t \in v(\varphi)) \Rightarrow t \in v(\psi)$.

The subscript stands here for "Belnap's logic", to avoid confusion with the pure first-degree entailment of Definition 1.1.[1] The property of a non-falsity preservation holds for $\vDash_{\mathrm{bl}}$ as well:

Lemma 2.2. $\Gamma \vDash_{\mathrm{bl}} \psi \Leftrightarrow \forall v : (\forall \varphi \in \Gamma : f \notin v(\varphi)) \Rightarrow f \notin v(\psi)$.

Proof. Analogously as the proof of Proposition 4 in [14]. Define for every valuation v its dual v^*, such that $t \in v^*(p) \Leftrightarrow f \notin v(p)$, and $f \in v^*(p) \Leftrightarrow t \notin v(p)$. A direct induction extends this valuation to any formula of the language. Now, assume $\Gamma \vDash_{\mathrm{bl}} \psi$. Consider an arbitrary valuation v, such that $\forall \varphi \in \Gamma : f \notin v(\varphi)$. We have then $\forall \varphi \in \Gamma : t \in v^*(\varphi)$, and hence, $t \in v^*(\psi)$. Thus, $f \notin v(\psi)$. The proof of the converse is similar. ■

Logical systems associated with $\vDash_{\mathrm{bl}}$ were studied, e.g., by Font in [16]. It is worth mentioning that Font's definition of $\vDash_{\mathrm{bl}}$ (see [16, p. 5]) differs from Definition 2.1, in that the former does not refer explicitly to *all* the sentences from the (maybe infinite) set Γ, but is limited to some of its finite subsets.

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[1] In the literature, the names "Belnap–Dunn logic" and "Dunn–Belnap logic" are also in circulation, but we retain here the term from [16]. Moreover, the term *first-degree entailment* is occasionally used in an extended sense for labeling *any* logic based on the four Belnapian truth values, see, e.g., [22,23]. However, for our purposes in this paper it is important to differentiate between **FDE** properly understood as the relation between single formulas, and other kinds of logical systems generally resting on Dunn's and Belnap's semantical framework.

Both definitions are equivalent, however, in view of the following *compactness theorem* (cf., e.g., [9, pp. 8–9], [23, p. 286]):

THEOREM 2.3. $\Gamma \models_{\text{bl}} \psi$ *iff there is a finite* $\Gamma' \subseteq \Gamma$, *such that* $\Gamma' \models_{\text{bl}} \psi$.

PROOF. $\Rightarrow$: Let $\Gamma \models_{\text{bl}} \psi$. Assume, for every finite $\Gamma' \subseteq \Gamma$, $\Gamma' \not\models_{\text{bl}} \psi$. That is, for every finite $\Gamma' \subseteq \Gamma$, there is a valuation v, such that $\forall \varphi \in \Gamma' : t \in v(\varphi)$, and $t \notin v(\psi)$. It can be demonstrated that there is a valuation v', such that for *every* finite $\Gamma' \subseteq \Gamma$, $\forall \varphi \in \Gamma' : t \in v'(\varphi)$, and $t \notin v'(\psi)$. Indeed, consider *some* finite $\Delta \subseteq \Gamma$, and the valuation v^{δ} in Δ, satisfying the assumption. We first observe that $v^{\delta}(\psi) = N$ or $v^{\delta}(\psi) = F$. Let us hold fix the ascription of the truth values to the propositional variables from ψ by the valuation v^{δ}.

Now, consider an *arbitrary* $\Gamma' \subseteq \Gamma$, and define a valuation v' for the formulas in Γ' as follows: (1) For any propositional variable q occurring in ψ, let $v'(q) = v^{\delta}(q)$. This would mean that $t \notin v'(\psi)$. Observe, that every valuation v defined on the four Belnapian truth values has the following property for any φ: If $t \in v(\varphi)$, then by changing for an arbitrary propositional variable occurring in φ its truth value to B, still $t \in v(\varphi)$ (this is clear from a direct check of the Belnapian truth tables). Thus, we can round off the definition of v' by the second clause: (2) For every propositional variable p not occurring in ψ, let $v'(p) = B$. Taking into account the above observation, we have $\forall \varphi \in \Gamma' : t \in v'(\varphi)$, and thus v' so defined is the required valuation. Note, that under this v' in *every* finite subset of Γ, for any formula φ, $t \in v'(\varphi)$. Since the set of all finite subsets of Γ is infinite, we may conclude that $\forall \varphi \in \Gamma : t \in v'(\varphi)$. Hence, $t \in v'(\psi)$. A contradiction.

$\Leftarrow$: Let $\Gamma' \models_{\text{bl}} \psi$ for some finite $\Gamma' \subseteq \Gamma$. Due to the monotonicity of $\models_{\text{bl}}$, for *any* Γ, such that $\Gamma' \subseteq \Gamma$, $\Gamma \models_{\text{bl}} \psi$. ■

We remark, that the property of compactness holds not only for $\models_{\text{bl}}$, but for every entailment relation considered in the present paper.

For a deductive formalization of $\models_{\text{bl}}$, Font constructs two proof systems. One [16, p. 7] is a usual Gentzen-type sequent calculus with an obvious restriction to a single formula in a succedent, and with the finite and non-empty Γ in antecedents of the sequents. For the purposes of the present paper, however, we are more interested in another system constructed by Font, which he characterizes as a "Hilbert-style axiomatization" of Belnap's logic [16, p. 10], denoting it by $\vdash_H$. This system consists only of the so-called direct rules of inferences of the form $\Gamma \vdash \psi$ (organized vertically in a two-level shape), and has no axioms. The set of rules for $\vdash_H$ is as follows:

(R1) $\dfrac{\varphi \wedge \psi}{\varphi}$ (R2) $\dfrac{\varphi \wedge \psi}{\psi}$ (R3) $\dfrac{\varphi, \psi}{\varphi \wedge \psi}$

(R4) $\dfrac{\varphi}{\varphi \vee \psi}$ (R5) $\dfrac{\varphi \vee \psi}{\psi \vee \varphi}$ (R6) $\dfrac{\varphi \vee \varphi}{\varphi}$

(R7) $\dfrac{\varphi \vee (\psi \vee \chi)}{(\varphi \vee \psi) \vee \chi}$ (R8) $\dfrac{\varphi \vee (\psi \wedge \chi)}{(\varphi \vee \psi) \wedge (\varphi \vee \chi)}$ (R9) $\dfrac{(\varphi \vee \psi) \wedge (\varphi \vee \chi}{\varphi \vee (\psi \wedge \chi)}$

(R10) $\dfrac{\varphi \vee \psi}{\sim\sim\varphi \vee \psi}$ (R11) $\dfrac{\sim\sim\varphi \vee \psi}{\varphi \vee \psi}$ (R12) $\dfrac{\sim(\varphi \vee \psi) \vee \chi}{(\sim\varphi \wedge \sim\psi) \vee \chi}$

(R13) $\dfrac{(\sim\varphi \wedge \sim\psi) \vee \chi}{\sim(\varphi \vee \psi) \vee \chi}$ (R14) $\dfrac{\sim(\varphi \wedge \psi) \vee \chi}{(\sim\varphi \vee \sim\psi) \vee \chi}$ (R15) $\dfrac{(\sim\varphi \vee \sim\psi) \vee \chi}{\sim(\varphi \wedge \psi) \vee \chi}$

A completeness theorem holds (see Theorem 3.11 in [16]):

THEOREM 2.4. *$\Gamma \vDash_{\text{bl}} \psi \Leftrightarrow \Gamma \vdash_H \psi$.*

Let us take a closer look at some deductive features of $\vdash_H$. This system is designed to establish valid consequences of the form $\Gamma \vdash_H \psi$. Although Font characterizes it as a "Hilbert-style presentation", he suggests a construction of its inferences in a tree-like form resembling natural deduction, see [16, p. 11]. By way of illustration, consider the following proofs of (a) $\varphi \vdash_H \sim\sim\varphi$ and (b) $\varphi \wedge \psi \vdash_H \sim\sim\varphi \wedge \psi$ in $\vdash_H$, inspired by Font's suggestion:

$$(a)\quad \dfrac{\dfrac{\dfrac{\varphi}{\varphi \vee \sim\sim\varphi}\,(\text{R4})}{\sim\sim\varphi \vee \sim\sim\varphi}\,(\text{R10})}{\sim\sim\varphi}\,(\text{R6}) \qquad (b)\quad \dfrac{\dfrac{\dfrac{\varphi \wedge \psi}{\varphi}\,(\text{R1})}{\sim\sim\varphi}\,(\text{a}) \qquad \dfrac{\varphi \wedge \psi}{\psi}\,(\text{R2})}{\sim\sim\varphi \wedge \psi}\,(\text{R3})$$

As one can see, these inferences are constructed as direct derivations in the form of trees, possibly branching upwards. The derived formula constitutes the root of a tree, whereas its leaves stand for the formulas from which the root is derived. Moreover, the derivations are conducted by employing some *implicit meta-rules*,[2] most crucially transitivity (cut) and contraction, ensuring thus the resulting derivability between the leaves and the root of a derivation tree, and removing redundant premises. The need for these rules becomes clear as soon as one reflects upon obtaining derivations of such elementary claims as $\varphi \wedge \psi \vdash_H \varphi \vee \psi$ or $\varphi \vdash_H \varphi \wedge \varphi$. Indeed, to obtain the former, we have to apply transitivity to R1 and R4, and the later is resulting from R3 by contraction. It will be shown below (Theorem 2.6) that weakening and permutation of the premises are also allowed in $\vdash_H$.

[2]For the inference rules like the ones in $\vdash_H$, which present direct consequence expressions, a meta-rule is an (indirect) principle that allows transition between such expressions, i.e., that allows one to state certain consequence expression based on some other(s) consequence expression(s).

We will now construct a proof system for Belnap's logic as a consequence system **BL**, to facilitate its direct comparison with **FDE**. Similar to **FDE**, **BL** has a number of initial consequence expressions taken as axioms, and certain inference rules for transforming consequences. However, as distinct from **FDE**, **BL** is not limited to binary consequences, and is a consequence system of the set–formula type, dealing more generally with expressions of the form $\Gamma \vdash \psi$.

System **BL**:

$a1_{\rm bl}$. $\varphi \wedge \psi \vdash \varphi$ $a2_{\rm bl}$. $\varphi \wedge \psi \vdash \psi$ $a3_{\rm bl}$. $\varphi, \psi \vdash \varphi \wedge \psi$

$a4_{\rm bl}$. $\varphi \vdash \varphi \vee \psi$ $a5_{\rm bl}$. $\varphi \vee \psi \vdash \psi \vee \varphi$ $a6_{\rm bl}$. $\varphi \vee \varphi \vdash \varphi$

$a7_{\rm bl}$. $\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi$

$a8_{\rm bl}$. $\varphi \vee (\psi \wedge \chi) \vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi)$

$a9_{\rm bl}$. $(\varphi \vee \psi) \wedge (\varphi \vee \chi) \vdash \varphi \vee (\psi \wedge \chi)$

$a10_{\rm bl}$. $\varphi \vee \psi \vdash {\sim}{\sim}\varphi \vee \psi$ $a11_{\rm bl}$. ${\sim}{\sim}\varphi \vee \psi \vdash \varphi \vee \psi$

$a12_{\rm bl}$. ${\sim}(\varphi \vee \psi) \vee \chi \vdash ({\sim}\varphi \wedge {\sim}\psi) \vee \chi$

$a13_{\rm bl}$. $({\sim}\varphi \wedge {\sim}\psi) \vee \chi \vdash {\sim}(\varphi \vee \psi) \vee \chi$

$a14_{\rm bl}$. ${\sim}(\varphi \wedge \psi) \vee \chi \vdash ({\sim}\varphi \vee {\sim}\psi) \vee \chi$

$a15_{\rm bl}$. $({\sim}\varphi \vee {\sim}\psi) \vee \chi \vdash {\sim}(\varphi \wedge \psi) \vee \chi$

$r1_{\rm bl}$. $\Gamma \vdash \varphi\,;\, \Delta, \varphi \vdash \psi \,/\, \Gamma, \Delta \vdash \psi$ $r2_{\rm bl}$. $\Gamma \vdash \psi \,/\, \Gamma, \varphi \vdash \psi$

$r3_{\rm bl}$. $\Gamma, \varphi, \varphi \vdash \psi \,/\, \Gamma, \varphi \vdash \psi$ $r4_{\rm bl}$. $\Gamma, \varphi, \psi \vdash \chi \,/\, \Gamma, \psi, \varphi \vdash \chi$.

It is easy to see that all the axiom schemata $a1_{\rm bl}$–$a15_{\rm bl}$ are simply direct linear consequence reformulations of the rules R1–R15 of $\vdash_H$. The only axiom schema of **BL** that is not a binary consequence, allowing thus multiple premises, is $a3_{\rm bl}$. Unlike **FDE**, **BL** has only structural rules among its initial inference rules. As observed above, these rules are implicit in $\vdash_H$ as meta-rules. The rules of *contraction* ($r3_{\rm bl}$) and *exchange* ($r4_{\rm bl}$) enable to treat the expressions to the left of $\vdash$ as lists, and they can be omitted if we consider Γ and Δ to be genuine sets. The inference rules of **FDE** ($r1_{\rm fde}$–$r4_{\rm fde}$) are admissible in **BL**, as will be demonstrated below.

An inference in **BL** is a finite sequence of consequences, in which every consequence is either an axiom, or is obtained from the preceding consequences by an inference rule. If $\Gamma \vdash \varphi$ is the last consequence in some inference in **BL**, then we say that this consequence is derivable in **BL** and mark this by $\Gamma \vdash_{\rm bl} \varphi$.

The following lemma allows to eliminate extraneous disjunctions and to turn disjunctions into conjunctions where appropriate (cf. Proposition 3.2. in [16]):

LEMMA 2.5. *For every schema* $a10_{bl}$–$a15_{bl}$ *of the form* $\varphi \vee \chi \vdash \psi \vee \chi$*:*
(a) $\varphi \vdash_{\text{bl}} \psi$*;*
(b) $\varphi \wedge \chi \vdash_{\text{bl}} \psi \wedge \chi$.

PROOF. (a) We have the following schema of an inference in **BL**:

1. $\varphi \vdash_{\text{bl}} \varphi \vee \psi$ $(a4_{\text{bl}})$
2. $\varphi \vee \psi \vdash_{\text{bl}} \psi \vee \psi$ $(ai_{\text{bl}},\ 10 \le i \le 15)$
3. $\psi \vee \psi \vdash_{\text{bl}} \psi$ $(a6_{\text{bl}})$
4. $\varphi \vee \psi \vdash_{\text{bl}} \psi$ (2, 3: $r1_{\text{bl}}$)
5. $\varphi \vdash_{\text{bl}} \psi$ (1, 4: $r1_{\text{bl}}$)

Having this schema, it is not difficult to obtain particular inferences for each case, by substituting φ and ψ with required formulas. For example, by substitution $\sim\sim\varphi$ for ψ, the above schema turns into the inference of $\varphi \vdash_{\text{bl}} \sim\sim\varphi$, where ai_{bl} will be $a10_{\text{bl}}$, etc.

(b) We have the following schema of an inference in **BL**:

1. $\varphi \wedge \chi \vdash_{\text{bl}} \varphi$ $(a1_{\text{bl}})$
2. $\varphi \vdash_{\text{bl}} \psi$ (a)
3. $\varphi \wedge \chi \vdash_{\text{bl}} \psi$ (1, 2: $r1_{\text{bl}}$)
4. $\varphi \wedge \chi \vdash_{\text{bl}} \chi$ $(a2_{\text{bl}})$
5. $\psi, \chi \vdash_{\text{bl}} \psi \wedge \chi$ $(a3_{\text{bl}})$
6. $\varphi \wedge \chi, \chi \vdash_{\text{bl}} \psi \wedge \chi$ (3, 5: $r1_{\text{bl}}$)
7. $\varphi \wedge \chi, \varphi \wedge \chi \vdash_{\text{bl}} \psi \wedge \chi$ (4, 6: $r1_{\text{bl}}$)
8. $\varphi \wedge \chi \vdash_{\text{bl}} \psi \wedge \chi$ (7: $r3_{\text{bl}}$) ■

BL and $\vdash_H$ are deductively equivalent in the sense that they determine one and the same set of consequences of the set–formula type:

THEOREM 2.6. $\Gamma \vdash_{\text{bl}} \psi \Leftrightarrow \Gamma \vdash_H \psi$.

PROOF. As already observed, all axioms of **BL** are direct reformulations of rules of $\vdash_H$, and *vice versa*. It remains to show that rules of inference of **BL** are justifiable within $\vdash_H$.

Consider $r1_{\text{bl}}$. We have to show, that if in $\vdash_H$ there is an inference of φ from Γ, and ψ from Δ, φ, then there is an inference of ψ from Γ, Δ. And it really is, as the following schema of inference in $\vdash_H$ shows:

$$\dfrac{\Delta \qquad \dfrac{\Gamma}{\varphi}\ (\textit{assumption})}{\psi}\ (\textit{assumption})$$

Consider $r2_{\mathrm{bl}}$. We have to show that if in $\vdash_H$ there is an inference of ψ from Γ, then there an inference of ψ from Γ, φ. We have the following schema of inference in $\vdash_H$:

$$\dfrac{\dfrac{\varphi \qquad \dfrac{\Gamma}{\psi}\ (\textit{assumption})}{\varphi \wedge \psi}\ (\mathrm{R3})}{\psi}\ (\mathrm{R2})$$

As to $r3_{\mathrm{bl}}$ and $r4_{\mathrm{bl}}$, they are trivially justifiable, since contraction and exchange are implicit in the notion of an inference accepted in $\vdash_H$, given that Γ in a consequence $\Gamma \vdash \psi$ is treated as a set of formulas. ■

Taking into account Theorem 2.4, this implies also the completeness of **BL** with respect to Definition 2.1. For another thing, **BL** is coincident with **FDE** in the sets of proven binary consequences:

THEOREM 2.7. $\varphi \vdash_{\mathrm{fde}} \psi \Leftrightarrow \varphi \vdash_{\mathrm{bl}} \psi$.

PROOF. By employing the completeness of **FDE** with respect to Definition 1.1, and completeness of **BL** with respect to Definition 2.1, and by the fact that the latter definition being restricted to the formula–formula consequences turns into the former. ■

For the record, all the inference rules of **FDE** are admissible in **BL**:

LEMMA 2.8. *Rules $r1_{\mathrm{fde}}$–$r4_{\mathrm{fde}}$ are admissible in* **BL**. *That is:*

(1) *If $\varphi \vdash_{\mathrm{bl}} \psi$ and $\psi \vdash_{\mathrm{bl}} \chi$, then $\varphi \vdash_{\mathrm{bl}} \chi$;*

(2) *If $\varphi \vdash_{\mathrm{bl}} \psi$ and $\varphi \vdash_{\mathrm{bl}} \chi$, then $\varphi \vdash_{\mathrm{bl}} \psi \wedge \chi$;*

(3) *If $\varphi \vdash_{\mathrm{bl}} \chi$ and $\psi \vdash_{\mathrm{bl}} \chi$, then $\varphi \vee \psi \vdash_{\mathrm{bl}} \chi$;*

(4) *If $\varphi \vdash_{\mathrm{bl}} \psi$, then $\sim\psi \vdash_{\mathrm{bl}} \sim\varphi$.*

PROOF. For $r1_{\mathrm{fde}}$ and $r2_{\mathrm{fde}}$ not just admissibility, but even derivability can be demonstrated:

(1) Rule $r1_{\mathrm{fde}}$ is just a particular case of $r1_{\mathrm{bl}}$, when Γ is a singleton, and Δ is empty.

(2) We have the following schema of an inference in **BL**:

1. $\varphi \vdash_{\mathrm{bl}} \psi$ (*assumption*)

2. $\varphi \vdash_{\mathrm{bl}} \chi$ (*assumption*)
3. $\psi, \chi \vdash_{\mathrm{bl}} \psi \wedge \chi$ ($a3_{\mathrm{bl}}$)
4. $\varphi, \chi \vdash_{\mathrm{bl}} \psi \wedge \chi$ (1, 3: $r1_{\mathrm{bl}}$)
5. $\varphi, \varphi \vdash_{\mathrm{bl}} \psi \wedge \chi$ (2, 4: $r1_{\mathrm{bl}}$)
6. $\varphi \vdash_{\mathrm{bl}} \psi \wedge \chi$ (5: $r3_{\mathrm{bl}}$)

(3) To prove an admissibility of $r3_{\mathrm{fde}}$ we first prove that the following rule is admissible in **BL**: if $\varphi \vdash_{\mathrm{bl}} \psi$, then $\varphi \vee \chi \vdash_{\mathrm{bl}} \psi \vee \chi$. To demonstrate this we have to show how an inference of $\varphi \vdash \psi$ in **BL** can be transformed into the inference of $\varphi \vee \chi \vdash \psi \vee \chi$. This can be proved by induction on the length of the initial inference of $\varphi \vdash \psi$. Let $\Gamma = \xi_1, \ldots, \xi_n$. Then $\Gamma^{\wedge}$ stands for $\xi_1 \wedge \ldots \wedge \xi_n$. We now can show that replacing every step of the form $\Gamma \vdash \xi$ in the given inference of $\varphi \vdash \psi$ by a derivation of the consequence $\Gamma^{\wedge} \vee \chi \vdash \xi \vee \chi$ will give an inference of $\varphi \vee \chi \vdash \psi \vee \chi$ in **BL**.

Indeed, it is not difficult to see that for every axiom schema $a1_{\mathrm{bl}}$–$a15_{\mathrm{bl}}$ of the form $\Gamma \vdash \xi$ the consequence $\Gamma^{\wedge} \vee \chi \vdash \xi \vee \chi$ is derivable in **BL** (in particular, for $a3_{\mathrm{bl}}$ we just have $(\varphi \wedge \psi) \vee \chi \vdash_{\mathrm{bl}} (\varphi \wedge \psi) \vee \chi$).

Next, for each inference rule $r1_{\mathrm{bl}}$–$r4_{\mathrm{bl}}$ it can be demonstrated that if we transform every premise of the form $\Gamma \vdash \varphi$ into the consequence $\Gamma^{\wedge} \vee \chi \vdash \varphi \vee \chi$, then provided that premises so transformed are derivable in **BL**, the analogous transformation $\Delta^{\wedge} \vee \chi \vdash \psi \vee \chi$ of the conclusion of the from $\Delta \vdash \psi$ will be derivable as well.

Consider $r1_{\mathrm{bl}}$. By inductive hypothesis assume $\Gamma^{\wedge} \vee \chi \vdash_{\mathrm{bl}} \varphi \vee \chi$ and $(\Delta^{\wedge} \wedge \varphi) \vee \chi \vdash_{\mathrm{bl}} \psi \vee \chi$. From the first assumption, using $a2_{\mathrm{bl}}$ and $r1_{\mathrm{bl}}$, we obtain $\Delta^{\wedge} \wedge (\Gamma^{\wedge} \vee \chi) \vdash_{\mathrm{bl}} \varphi \vee \chi$. By $a1_{\mathrm{bl}}$ we also have $\Delta^{\wedge} \wedge (\Gamma^{\wedge} \vee \chi) \vdash_{\mathrm{bl}} \Delta^{\wedge}$. Thus, by (2) we get $\Delta^{\wedge} \wedge (\Gamma^{\wedge} \vee \chi) \vdash_{\mathrm{bl}} \Delta^{\wedge} \wedge (\varphi \vee \chi)$. From the second assumption, using $a8_{\mathrm{bl}}$ and $a9_{\mathrm{bl}}$, we obtain $\Delta^{\wedge} \wedge (\varphi \vee \chi) \vdash_{\mathrm{bl}} \psi \vee \chi$, and then, by $r1_{\mathrm{bl}}$ $\Delta^{\wedge} \wedge (\Gamma^{\wedge} \vee \chi) \vdash_{\mathrm{bl}} \psi \vee \chi$. Using $a8_{\mathrm{bl}}$ and $a9_{\mathrm{bl}}$, we obtain $(\Delta^{\wedge} \wedge \Gamma^{\wedge}) \vee \chi \vdash_{\mathrm{bl}} \psi \vee \chi$, which was to be proved.

The cases with $r2_{\mathrm{bl}}$–$r4_{\mathrm{bl}}$ are rather straightforward, and are left to the reader.

In honor of Font, who observes an admissibility of this rule in his $\vdash_H$ (see the corresponding remark in [16], Proposition 3.3) we call it *Font's rule*.

Now, we have the following schema of an inference in **BL**:

1. $\varphi \vdash_{\mathrm{bl}} \chi$ (*assumption*)
2. $\psi \vdash_{\mathrm{bl}} \chi$ (*assumption*)
3. $\varphi \vee \psi \vdash_{\mathrm{bl}} \chi \vee \psi$ (1: Font's rule)

4. $\chi \vee \psi \vdash_{\text{bl}} \chi \vee \chi$ (2: Font's rule)
5. $\chi \vee \chi \vdash_{\text{bl}} \chi$ ($a6_{\text{bl}}$)
6. $(\varphi \vee \psi) \vdash_{\text{bl}} \chi$ (3, 4, 5: $r1_{\text{bl}}$, twice).

(4) To show an admissibility of $r4_{\text{fde}}$ (contraposition) we have to demonstrate, how an inference of $\varphi \vdash_{\text{bl}} \psi$ can be transformed into the inference of $\sim\psi \vdash_{\text{bl}} \sim\varphi$. This can be proved by induction on the length of the initial inference of $\varphi \vdash_{\text{bl}} \psi$. Let $\Gamma = \varphi_1, \ldots, \varphi_n$, and let $\Gamma^{\sim}$ stands for $\sim\varphi_1 \vee \ldots \vee \sim\varphi_n$. Then for a consequence $\Gamma \vdash \varphi$ we call $\sim\varphi \vdash \Gamma^{\sim}$ its *contrapositive image*. We have to show that replacing every step in the given inference of $\varphi \vdash \psi$ by a derivation of its contrapositive image will give an inference of $\sim\psi \vdash_{\text{bl}} \sim\varphi$ (cf. the proof of Proposition 11 in [14]).

For axiom schemata it is not difficult to see that their contrapositive images are derivable in **BL** (in particular, for $a3_{\text{bl}}$ its contrapositive image $\sim(\varphi \wedge \psi) \vdash \sim\varphi \vee \sim\psi$ is one of the De Morgan laws, derivable by Lemma 2.5 from $a14_{\text{bl}}$).

Moreover, for each inference rule we can show, that if the contrapositive images of its premises are derivable in **BL**, so is the contrapositive image of the conclusion.

Consider $r1_{\text{bl}}$. By inductive hypothesis assume $\sim\varphi \vdash_{\text{bl}} \Gamma^{\sim}$ and $\sim\psi \vdash_{\text{bl}} \Delta^{\sim} \vee \sim\varphi$. By $a4_{\text{bl}}$ we obtain $\sim\varphi \vdash_{\text{bl}} \Gamma^{\sim} \vee \Delta^{\sim}$. By $a4_{\text{bl}}$ and $a5_{\text{bl}}$ we also have $\Delta^{\sim} \vdash_{\text{bl}} \Gamma^{\sim} \vee \Delta^{\sim}$. Hence, by (3) $\Delta^{\sim} \vee \sim\varphi \vdash_{\text{bl}} \Gamma^{\sim} \vee \Delta^{\sim}$. By $r1_{\text{bl}}$ we obtain $\sim\psi \vdash_{\text{bl}} \Gamma^{\sim} \vee \Delta^{\sim}$.

Consider $r2_{\text{bl}}$. By inductive hypothesis assume $\sim\psi \vdash_{\text{bl}} \Gamma^{\sim}$. Using $a4_{\text{bl}}$ and $r1_{\text{bl}}$, we easily obtain $\sim\psi \vdash_{\text{bl}} \Gamma^{\sim} \vee \sim\varphi$.

Consider $r3_{\text{bl}}$. By inductive hypothesis assume $\sim\psi \vdash_{\text{bl}} \Gamma^{\sim} \vee \sim\varphi \vee \sim\varphi$. Since in **BL** one can derive $\Gamma^{\sim} \vee \sim\varphi \vee \sim\varphi \vdash_{\text{bl}} \Gamma^{\sim} \vee \sim\varphi$ (for this fact consult, e.g., [17, p. 125]), we get $\sim\psi \vdash_{\text{bl}} \Gamma^{\sim} \vee \sim\varphi$, using $r1_{\text{bl}}$.

Consider $r4_{\text{bl}}$. By inductive hypothesis assume $\sim\chi \vdash_{\text{bl}} \Gamma^{\sim} \vee \sim\varphi \vee \sim\psi$. Using $a5_{\text{bl}}$ and $r1_{\text{bl}}$ we easily get $\sim\chi \vdash_{\text{bl}} \Gamma^{\sim} \vee \sim\psi \vee \sim\varphi$. ■

BL is a consequence system of the set–formula type for establishing *provability* of formulas from certain assumptions (cf., [27, p. 487]). Provability can be regarded as a kind of *logical verification*. Let Γ generally stand for some data set (this may be some theory, or a set of empirical observations, or some computer database). Then a sentence ψ can be considered to be *logically verified with respect to* Γ, if and only if Γ *entails* ψ.

3. Nothing But the Truth

Thus, in view of the above observations, a logical verification of a sentence ψ with respect to some data (set of assumptions) Γ consists simply of inferring this sentence from Γ. If successful, one can state a logical coherence of ψ relative to Γ, to the point that ψ can be added to Γ in a coherent way, provided Γ is self-coherent (in the sense that it is at least internally consistent).

However, in accordance to Dunn and Belnap's relevantist methodology, not only data sets, but even some sentences from the data can be self-incoherent (which is not infrequent in a scientific practice). Such sentences may be regarded as being both true and false, thus obtaining the truth value B. Pietz and Rivieccio [22, p. 134] reasonably observe in this respect:

> [I]f a data set contains contradictory information, then something went wrong. It is exactly the contradictory, corrupted pieces of information that show that not all is well with the data set. Thus, we are well advised to keep the underdetermined value N and the overdetermined value B apart; it is of interest to know whether no information or too much information has been given. However, it is clear that if one decides to place one's trust in the data set, one should treat the corrupted data with suspicion.

This observation suggests an idea of considering T as the only designated truth value. Indeed, speaking of a provability in a strict sense, one might wish to exclude self-incoherent sentences from a set of assumptions and rely only upon the *exactly true* sentences. In this way we arrive at what Pietz and Rivieccio [22] call *exactly true logic*—**ETL**, demanding "nothing but the truth". The entailment relation of this logic is determined by the following definition:

Definition 3.1. $\Gamma \vDash_{\text{etl}} \psi =_{df} \forall v : (\forall \varphi \in \Gamma : v(\varphi) = T) \Rightarrow v(\psi) = T$.

A consequence system for **ETL** can be obtained by adding to **BL** the following axiom schema (with the corresponding subscript change in the other axiom schemata and inference rules):

$a16_{\text{etl}}$. $\sim\varphi \wedge (\varphi \vee \psi) \vdash \psi$.

ETL has certain interesting properties that might seem unusual, and which have been investigated in [22]. To take a closer look at these properties, we first observe that among the initial inference rules of **FDE**, only

$r1_{\text{fde}}$ and $r2_{\text{fde}}$ remain admissible in **ETL**, whereas $r3_{\text{fde}}$ and $r4_{\text{fde}}$ lose their admissibility:

LEMMA 3.2. *Rules $r1_{\text{fde}}$ and $r2_{\text{fde}}$ are admissible in **ETL**. That is:*

1. *If $\varphi \vdash_{\text{etl}} \psi$ and $\psi \vdash_{\text{etl}} \chi$, then $\varphi \vdash_{\text{etl}} \chi$;*
2. *If $\varphi \vdash_{\text{etl}} \psi$ and $\varphi \vdash_{\text{etl}} \chi$, then $\varphi \vdash_{\text{etl}} \psi \wedge \chi$.*

*However, rules $r3_{\text{fde}}$ and $r4_{\text{fde}}$ are inadmissible in **ETL**. That is:*

3. *It can be that $\varphi \vdash_{\text{etl}} \chi$ and $\psi \vdash_{\text{etl}} \chi$, but $\varphi \vee \psi \nvdash_{\text{etl}} \chi$;*
4. *It can be that $\varphi \vdash_{\text{etl}} \psi$, but $\sim\psi \nvdash_{\text{etl}} \sim\varphi$.*

PROOF. For $r1_{\text{fde}}$ and $r2_{\text{fde}}$ the proof is the same as in Lemma 2.8. For $r3_{\text{fde}}$ we remark that $\varphi\wedge\sim\varphi \vdash_{\text{etl}} \chi$ and $\psi\wedge\sim\psi \vdash_{\text{etl}} \chi$, but $(\varphi\wedge\sim\varphi)\vee(\psi\wedge\sim\psi) \nvdash_{\text{etl}} \chi$ (see [22, p. 129]). For $r4_{fde}$ we observe that the contrapositive of $a16_{\text{etl}}$ is not derivable in **ETL**. ■

Now, it should be noted that $a16_{\text{etl}}$ is a principle called *disjunctive syllogism*, which is much debated in the literature on relevant logic see, e.g., [8].[3] The point is that adding this principle to **BL** allows one to derive in the resulting system (**ETL**) the famous *negative paradox of relevance* (*ex contradictione quodlibet*):

1. $\varphi \wedge \sim\varphi \vdash \sim\varphi$ $(a2_{\text{etl}})$
2. $\varphi \wedge \sim\varphi \vdash \varphi$ $(a1_{\text{etl}})$
3. $\varphi \vdash \varphi \vee \psi$ $(a4_{\text{etl}})$
4. $\varphi \wedge \sim\varphi \vdash \varphi \vee \psi$ (2, 3: $r1_{\text{etl}}$)
5. $\sim\varphi, (\varphi \vee \psi) \vdash \sim\varphi \wedge (\varphi \vee \psi)$ $(a3_{\text{etl}})$
6. $(\varphi \vee \psi), \sim\varphi \vdash \sim\varphi \wedge (\varphi \vee \psi)$ (5: $r4_{\text{etl}}$)
7. $\varphi \wedge \sim\varphi, (\varphi \vee \psi) \vdash \sim\varphi \wedge (\varphi \vee \psi)$ (1, 6: $r1_{\text{etl}}$)
8. $\varphi \wedge \sim\varphi, \varphi \vdash \sim\varphi \wedge (\varphi \vee \psi)$ (3, 7: $r1_{\text{etl}}$)
9. $\varphi \wedge \sim\varphi, \varphi \wedge \sim\varphi \vdash \sim\varphi \wedge (\varphi \vee \psi)$ (2, 8: $r1_{\text{etl}}$)
10. $\varphi \wedge \sim\varphi \vdash \sim\varphi \wedge (\varphi \vee \psi)$ (9: $r3_{\text{etl}}$)
11. $\sim\varphi \wedge (\varphi \vee \psi) \vdash \psi$ $(a16_{\text{etl}})$
12. $\varphi \wedge \sim\varphi \vdash \psi$ (10, 11: $r1_{\text{etl}}$)

[3] We take this principle in a canonical form of *modus tollendo ponens*, whereas Pietz and Rivieccio [22, p. 130] prefer to employ it in a form of *modus ponendo ponens* for a material conditional $\varphi \wedge (\sim\varphi \vee \psi) \vdash \psi$, known also as Ackermann's rule γ from [1].

Although adding the negative paradox (and hence, disjunctive syllogism) to **FDE** collapses it into classical logic,[4] it is not the case with **BL**. Namely, as observed in the proof of Lemma 3.2, the consequence $(\varphi \wedge {\sim}\varphi) \vee (\psi \wedge {\sim}\psi) \vdash \chi$ is not derivable in **ETL**, even though both $\varphi \wedge {\sim}\varphi \vdash \chi$ and $\psi \wedge {\sim}\psi \vdash \chi$ are. This is simply due to **BL** not having disjunction elimination among its initial rules. Therefore, $r3_{\text{fde}}$ can become (and really is) inadmissible in **ETL**, even though the latter is a direct extension of the former.

Thus, system **BL** from the previous section is indeed an interesting multi-premise expansion of **FDE**, not only because it structurally suits better the idea of logical verification as a provability from the (true) assumptions, but also in that it allows a non-trivial extension leading to a strict verificationistic logic (sensitive to the self-incoherent assumptions in the data sets) presented by **ETL**.

Theorem 2.6 is straightforwardly extendable to the relationship between the Font-style version of **ETL** from [22] and its consequence formulation from this section, stating the deductive equivalency between them (one has only to remark that $a16_{\text{etl}}$ is just a reformulation of (R16) from [22]). Taking into account the completeness of Pietz & Rivieccio's system with respect to Definition 3.1 (see Theorem 3.4 in [22]), we obtain then the following:

Theorem 3.3. *$\Gamma \vDash_{\text{etl}} \psi \Leftrightarrow \Gamma \vdash_{\text{etl}} \psi$.*

4. Dual Belnap's Logic

Wansing in [27, p. 487], by distinguishing inferential relations of different kinds, considers not only provability from assumptions, but also *reducibility to absurdity from counterassumptions*. One can treat such a reducibility as a kind of *logical falsification* with respect to some data. From a pure structural standpoint, logical falsification is simply the dual to logical verification. This duality can be expressed by reversing the entailment relation so that $\Gamma \dashv \varphi$ (or more conventionally $\varphi \vDash \Gamma$) means that φ is falsified with respect to the data set Γ, or in Wansing's terminology, φ is reducible to absurdity from counterassumptions Γ. In this way, we deal with entailment relation of the formula–set type, which is structurally dual to that of Belnap's logic:

Definition 4.1. $\varphi \vDash_{\text{dbl}} \Gamma =_{df} \forall v : (\forall \psi \in \Gamma : f \in v(\psi)) \Rightarrow f \in v(\varphi)$.

[4] This holds for the formulation of **FDE** with the initial rule of contraposition as accepted in this paper. If one takes the formulation with De Morgan laws instead of contraposition, then adding the negative paradox results in the first-degree entailment fragment of Kleene's three-valued logic, see [14, p. 15].

To stress the falsificationistic point, this definition is formulated in terms of a backward falsity preservation from conclusions to the premise. But the property of a truth preservation in the forward direction holds for $\vDash_{\text{dbl}}$ as well:

LEMMA 4.2. $\varphi \vDash_{\text{dbl}} \Gamma \Leftrightarrow \forall v : t \in v(\varphi) \Rightarrow (\exists \psi \in \Gamma : t \in v(\psi))$.

PROOF. *Mutatis mutandis* like the proof of Lemma 2.2. ■

The entailment relation determined by Definition 4.1 can be formalized by a Font-style proof system, which we denote by $\vdash_{dH}$ and characterize by the following set of rules:

$$(\text{R1}_d)\ \frac{\varphi}{\varphi \vee \psi} \qquad (\text{R2}_d)\ \frac{\psi}{\varphi \vee \psi} \qquad (\text{R3}_d)\ \frac{\varphi \vee \psi}{\varphi, \psi}$$

$$(\text{R4}_d)\ \frac{\varphi \wedge \psi}{\varphi} \qquad (\text{R5}_d)\ \frac{\varphi \wedge \psi}{\psi \wedge \varphi} \qquad (\text{R6}_d)\ \frac{\varphi}{\varphi \wedge \varphi}$$

$$(\text{R7}_d)\ \frac{(\varphi \wedge \psi) \wedge \chi}{\varphi \wedge (\psi \wedge \chi)} \qquad (\text{R8}_d)\ \frac{(\varphi \wedge \psi) \vee (\varphi \wedge \chi)}{\varphi \wedge (\psi \vee \chi)} \qquad (\text{R9}_d)\ \frac{\varphi \wedge (\psi \vee \chi)}{(\varphi \wedge \psi) \vee (\varphi \wedge \chi}$$

$$(\text{R10}_d)\ \frac{\sim\sim\varphi \wedge \psi}{\varphi \wedge \psi} \qquad (\text{R11}_d)\ \frac{\varphi \wedge \psi}{\sim\sim\varphi \wedge \psi} \qquad (\text{R12}_d)\ \frac{(\sim\varphi \vee \sim\psi) \wedge \chi}{\sim(\varphi \wedge \psi) \wedge \chi}$$

$$(\text{R13}_d)\ \frac{\sim(\varphi \wedge \psi) \wedge \chi}{(\sim\varphi \vee \sim\psi) \wedge \chi} \qquad (\text{R14}_d)\ \frac{(\sim\varphi \wedge \sim\psi) \wedge \chi}{\sim(\varphi \vee \psi) \wedge \chi} \qquad (\text{R15}_d)\ \frac{\sim(\varphi \vee \psi) \wedge \chi}{(\sim\varphi \wedge \sim\psi) \wedge \chi}$$

This system is obtained by a direct dualization of the rules of $\vdash_H$. Dually to $\vdash_H$, inferences in $\vdash_{dH}$ constitute derivation trees branching downwards. Proposition 3.2 from [16] can be dualized as follows:

LEMMA 4.3. *For each rule* $(\text{R}i_d)$ $(10 \leq i \leq 15)$ *of the form* $\dfrac{\varphi \wedge \chi}{\psi \wedge \chi}$, *the following rules hold: (a)* $\dfrac{\varphi}{\psi}$, *and (b)* $\dfrac{\varphi \vee \chi}{\psi \vee \chi}$.

PROOF. (a) From φ we obtain $\varphi \wedge \varphi$ by (R6_d). Then, by $(\text{R}i_d)$ we get $\psi \wedge \varphi$, and finally, by (R4_d) we obtain ψ.

(b) From $\varphi \vee \chi$ by R3_d we get φ, χ. Then, using (a), we get ψ from φ, and by R1_d we obtain $\psi \vee \chi$. From χ we also obtain $\psi \vee \chi$, by R2_d. ■

We next axiomatize dual Belnap's logic by a consequence system of the formula-set type.

System **DBL**:

$a1_{\text{dbl}}.\ \varphi \vdash \varphi \vee \psi$ $\qquad a2_{\text{dbl}}.\ \psi \vdash \varphi \vee \psi$ $\qquad a3_{\text{dbl}}.\ \varphi \vee \psi \vdash \varphi, \psi$

$a4_{\rm dbl}$. $\varphi \wedge \psi \vdash \varphi$ $a5_{\rm dbl}$. $\varphi \wedge \psi \vdash \psi \wedge \varphi$ $a6_{\rm dbl}$. $\varphi \vdash \varphi \wedge \varphi$

$a7_{\rm dbl}$. $(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$

$a8_{\rm dbl}$. $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \vdash \varphi \wedge (\psi \vee \chi)$

$a9_{\rm dbl}$. $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

$a10_{\rm dbl}$. $\sim\sim\varphi \wedge \psi \vdash \varphi \wedge \psi$ $a11_{\rm dbl}$. $\varphi \wedge \psi \vdash \sim\sim\varphi \wedge \psi$

$a12_{\rm dbl}$. $(\sim\varphi \vee \sim\psi) \wedge \chi \vdash \sim(\varphi \wedge \psi) \wedge \chi$

$a13_{\rm dbl}$. $\sim(\varphi \wedge \psi) \wedge \chi \vdash (\sim\varphi \vee \sim\psi) \wedge \chi$

$a14_{\rm dbl}$. $(\sim\varphi \wedge \sim\psi) \wedge \chi \vdash \sim(\varphi \vee \psi) \wedge \chi$

$a15_{\rm dbl}$. $\sim(\varphi \vee \psi) \wedge \chi \vdash (\sim\varphi \wedge \sim\psi) \wedge \chi$

$r1_{\rm dbl}$. $\varphi \vdash \Gamma\,;\, \psi \vdash \varphi, \Delta \,/\, \psi \vdash \Gamma, \Delta$ $r2_{\rm dbl}$. $\varphi \vdash \Gamma \,/\, \varphi \vdash \psi, \Gamma$

$r3_{\rm dbl}$. $\varphi \vdash \psi, \psi, \Gamma \,/\, \varphi \vdash \psi, \Gamma$ $r4_{\rm dbl}$. $\varphi, \vdash \psi, \chi, \Gamma \,/\, \varphi \vdash \chi, \psi, \Gamma$.

DBL and $\vdash_{dH}$ are deductively equivalent in the sense that they determine one and the same set of consequences of the formula–set type:

Theorem 4.4. $\varphi \vdash_{\rm dbl} \Gamma \Leftrightarrow \varphi \vdash_{dH} \Gamma$.

Proof. *Mutatis mutandis* like the proof of Theorem 2.6. ■

In particular, it is easy to obtain for **DBL** the facts established in Lemma 4.3 with respect to $\vdash_{dH}$:

Lemma 4.5. *For every schema* $a10_{\rm dbl}$*–*$a15_{\rm dbl}$ *of the form* $\varphi \wedge \chi \vdash \psi \wedge \chi$*:*

(a) $\varphi \vdash_{\rm dbl} \psi$*;*

(b) $\varphi \vee \chi \vdash_{\rm dbl} \psi \vee \chi$*.*

Proof. Analogously as in the proof of Lemma 4.3. ■

Lemma 4.6. *Rules* $r1_{\rm fde}$*–*$r4_{\rm fde}$ *are admissible in* **DBL** *just like in* **BL** *(in the sense on Lemma* 2.8*).*

Proof. See *mutatis mutandis* (with appropriate dualizations) the proof of Lemma 2.8. ■

We now establish the completeness of **DBL** with respect to Definition 4.1:

Theorem 4.7. $\varphi \vDash_{\rm dbl} \Gamma \Leftrightarrow \varphi \vdash_{\rm dbl} \Gamma$.

Proof. First, the completeness of $\vdash_{dH}$ with respect to Definition 4.1 can be obtained by a direct dualization of the corresponding proof from [16]. Namely, in Definition 3.6 from [16] we change the definition of the set of *clauses*, as the least set of formulas containing the set of *literals* and closed under $\wedge$, defining thus clauses just as *conjunctions* of literals. Then consider the closure operator $\mathbf{C}_{dH}$ defined so that $\varphi \vdash_{dH} \Gamma \Leftrightarrow \varphi \in \mathbf{C}_{dH}(\Gamma)$. One easily obtains the dual version of Lemma 3.7 from [16] for the claim that $\mathbf{C}_{dH}(\varphi \wedge \psi) = \mathbf{C}_{dH}(\{\gamma \wedge \psi : \gamma \in \Gamma\})$. The corresponding dualization of Proposition 3.8 from [16] is also obtainable for $\mathbf{C}_{dH}$ instead of $\mathbf{C}_H$. The dual version of Theorem 3.9 from [16] holds for the normal forms defined as disjunctions of clauses as well. The dual analogue of Proposition 3.10 from [16] can then be proved, what gives us the targeted completeness of $\vdash_{dH}$ with respect to Definition 4.1. Together with Theorem 4.4 we immediately get the claim of the present theorem. ■

Finally, observe that exactly as **BL**, **DBL** is coincident with **FDE** in the sets of proven binary consequences:

THEOREM 4.8. *$\varphi \vdash_{\mathrm{fde}} \psi \Leftrightarrow \varphi \vdash_{\mathrm{dbl}} \psi$.*

PROOF. Analogously like the proof of Theorem 2.7. ■

5. Anything But the Falsehood

Throughout this paper we deal with logics based on the set of four Belnapian truth values—$\{N, F, T, B\}$. So far, we have considered two possibilities of picking out the designated elements from this set. The first one is to take as designated the elements, which are *at least true*, i.e. to distinguish the subset $\{T, B\}$. This is the standard option for the first-degree entailment, Belnap's logic and its dual. It is worth noticing that taking as designated the subset $\{N, T\}$ leads to the same outcome, since

> the two choices of designated values are perfectly symmetrical, and it is therefore no surprise that the two logics thus characterized are identical with respect to the inferences they validate [22, p. 128].

Speaking algebraically, both subsets constitute prime filters on the four-element De Morgan lattice defined on $\{N, F, T, B\}$, and Belnap's logic (as well as its dual) is equally characterizable by either of these two prime filters alone (cf. Proposition 2.3 in [16]).

The second option is to distinguish only sentences, which are *exactly true*. We obtain then Pietz and Rivieccio's *exactly true logic* with T as the only designated value, which may serve as a logical basis for a strict verificationism.

There is also another possibility deserving consideration—to allow as designated all the truth values *except the worst one*. According to Belnap, "the worst thing to be told is that something you cling to is false, *simpliciter*" [3, p. 516]. So, T is the "best of all" [ibid], N and B still hold out hope of a better outcome, and only F is irrecoverable. Hence, it is not unreasonable to take $\{N, T, B\}$ as the subset of distinguished elements among the four Belnap's truth values, i.e., to allow anything but the (outright) falsehood.[5]

Thus, Wansing's description of counterassumptions as "sentences assumed not to be true" [27, p. 487] can be extended as follows: "and to be false". In accordance with this understanding, the set of counterassumptions should involve only sentences marked by the truth value F. One thus obtains the *non-falsity logic* with entailment relation, which ensures preservation of any truth value except of F (or equally, the backward preservation of F from conclusions to the premise):

Definition 5.1. $\varphi \vDash_{\text{nfl}} \Gamma =_{df} \forall v : (\forall \psi \in \Gamma : v(\psi) = F) \Rightarrow v(\varphi) = F.$

To formalize this relation, one has to extend dual Belnap's logic with the *dual disjunctive syllogism*, either in a form of inference rule:

$$(\text{R16}_d)\ \frac{\psi}{{\sim}\varphi \vee (\varphi \wedge \psi)},$$

to obtain a Font-style system $\vdash_{dH'}$, or in a form of axiom schema:

$$a16_{\text{nfl}}.\psi \vdash {\sim}\varphi \vee (\varphi \wedge \psi),$$

to obtain consequence system **NFL**.

We invite an interested reader to dualize the corresponding proofs from Section 3, to make sure that **NFL** has indeed the properties dual to **ETL**. In particular, the rules $r1_{\text{fde}}$ and $r3_{\text{fde}}$ are admissible in **NFL**, whereas $r2_{\text{fde}}$ and $r4_{\text{fde}}$ are not. The *positive paradox of relevance* (*verum ex quodlibet*) $\varphi \vdash {\sim}\psi \vee \psi$ is derivable in **NFL**, but its conjunctive extension $\varphi \vdash ({\sim}\psi \vee \psi) \wedge ({\sim}\chi \vee \chi)$ is not.

Theorem 5.2. $\varphi \vdash_{\text{nfl}} \Gamma \Leftrightarrow \varphi \vdash_{dH'} \Gamma$.

Proof. By a straightforward extension of the proof of Theorem 4.4 to the cases with (R16_d) and $a16_{\text{nfl}}$. ■

[5]Marcos in [21] develops a uniform semantic approach to entailment relations based on different subsets of designated truth values from $\{N, F, T, B\}$ the subsets $\{T\}$ and $\{N, T, B\}$ among them. He formulates semantic constructions in terms of only two classical-like truth values, and naturally extracts from them the two-signed tableau systems for characterizing the corresponding entailment relations.

THEOREM 5.3. $\varphi \models_{\mathrm{nfl}} \Gamma \Leftrightarrow \varphi \vdash_{\mathrm{nfl}} \Gamma$.

PROOF. By the above theorem and an appropriate dualization of Theorem 3.4 from [22]. ■

6. Generalized Entailment Relations: A Double Diamond of Consequence Systems

Fitting once characterized first-degree entailment as a "child" of Kleene's three-valued logics [15]. It develops that **FDE** in turn has a rich brood of systems descend from its noble stem. They all are consequence systems, manipulating with (nondegenerated) *consequences* as their primary formal objects. Generally, a consequence is an expression of the form $\Gamma \vdash \Delta$, where neither Γ nor Δ is empty. Additional restrictions may apply, narrowing either Γ, or Δ (or both) to a singleton. We thus may have consequences (and the corresponding systems) of formula–formula, set–formula, formula–set, and set–set type.

The consequence systems considered in the present paper are in a sense of a hybrid nature. On the one hand, they resemble usual Gentzen-style sequent calculi, in that they are designed to establish validity of consequence expressions ("sequents", in Gentzen's terminology[6]). On the other hand, they are constructed *not* as standard sequent systems with an identity axiom, structural and logical inference rules for operating with expression on the left/right of the turnstile. Rather, the consequence systems developed in this paper are built in a manner very close to axiomatic Hilbert-style formulations, with certain (consequence) expressions taken as axioms, and with inference rules (both logical and structural) for their direct transformations. Moreover, an inference in such systems is defined precisely as in standard Hilbert systems (and this is the crucial dissimilarity from Gentzen systems), namely, as a finite *consecutive* list of (occurrences of) consequence expressions, each of which either is an axiom or comes by an inference rule from some consequence expressions preceding it in the list. If an inference of a given consequence exists, one says it is (formally) derivable in the system (cf. [19, p. 34]).

From this perspective, consequence systems are much more "Hilbert-style" than Font's system $\vdash_H$ from [16], and **FDE** is the paradigmatic *binary*

[6] As Kleene explains: "Gentzen says "Sequenz", which we translate as "sequent", because we have already used "sequence" for any succession of objects, where the German is "Folge"." [18, p. 441].

consequence system of the kind described above. Interestingly, system $\mathbf{E}_{\text{fde}}$, which is essentially a consequence system for the first-degree entailment (with arrows instead of turnstiles), is specified in [2, p. 158] as a "Hilbert-style formalism". Dunn's characterization of the system $\mathbf{R}_{\text{fde}}$ in [12, p. 146] as a "Hilbert-style presentation" is also remarkable in this respect. In the present paper we generalize and expand the binary approach to consequence systems, and allow *any* consequences of a required kind (and not only the binary ones) as axioms and theorems of the constructed systems.

The structure of consequence systems of different types may suit various purposes. For one thing, the binary consequence systems of the formula–formula type can be used to express a pure entailment relation between (single) sentences, explicated as a set of pairs of formulas. The set–formula consequence systems may serve as a general logical framework for a logical verification of formulas with respect to certain assumptions. Dually, the consequence systems of the formula–set type fits well an idea of falsifying formulas with respect to counterassumptions. Finally, the consequence systems of the set–set type represent a generalized logical entailment as the relation between sets of sentences, i.e., as a set of pairs of sets.

Thus, if we take **FDE** as the basic consequence system of a formula–formula type in its standard formulation from [2, p. 158], with the rules for conjunction introduction ($r2_{\text{fde}}$), disjunction elimination ($r3_{\text{fde}}$), and most crucially, contraposition ($r4_{\text{fde}}$), we can obtain its three natural descendants by a mere abandoning the singleton restriction in the appropriate places. We can thus transit either to Belnap's logic (of the set–formula type), or to its dual (of the formula–set type), and then proceed to what can be called *De Morgan logic*, which is the union of the two and handles consequences of the form $\Gamma \vdash \Delta$. Syntactically, the corresponding transitions might be accomplished by admitting certain structural inference rules, most crucially, Cut and Weakening (either leftsided, or rightsided, or symmetrical), and by transforming appropriate inference rules to direct axioms either for conjunction introduction only (in the case of Belnap's logic), or for disjunction elimination only (dual Belnap's logic), or for both (De Morgan logic).[7] Semantically one needs only to modify accordingly the definition of entailment relation, to allow multiple formulas either to the right, or to the left, or on both sides of the entailment sign $\vDash$.

[7] The rule $r2_{\text{fde}}$ is transformed into the axiom schema $a3_{\text{bl}}$, and $r3_{\text{fde}}$ into $a3_{\text{dbl}}$.

However, a selection of axiom schemata and rules for Belnap's logic (and its dual) can be exercised along a different path, as exemplified by the selection of rules for system $\vdash_H$ in [16]. It turns out that exactly this selection allows some further interesting extensions not collapsing into classical logic. By such formulations, the rule of contraposition is replaced by De Morgan laws; and then the inference rule of disjunction elimination in the case of Belnap's logic, and the inference rule of conjunction introduction for dual Belnap's logic are also excluded from the corresponding systems, being compensated for by additional axiom schemata. This gives a possibility to transit to systems that do not admit the corresponding rule of inference any more, and remain, therefore, distinct from classical logic.[8] The picture is then finished with various versions of classical consequence systems: either in verificationistic or in falsificationistic flavor, or as a direct extension of De Morgan logic.

To concretize these general remarks, and to summarize considerations of the previous sections, let us recapitulate the consequences and rules used so far, reorganizing and expanding them as applicable, and removing unnecessary repetitions:

Axiom schemata:

(1) $\varphi \wedge \psi \vdash \varphi$ (2) $\varphi \wedge \psi \vdash \psi$ (3) $\varphi \vdash \varphi \vee \psi$ (4) $\psi \vdash \varphi \vee \psi$
(5) $\varphi \vee \varphi \vdash \varphi$ (6) $\varphi, \psi \vdash \varphi \wedge \psi$ (7) $\varphi \vdash \varphi \wedge \varphi$ (8) $\varphi \vee \psi \vdash \varphi, \psi$
(9) $(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$ (10) $\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi$
(11) $\varphi \vee (\psi \wedge \chi) \vdash (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
(12) $(\varphi \vee \psi) \wedge (\varphi \vee \chi) \vdash \varphi \vee (\psi \wedge \chi)$
(13) $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$
(14) $(\varphi \wedge \psi) \vee (\varphi \wedge \chi) \vdash \varphi \wedge (\psi \vee \chi)$
(15) $\varphi \vdash {\sim}{\sim}\varphi$ (16) $\varphi \vee \psi \vdash {\sim}{\sim}\varphi \vee \psi$ (17) $\varphi \wedge \psi \vdash {\sim}{\sim}\varphi \wedge \psi$
(18) ${\sim}{\sim}\varphi \vdash \varphi$ (19) ${\sim}{\sim}\varphi \vee \psi \vdash \varphi \vee \psi$ (20) ${\sim}{\sim}\varphi \wedge \psi \vdash \varphi \wedge \psi$
(21) ${\sim}(\varphi \vee \psi) \vee \chi \vdash ({\sim}\varphi \wedge {\sim}\psi) \vee \chi$ (22) $({\sim}\varphi \wedge {\sim}\psi) \vee \chi \vdash {\sim}(\varphi \vee \psi) \vee \chi$
(23) ${\sim}(\varphi \wedge \psi) \vee \chi \vdash ({\sim}\varphi \vee {\sim}\psi) \vee \chi$ (24) $({\sim}\varphi \vee {\sim}\psi) \vee \chi \vdash {\sim}(\varphi \wedge \psi) \vee \chi$
(25) ${\sim}(\varphi \vee \psi) \wedge \chi \vdash ({\sim}\varphi \wedge {\sim}\psi) \wedge \chi$ (26) $({\sim}\varphi \wedge {\sim}\psi) \wedge \chi \vdash {\sim}(\varphi \vee \psi) \wedge \chi$
(27) ${\sim}(\varphi \wedge \psi) \wedge \chi \vdash ({\sim}\varphi \vee {\sim}\psi) \wedge \chi$ (28) $({\sim}\varphi \vee {\sim}\psi) \wedge \chi \vdash {\sim}(\varphi \wedge \psi) \wedge \chi$
(29) ${\sim}\varphi \wedge (\varphi \vee \psi) \vdash \psi$ (30) $\psi \vdash {\sim}\varphi \vee (\varphi \wedge \psi)$

[8] As Rivieccio demonstrates in [24], there are infinitely many such logics for Belnap's case. This result can be straightforwardly extended to dual Belnap's logic.

Inference rules:

(31) $\varphi \vdash \psi;\ \psi \vdash \chi \,/\, \varphi \vdash \chi$	(32) $\varphi \vdash \psi;\ \varphi \vdash \chi \,/\, \varphi \vdash \psi \wedge \chi$
(33) $\varphi \vdash \chi;\ \psi \vdash \chi \,/\, \varphi \vee \psi \vdash \chi$	(34) $\varphi \vdash \psi \,/\, {\sim}\psi \vdash {\sim}\varphi$
(35) $\Gamma \vdash \varphi\,;\ \Delta, \varphi \vdash \psi \,/\, \Gamma, \Delta \vdash \psi$	(36) $\Gamma \vdash \psi \,/\, \Gamma, \varphi \vdash \psi$
(37) $\Gamma, \varphi, \varphi \vdash \psi \,/\, \Gamma, \varphi \vdash \psi$	(38) $\Gamma, \varphi, \psi \vdash \chi \,/\, \Gamma, \psi, \varphi \vdash \chi$
(39) $\varphi \vdash \Gamma\,;\ \psi \vdash \varphi, \Delta \,/\, \psi \vdash \Gamma, \Delta$	(40) $\varphi \vdash \Gamma \,/\, \varphi \vdash \psi, \Gamma$
(41) $\varphi \vdash \psi, \psi, \Gamma \,/\, \varphi \vdash \psi, \Gamma$	(42) $\varphi, \vdash \psi, \chi, \Gamma \,/\, \varphi \vdash \chi, \psi, \Gamma$
(43) $\Gamma \vdash \Delta \,/\, \Gamma, \varphi \vdash \Delta$	(44) $\Gamma \vdash \Delta \,/\, \Gamma \vdash \varphi, \Delta$
(45) $\Gamma, \varphi, \varphi \vdash \Delta \,/\, \Gamma, \varphi \vdash \Delta$	(46) $\Gamma \vdash \varphi, \varphi, \Delta \,/\, \Gamma \vdash \varphi, \Delta$
(47) $\Gamma, \varphi, \psi \vdash \Delta \,/\, \Gamma, \psi, \varphi \vdash \Delta$	(48) $\Gamma \vdash \varphi, \psi, \Delta \,/\, \Gamma \vdash \psi, \varphi, \Delta$
(49) $\Gamma \vdash \varphi, \Delta\,;\ \Lambda, \varphi \vdash \Pi \,/\, \Gamma, \Lambda \vdash \Delta, \Pi$	

We have then the following systems:

FDE : *Axioms*: (1)–(4), (11), (15), (18); *Rules*: (31)–(34).

BL : *Axioms*: (1)–(6), (10)–(12), (16), (19), (21)–(24); *Rules*: (35)–(38).

DBL : *Axioms*: (1)–(4), (7)–(9), (13), (14), (17), (20), (25)–(28); *Rules*: (39)–(42).

DML : *Axioms*: (1)–(4), (6), (8), (11), (15), (18); *Rules*: (34), (43)–(49).

ETL : **BL** + (29).

NFL : **DBL** + (30).

VCL : *Axioms*: (1)–(4), (6), (11), (15), (18), (29); *Rules*: (33)–(38).

FCL : *Axioms*: (1)–(4), (8), (11), (15), (18), (30); *Rules*: (32), (34), (39)–(42).

CL : **DML** + (29).

Systems **FDE**, **BL**, **DBL**, **ETL** and **NFL** were presented in the previous sections, although here, some of them received a slightly different (but an equivalent) axiomatization. **DML**, **VCL**, **FCL** and **CL** stand correspondingly for *De Morgan logic*, *verificationistic classical logic*, *falsificationistic classical logic*, and *classical logic* as such. Entailment relations for these logics are defined as follows:

DEFINITION 6.1. $\Gamma \vDash_{\text{dml}} \Delta =_{df} \forall v : (\forall\varphi \in \Gamma : t \in v(\varphi)) \Rightarrow (\exists\psi \in \Delta : t \in v(\psi))$.

DEFINITION 6.2. $\Gamma \vDash_{\text{vcl}} \psi =_{df} \forall v : [(\forall\varphi \in \Gamma : v(\varphi) = T) \Rightarrow v(\psi) = T$ and $v(\psi) = F \Rightarrow (\exists\varphi \in \Gamma : v(\varphi) = F)]$.

Definition 6.3. $\varphi \vDash_{\text{fcl}} \Gamma =_{df} \forall v : [v(\varphi) = T \Rightarrow (\exists\psi \in \Gamma : v(\psi) = T)$ and $(\forall\psi \in \Gamma : v(\psi) = F) \Rightarrow v(\varphi) = F]$.

Definition 6.4. $\Gamma \vDash_{\text{cl}} \Delta =_{df} \forall v : [(\forall\varphi \in \Gamma : v(\varphi) = T \Rightarrow \exists\psi \in \Gamma : v(\psi) = T)$ and $(\forall\psi \in \Gamma : v(\psi) = F) \Rightarrow (\exists\varphi \in \Gamma : v(\varphi) = F)]$.

The distinction between the three versions of the classical consequence systems of different types is worth noticing. In this way, we are able to characterize (1) classical entailment relation of a set–formula type when employed in a verificationistic context, (2) classical entailment relation of a formula–set type suitable for a falsificationistic framework, and (3) classical entailment as a generalized relation between sets of sentences. Entailment is regarded as *classical* if it validates both principles of double negation, contraposition, ex contradictione quodlibet, and verum ex quodlibet (along with standard lattice-theoretic properties of conjunction and disjunction).

The following definition will be helpful for establishing the exact relationships between the systems of the **FDE**-family.

Definition 6.5. Let $S_{f\vdash f}$, $S_{s\vdash f}$, $S_{f\vdash s}$ and $S_{s\vdash s}$ be systems of the formula–formula, set–formula, formula–set and set–set type correspondingly. Let $S_x \subseteq S_y$ means that the set of consequences derivable in system S_x is a subset of the set of consequences derivable in system S_y. Then:

1. $S_{s\vdash f}(S_{f\vdash s}, S_{s\vdash s})$ is said to be a conservative extension of $S_{f\vdash f}$ iff
   (a) $S_{f\vdash f} \subseteq S_{s\vdash f}(S_{f\vdash s}, S_{s\vdash s})$; (b) $S_{s\vdash f}(S_{f\vdash s}, S_{s\vdash s})$ and $S_{f\vdash f}$ have the same set of derivable binary consequences of the form $\varphi \vdash \psi$.
2. $S_{s\vdash s}$ is said to be a conservative extension of $S_{s\vdash f}(S_{f\vdash s},)$ iff
   (a) $S_{s\vdash f}(S_{f\vdash s}) \subseteq S_{s\vdash s}$; (b) $S_{s\vdash f}(S_{f\vdash s})$ and $S_{s\vdash s}$ have the same set of derivable consequences of the form $\Gamma \vdash \varphi$ $(\varphi \vdash \Gamma)$.

The relationships between consequence systems defined above can be represented then by a Hasse diagram as shown in the Fig. 1, where solid lines stand for conservative extensions and dotted lines signify non-conservative extensions of the systems. If one compares these relationships with definitions of entailment for corresponding systems, one can notice that by a conservative extension the definition of entailment undergoes merely structural modification, whereas a non-conservative extension requires more significant transformations in the truth- or falsity -conditions.

We would like to note that the idea of verification and falsification involved in this paper are taken most abstractly from their purely *structural* aspect. In this sense, even classical logic can be put into a verificationistic or falsificationistic perspective, although, in accordance with an established

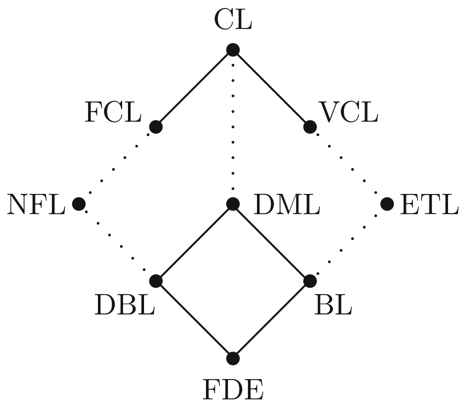

Figure 1. Double diamond: relationships between **FDE** and its relatives

view, classical logic is not really fit for such purposes. If we wish to pursue a more substantial idea of a verification or falsification, we might need to involve the machinery of a constructive logic or its dual.[9]

Interestingly, **FDE** (formulated without contraposition but with De Morgan laws) may serve as a genuine basis for Nelson's logic of constructible falsity [20], comprising simultaneously a verificationistic and falsificationistic idea within a homogeneous logical framework, see [14]. However, from a *purely* verificationistic standpoint, we might wish to consider an *intuitionistic* version of the first-degree entailment, which can be obtained from **FDE** by taking as an axiom schema $\varphi \vdash \varphi$ instead of (18), cf. [25, p. 181]. To implement a falsificationistic idea alone, the dual version of this system can be formulated with (18) instead of (15) as an axiom. We leave for future work the task of a systematic developing an *intuitionistic Belnap's logic* and its dual, as well as their possible extensions. Incidentally, note the deep affinity between an intuitionistic sequent system and **BL**, consisting in the singleton-on-the-right restriction in the accepted consequences.

We finish this paper with posing an open question about a possibility of deductive formalizations of the *exactly true logic* and its dual by means of the binary consequence systems. Consider the following two definitions:

Definition 6.6. $\varphi \vDash_{\mathrm{etl}^{\circ}} \psi =_{df} \forall v : v(\varphi) = T \Rightarrow v(\psi) = T$.

Definition 6.7. $\varphi \vDash_{\mathrm{nfl}^{\circ}} \psi =_{df} \forall v : v(\psi) = F \Rightarrow v(\varphi) = F$.

[9] On the interconnections of the notions of verification and falsification with the positive and negative information embodied in sentences, the role of different kinds of negations in expressing such information see, e.g. [28].

What kind of systems axiomatize the relations determined by these definitions? Clearly, one cannot obtain such axiomatizations by adding correspondingly disjunctive syllogism and its dual to **FDE**. Does **FDE** have a formalization that allow a non-trivial extensions to the *binary* exactly true logic and non-falsity logic? In particular, does the system determined by axioms (1)–(5), (10)–(12), (16), (19), (21)–(24), and rules (31), (32), as well the system determined by axioms (1)–(4), (7), (9), (13), (14), (17), (20), (25)–(28), and rules (31), (33) are equivalent to **FDE**? These questions require further investigations.

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