Lattice of Three-Valued Literal Paralogics

Alexander S. Karpenko, Institute of Philosophy of RAS
as.karpenko@gmail.com
Natalya E. Tomova, Institute of Philosophy of RAS
natalya-tomova@yandex.ru

Let V_3 be a set of truth values $\{0, 1/2, 1\}$ and D be a set of designated values. Implication is called *natural* if it satisfies the following properties:

(1) **C**-extending, i.e. restrictions to the subset $\{0,1\}$ of V_3 coincide with the classical implication;
(2) If $p \to q \in D$ and $p \in D$, then $q \in D$, i.e. the matrices for implication need to be normal in the sense of Łukasiewicz-Tarski (condition sufficient for the verification of *modus ponens*);
(3) Let $p \leq q$, then $p \to q \in D$;
(4) $p \to q \in V_3$, in other cases.

In [9] it is shown that in the class of three-valued logics there are only 3 *natural* implications, that are the extensions of weak Kleene logic $\mathbf{K}_3^w$ with connectives $\{\sim, \cap, \cup\}$ [4] and which generate 3 logics, which are functionally equivalent to the Bochvar's logic of nonsense $\mathbf{B}_3$. Let's define the tables for these implications and involution $\sim$:

	$\sim$
1	0
1/2	1/2
0	1

$\to_1$	1	1/2	0
1	1	1	0
1/2	1	1	0
0	1	1	1

$\to_2$	1	1/2	0
1	1	0	0
1/2	1	1	1
0	1	1	1

$\to_3$	1	1/2	0
1	1	0	0
1/2	1	1	0
0	1	1	1

Let's consider the following matrices:

$$\mathfrak{M}_1 = \langle\{0, 1/2, 1\}, \sim, \to_1, \{1, 1/2\}\rangle,$$

$$\mathfrak{M}_2 = \langle\{0, 1/2, 1\}, \sim, \to_2, \{1\}\rangle,$$

$$\mathfrak{M}_3 = \langle\{0, 1/2, 1\}, \sim, \to_3, \{1\}\rangle.$$

Matrice $\mathfrak{M}_1$ is the characteristic matrix for *paraconsistent* logic $\mathbf{P}_2^1$ and matrice $\mathfrak{M}_2$ is the characteristic matrix for *paracomplete* logic $\mathbf{I}_2^1$. Note, that $\mathbf{P}_2^1$ is the extension of paraconsistent logic $\mathbf{P}^1$ [6] by adding $\sim$, and $\mathbf{I}_2^1$ is the extension of paracomplete logic $\mathbf{I}^1$ [7], which was constructed as dual for $\mathbf{P}^1$. Hilbert-style axiomatic systems for all these four logics are given in [5]. Notice, that logics $\mathbf{P}^1$ and $\mathbf{I}^1$ are a combination of two three-valued isomorphs of $\mathbf{C}_2$ [2].

Matrice $\mathfrak{M}_3$, whether, $D = \{1\}$, or $D = \{1, 1/2\}$, defines *paranormal* logic $\mathbf{TK}^1$, i.e. logic, which is paraconsistent and paracomplete.

Notice, that in definition of *natural* implication we used the *strong* formulation of *modus ponens* rule, asserting preserving of designated truth-values:

$$(i)\forall\mathfrak{M}\forall v[(|A|_v^{\mathfrak{M}} \in D \,\&\, |A \to B|_v^{\mathfrak{M}} \in D) \Rightarrow (|B|_v^{\mathfrak{M}} \in D)],$$

where $|A|_v^{\mathfrak{M}}$ is a valuation v of some formula A in matrix $\mathfrak{M}$.

But if we accept the *weak* formulation of *modus ponens* rule, asserting preserving tautologies:

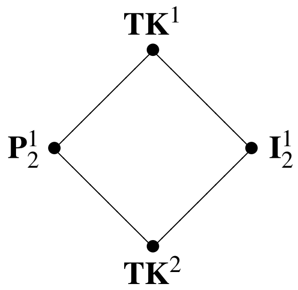
$(ii)\forall\mathfrak{M}\,[\forall v(|A|_v^{\mathfrak{M}} \in D)\,\&\,\forall v(|A \to B|_v^{\mathfrak{M}} \in D) \Rightarrow \forall v(|B|_v^{\mathfrak{M}} \in D)]$,

then the class of Bochvar's logics is complimented by one more logic, this time with implication $\to_4$ (see [10, p. 123], there it is a logic with implication $\to_{29}$), which is defined as follows:

$$x \to_4 y = \begin{cases} 0, \text{if } x = 1 \text{ and } y = 0, \\ 1, \text{otherwise.} \end{cases}$$

Note, that in [3, p. 27] Bochvarian's implications are lattice ordered with respect to the property of *strong* / *weak* modus ponens and the set of designated values $\{1\}/\{1, 1/2\}$.

Let's consider the matrix $\mathfrak{M}_4 = \langle\{0, 1/2, 1\}, \sim, \to_4, \{1, 1/2\}\rangle$, which characterizes the logic $\mathbf{TK}^2$, dual to $\mathbf{TK}^1$, because $\mathbf{TK}^2$ is neither paraconsistent nor paracomplete. This allows us to construct a lattice of logics (denoted by TK), with respect to the possession of one of the *paraproperties*:



THEOREM 1. *Logics* $\mathbf{P}^1_2$, $\mathbf{I}^1_2$, $\mathbf{TK}^1$ *and* $\mathbf{TK}^2$ *are pairwise fuctionally equivalent.*

THEOREM 2. *Let* $\mathbf{B}_1^{\sim}$ *be the class of all external formulas (i.e. the only possible values of these formulas are 1 or 0) of three-valued Bochvar's logic* $\mathbf{B}_3$. *Let this class be defined by the Peirce's arrow* γ *[8] and extended by the connective* $\sim$. *Then logic* $\mathbf{I}^1_2$ *with connectives* $\{\sim, \to_2\}$ *and logic* $\mathbf{B}_1^{\sim}$ *with connectives* $\{\sim, \gamma\}$ *are fuctionally equivalent.*

COROLLARY 1. *Logics* $\mathbf{P}^1_2$, $\mathbf{I}^1_2$, $\mathbf{TK}^1$ *and* $\mathbf{TK}^2$ *are fuctionally equivalent to* $\mathbf{B}_1^{\sim}$.

The research was supported by Russian Foundation for Humanities, project № 14-03-00341a.

References

1. Bochvar A.D. (1938) On a three-valued calculus and its application to analysis of paradoxes of classical extended functional calculus // *History and Philosophy of Logic*, 2, 1981, pp. 87–112

2. Karpenko A.S. A maximal paraconsistent logic: The combination of two three-valued isomorphs of classical propositional logic, in D. Batens, C. Mortensen, G. Priest and J.-P. van Bendegem (eds.), *Frontiers of Paraconsistent Logic*, Baldock Research Studies Press, 2000, pp. 181–187

3. Karpenko A.S. Foreword. The variety of three-valuedness (in Russian) // L. Devyatkin, N. Prelovskiy, N. Tomova. *Within the limits of three-valuedness*. Moscow: IF RAS, 2015, pp. 9–33

4. Kleene S.C. On a notation for ordinal numbers // *The Journal of Symbolic Logic*, 3, 1938, pp. 150–155

5. Lewin R.A., Mikenberg I.F. Literal-paraconsistent and literal-paracomplete matrices // *Math. Log. Quart.*, 52(5), 2006, pp. 478–493

6. Sette A.M. On propositional calculus P^1 // *Mathematica Japonica*, 18, 1973, pp. 173–180

7. Sette A.M., Carnielli W.A. Maximal weakly-intuitionistic logics // *Studia Logica*, 55, 1995, pp. 181–203

8. Shestakov V.I. On one fragment of D.A. Bochvar's calculus (in Russian) // *Information issues of semiotics, linguistics and automatic translation. VINITI*, vol. 1, 1971, pp. 102–115

9. Tomova N.E. A Lattice of implicative extensions of regular Kleene's logics // *Report on Mathematical Logic*, 47, 2012, pp. 173–182

10. Tomova N.E. On the extension of the class of natural three-valued logics: the new classification (in Russian) // L. Devyatkin, N. Prelovskiy, N. Tomova. *Within the limits of three-valuedness*. Moscow: IF RAS, 2015, pp. 97–130