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CONTROL THEORY

Generalization of the Notion of Relative Degree and Its Properties

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Abstract— We suggest a generalization of the classical notion of relative degree for linear dynamical MIMO systems. We study the properties of that notion, its relationship with the zero dynamics of the system, and the possibility of reduction of the system to a special form.

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1. INTRODUCTION

Consider a standard linear stationary MIMO system. In the present paper, we study a system described by differential equations; nevertheless, all the main results can be almost word for word stated for systems described by difference equations.

The above-mentioned system has the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, and $C \in \mathbb{R}^{l\times n}$ are known constant matrices, $x(t)$ is the state vector of the system, $y(t)$ is the output, and $u(t)$ is the input. We consider a "square" system; i.e., the dimensions of the input and the output coincide. In addition, we assume that $\operatorname{rank} B = \operatorname{rank} C = l$.

The notion of zero dynamics, that is, the dynamics of the system for $y(t) \equiv 0$, plays an important role in automated control theory (especially in the solution of stabilization problems). This is related to the fact that the stability of the zero dynamics essentially guarantees that once the output is stabilized at zero, the system is globally stabilized. To describe the zero dynamics, it is convenient to use the Rosenbrock matrix

$$R(s) = \begin{pmatrix} sI - A & -B \\ C & 0 \end{pmatrix}, \qquad s \in \mathbb{C}.$$

It is well known (see [1, p. 67]) that the values of s^* for which the determinant $R(s)$ is zero (the so-called invariant zeros of the system) define the spectrum of the zero dynamics. Set $\beta(s) = |R(s)|$. (From now on, $|A|$ is the determinant of a matrix A.) The polynomial $\beta(s)$ is sometimes called the *characteristic polynomial* of the zero dynamics. It follows from the definition of the matrix $R(s)$ that $\deg \beta(s) \le n - l$.

The notion of relative degree is closely related to the zero dynamics of the system. Let us introduce this notion in accordance with [2, p. 220] and with regard of the linearity of the system.

Definition 1. A vector $r = (r_1, \dots, r_l) \in \mathbb{N}^l$ is called a *relative degree vector* of system (1) if the following conditions are satisfied.

1. $C_iB = 0$, $C_iAB = 0$, $\dots$, $C_iA^{r_i-2}B = 0$, $C_iA^{r_i-1}B \ne 0$.
2. $|H(r)| \ne 0$.

Here the C_i are the rows of the matrix C, $i = 1, \dots, l$, and

$$H(r) = \begin{pmatrix} C_1 A^{r_1 - 1} B \\ \dots \\ C_l A^{r_l - 1} B \end{pmatrix}.$$

Condition 1 of the definition indicates which derivative of the ith output depends explicitly on the inputs; i.e.,

$$y_i(t) = C_i x(t), \quad \frac{dy_i(t)}{dt} = C_i A x(t), \ \dots, \quad \frac{d^{r_i - 1} y_i(t)}{dt^{r_i - 1}} = C_i A^{r_i - 1} x(t),$$
$$\frac{d^{r_i} y_i(t)}{dt^{r_i}} = C_i A^{r_i} x(t) + C_i A^{r_i - 1} B u(t);$$

condition 2 implies that the outputs "depend on all inputs."

In the scalar case ($l = 1$), the relative degree of system (1) in general position (i.e., for a controllable and observable system) is uniquely determined. In this case, there is a close relationship between the dimension of the zero dynamics and the relative degree; namely, $\deg \beta(s) = n - r$, where r is the relative degree for $l = 1$.

In the vector case, the situation is much more complicated. It was shown in the monograph [1, p. 72] and in [3–5] that conditions 1 and 2 of Definition 1 are not necessarily consistent. In addition, a nonsingular change of coordinates does not change the components of the vector r, but a nonsingular change of the outputs can change them and, in a number of cases, permits one to pass from the triple $\{A, B, C\}$ to a triple $\{A, B, \tilde{C}\}$, where $\tilde{C} = TC$, $T \in \mathbb{R}^{l \times l}$, and $|T| \neq 0$, for which the conditions of Definition 1 are true. However, there exist systems (see [3, 6]) for which there is no output transformation providing the validity of conditions of Definition 1.

The validity of conditions of Definition 1 is important for the solution of various control problems, because this is (precisely) the case in which system (1) can be reduced to a form in which the zero dynamics is singled out (see [1, p. 91]). Moreover, in this case, the dimension of the zero dynamics is equal to $\deg \beta(s) = n - |r|$, where $|r| = r_1 + \dots + r_l$.

We encounter the problem on conditions providing the existence of an output transformation that reduces a system to a form with consistent definition 1 (below, a system with relative degree). A complete solution of this problem was obtained in [6], where a generalization of relative degree was obtained, more precisely, the notions of incomplete relative degree and leading incomplete relative degree were introduced. The aim of the present paper is to study further properties of leading incomplete relative degree and analyze the problem on the reduction of system (1) to a special form for the case in which the leading incomplete relative degree is not a relative degree.

2. GENERALIZATION OF THE NOTION OF RELATIVE DEGREE

Following [6], consider the following generalization of relative degree: since conditions 1 and 2 of Definition 1 may be incompatible, we only retain condition 1. Let a vector $r = (r_1, \dots, r_l) \in \mathbb{N}^l$ satisfy condition 1 of Definition 1. Note that the outputs can always be numbered in nondecreasing order of components of the vector r. Such an ordered vector satisfying condition 1 of Definition 1 is called an incomplete relative degree vector.

In this case, the components of the vector r split into "sections,"

$$r = (r_1^{(1)}, r_2^{(1)}, \dots, r_{n_1}^{(1)}, r_1^{(2)}, \dots, r_1^{(k)}, \dots, r_{n_k}^{(k)}),$$

where[1] $r_i^{(p)} = r_j^{(p)}$, $i, j \in \{1, 2, \dots, n_p\}$, and $r_i^{(p)} < r_j^{(q)}$ for $p < q$, $i \in \{1, 2, \dots, n_p\}$, and $j \in \{1, 2, \dots, n_q\}$. Thus, for the elements of a vector r we use two forms of numbering, the ordinary

[1] From now on, (i) in $f^{(i)}$ stands for the superscript i; the ith derivative of a function $f(t)$ is denoted by $d^i f(t)/dt^i$.

(successive) and the "double" numbering for which each element is specified by the index of a section (the superscript) and the "position" in it (the subscript). The double numbering can be used for the rows of the matrix H as well (if an element $r_j^{(i)}$ has index k for the ordinary numbering, then $H_j^{(i)} = H_k$), which permits one to operate with sections of rows of that matrix. A constructive algorithm generating output transformations was suggested in [6]; it ensures the linear independence of the rows $\{H_j^{(p)}\}_{j=1}^{n_p}$ for all $p = 1, \dots, k$ for the transformed system. (In this case, obviously, the values $r_j^{(p)}$ themselves can change under the transformations.) As a result, the rows in each section are linearly independent, but all rows of the matrix $H(r)$ can be linearly dependent.

The above-mentioned vector r arranged in nondecreasing order (for the linear independence of rows of the matrix $H(r)$ in each pth section) is called the leading incomplete relative degree vector. The formal definition of leading incomplete relative degree can be stated, for example, as follows.

Definition 2. A vector $r = (r_1, \dots, r_l) \in \mathbb{N}^l$ is called a leading incomplete relative degree vector if the following conditions are satisfied.

1. $C_iB = 0$, $C_iAB = 0$, $\dots$, $C_iA^{r_i-2}B = 0$, $C_iA^{r_i-1}B \neq 0$.
2. $r_i \leq r_{i+1}$, $i = 1, \dots, l-1$.
3. For any set of pairwise distinct indices $i_1, \dots, i_q \in \{1, 2, \dots, l\}$ such that $r_{i_1} = r_{i_2} = \dots = r_{i_q}$, the rows $H_{i_1}, \dots, H_{i_q}$ are linearly independent.

It was shown in [6] that this notion is invariant under nonsingular transformations of the outputs (if the leading incomplete relative degree vector is understood as the leading incomplete relative degree vector of the reduced system, that is, a system reduced by linear nonsingular transformations of the outputs to a form satisfying the conditions of Definition 2). In addition, if for a leading incomplete relative degree all rows of the matrix $H(r)$ are linearly independent, then a leading incomplete relative degree is a relative degree; otherwise, there is no transformation of outputs reducing the system to a form with a relative degree.

In what follows, we study the properties of leading incomplete relative degree and the possibility to reduce a system to a special form.

3. PROPERTIES OF LEADING INCOMPLETE RELATIVE DEGREE

Consider system (1). As was mentioned above, if the system has a relative degree vector r, then the dimension of the zero dynamics is $n - |r|$. The results obtained in [7] permit one to show that this relation holds only in the case of the existence of a relative degree of the system.

Assertion 1. *Let* $\beta(s) \not\equiv 0$, *and let* r *be the leading incomplete relative degree vector of system* (1). *Then the vector* r *is a relative degree vector of system* (1) *if and only if* $\deg \beta(s) = n - |r|$.

Proof. Necessity. The necessity was proved in [1, p. 92].

Sufficiency. Let $\deg \beta(s) = n - |r|$, and let r be not a relative degree vector in the sense of Definition 1. Then $|H(r)| = 0$. Consider the system $\tilde{S} = \{A, B, \tilde{C}\}$, where

$$\tilde{C} = \begin{pmatrix} C_1A^{r_1-1} \\ \dots \\ C_lA^{r_l-1} \end{pmatrix}.$$

By the corollary of Lemma 4 in [7], this matrix has full rank; i.e., the system $\tilde{S}$ satisfies the original assumptions (the input and output matrices have full rank). Note that $\tilde{C}B = H(r) = \tilde{H}(\tilde{r})$; i.e., the incomplete relative degree vector of the system $\tilde{S}$ is equal to $\tilde{r} = (1, 1, \dots, 1)$. At the same time, by Lemma 4 in [7], the characteristic polynomial of the zero dynamics $\tilde{\beta}(s)$ of the system $\tilde{S}$ satisfies the relation $\tilde{\beta}(s) = s^{|r|-l}\beta(s)$.

Since $\tilde{H}(\tilde{r})$ is a singular matrix and $|\tilde{r}| = l$, it follows that there exists a nonsingular matrix T such that the system $\hat{S} = \{A, B, \hat{C} = T\tilde{C}\}$ has the leading incomplete relative degree vector $\hat{r}$; moreover, $|\hat{r}| > l$. Since $|\hat{r}| > l$, it follows that one can consider the system $\check{S} = \{A, B, \check{C}\}$ (it differs from $\tilde{S}$), where

$$\check{C} = \begin{pmatrix} \hat{C}_1 A^{\hat{r}_1 - 1} \\ \dots \\ \hat{C}_l A^{\hat{r}_l - 1} \end{pmatrix}.$$

The characteristic polynomial of the zero dynamics of this system satisfies the relation

$$\check{\beta}(s) = s^{|\hat{r}|-l}\hat{\beta}(s) = s^{|\hat{r}|-l}|T|\tilde{\beta}(s) = s^{|\hat{r}|-l}|T|s^{|r|-l}\beta(s) = |T|s^{|r|+|\hat{r}|-2l}\beta(s). \tag{2}$$

The polynomial on the right-hand side in relation (2) has the degree

$$\deg \beta(s) + |r| - l + |\hat{r}| - l = n - |r| + |r| - l + |\hat{r}| - l = n - l + |\hat{r}| - l > n - l,$$

because $|\hat{r}| > l$. The obtained contradiction completes the proof.

Assertion 2. *Let $\beta(s) \not\equiv 0$, and let r be an incomplete relative degree vector of system* (1). *Then*

$$\deg \beta(s) \le n - |r|.$$

Proof. Suppose that $\deg \beta(s) > n - |r|$. Consider the system

$$\tilde{S} = \{A, B, \tilde{C}\},$$

where

$$\tilde{C} = \begin{pmatrix} C_1 A^{r_1 - 1} \\ \dots \\ C_l A^{r_l - 1} \end{pmatrix}.$$

Then $\deg \tilde{\beta}(s) = \deg(s^{|r|-l}\beta(s)) > n - l$. The obtained contradiction justifies the desired assertion.

Remark 1. It follows from Assertions 1 and 2 that if r is a relative degree vector, then $\deg \beta(s) = n - |r|$; but if r is a vector leading incomplete relative degree that is not a relative degree vector, then $\deg \beta(s) < n - |r|$.

Assertion 3. *Let $\beta(s) \not\equiv 0$ for system* (1), *and let*

$$\deg \beta(s) \ge n - l - 1.$$

Then there exists a nonsingular matrix T such that the system $\{A, B, \tilde{C} = TC\}$ has a relative degree (*in the sense of Definition* 1).

Proof. By [6, 7], if $\beta(s) \not\equiv 0$, then there exists a nonsingular matrix T such that the system $\{A, B, \tilde{C} = TC\}$ has a leading incomplete relative degree vector r. Let us show that this vector is a also relative degree vector. By assumption, $\deg \beta(s) \ge n - l - 1$. At the same time, by Assertion 2, $\deg \beta(s) \le n - |r|$; consequently, $|r| \le l + 1$. If $|r| = l$, then the system has a relative degree. (In this case, $r_1 = \dots = r_l = 1$; therefore, by the definition of leading incomplete relative degree, all rows of the matrix H are linearly independent.) Let $|r| = l + 1$, and let r be not a relative degree vector. Then, by Remark 1, $\deg \beta(s) < n - |r| = n - l - 1$, which contradicts the assumption. The proof of the assertion is complete.

Corollary 1. *If $\beta(s) \not\equiv 0$ and $l = n - 1$, then there exists a nonsingular matrix T such that the system $\{A, B, \tilde{C} = TC\}$ has relative degree.*

Proof. Since $\beta(s) \not\equiv 0$, we have $\deg \beta(s) \geq 0 = n - l - 1 = n - (n-1) - 1$; i.e., the assumptions of Assertion 3 are satisfied. The proof of the Corollary is complete.

Therefore, if $\beta(s) \not\equiv 0$, then the dimension of the zero dynamics can be estimated with the use of the leading incomplete relative degree vector.

CANONICAL FORM OF SYSTEMS OF GENERAL FORM

To estimate the dimension of the zero dynamics for the case in which $\beta(s) \equiv 0$ for system (1), one needs a different approach. (Such systems exist; examples can be found in the monograph [1, p. 79] and in the papers [4, 5].) Let us show that each system can be reduced to a form that permits one to estimate the dimension of the zero dynamics.

In this section, we consider less restrictive constraints on the output matrix C of system (1); more precisely, we assume that $C \neq 0$. (In this case, the rank of this matrix is not necessarily l.) The other assumptions for system (1) remain the same.

Definition 3. The vector $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_l) \in (\mathbb{N} \cup \{0\})^l$ whose components are defined as

$$\begin{aligned} &\varrho_i = 0 \quad \text{if} \quad C_i A^{q-1} B = 0, \quad q \in \mathbb{N}, \\ &\varrho_i = q \quad \text{if} \quad C_i A^{q-1} B \neq 0 \quad \text{and} \quad C_i A^{j-1} B = 0 \quad \text{for arbitrary} \quad j, \quad j < q, \quad j \in \mathbb{N}, \end{aligned}$$

is called the degenerate relative degree vector. Just as above, here C_i is the ith row of the matrix C.

Obviously, the degenerate relative degree vector is well defined for any system of the form (1). Along with the degenerate relative degree vector, by analogy with Definition 1, consider the matrix

$$\mathcal{H}(\varrho) = \begin{pmatrix} \mathcal{H}_1 \\ \ldots \\ \mathcal{H}_l \end{pmatrix},$$

where $\mathcal{H}_i = 0$ if $\varrho_i = 0$ and $\mathcal{H}_i = C_i A^{\varrho_i - 1} B$ if $\varrho_i \neq 0$.

Let $\varrho \neq 0$. (that is, there exists an i such that $\varrho_i \neq 0$) and $\operatorname{rank} \mathcal{H} = d$. (We assume that $d < l$, because if $d = l$, then the system satisfies the conditions of the definition of relative degree.) Let us show that in this case, by using a transformation of state coordinates (and, possible, inputs and outputs), one can reduce the system to a form that permits estimating the dimension of the zero dynamics. Set $\mathbb{I} = \{1, 2, \ldots, l\}$ and consider the following functional defined on the set $\mathbb{I}^d$:

$$\sigma(j_1, j_2, \ldots, j_d) = \sum_{k=1}^{d} \varrho_{j_k}, \qquad j_k \in \mathbb{I}.$$

In other words, $\sigma(j_1, j_2, \ldots, j_d)$ is the sum of components of the degenerate relative degree vector with indices $j_1, j_2, \ldots, j_d$. Among all sets of d linearly independent rows of the matrix $\mathcal{H}$, we choose the set with the maximum value of the functional σ for the indices of those rows (i.e., the sum of the corresponding components of the degenerate relative degree vector is maximal),

$$\max_{\substack{j_1, j_2, \ldots, j_d: \\ \mathcal{H}_{j_1}, \ldots, \mathcal{H}_{j_d} \text{ lin.indep.}}} \sigma(j_1, j_2, \ldots, j_d) = \sigma(\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_d) = \sigma_0. \tag{3}$$

Note that if $j_k = j_q$ in the set $(j_1, j_2, \ldots, j_d) \in \mathbb{I}^d$, then the rows $\{\mathcal{H}_{j_s}\}_{s=1}^d$ are necessarily linearly dependent, because two of them coincide. It follows that it suffices to look for the maximum in relation (3) on sets in $\mathbb{I}^d$, which do not contain coinciding elements.

Without loss of generality, one can assume that the maximum in relation (3) is attained on the set $(1, 2, \ldots, d) \in \mathbb{I}^d$; i.e., the rows $\{\mathcal{H}_{\bar{i}_j}\}_{j=1}^d$ occurring in (3) are the first d rows of the matrix $\mathcal{H}$.

(This can always be achieved by an appropriate renumbering of the system outputs.) We make the change of inputs $u(t) = T_1\tilde{u}(t)$, $T_1 \in \mathbb{R}^{l\times l}$, $|T_1| \neq 0$, where

$$T_1 = (\mathcal{H}^{*^T}(\mathcal{H}^*\mathcal{H}^{*^T})^{-1}; Z).$$

Here $\mathcal{H}^* = (\mathcal{H}_1^T, \dots, \mathcal{H}_d^T)^T$ is the matrix consisting of the first d rows of the matrix $\mathcal{H}$ (as was mentioned above, it has full rank), and Z is a matrix, whose columns form a basis of the kernel of the matrix $\mathcal{H}^*$. Under this transformation, the matrix B is transformed as follows: $\tilde{B} = BT_1$; consequently, vectors of the form C_iA^pB are multiplied on the right by the nonsingular matrix T_1; i.e., zero rows of the above-mentioned form become zero ones, and nonzero rows remain nonzero. It follows that the degenerate relative degree vector of the system does not change. The matrix $\mathcal{H}$ is transformed as follows: $\tilde{\mathcal{H}} = \mathcal{H}T_1$. By taking into account the form of the matrix T_1, we obtain

$$\tilde{\mathcal{H}} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \tilde{h}_{d+1,1} & \tilde{h}_{d+1,2} & \dots & \tilde{h}_{d+1,d} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \tilde{h}_{l1} & \tilde{h}_{l2} & \dots & \tilde{h}_{ld} & 0 & \dots & 0 \end{pmatrix}. \quad (4)$$

For convenience of the subsequent exposition, for the system thus transformed we use the notation of the original one. [In other words, we assume that the system $\{A, B, C\}$ has a matrix $\mathcal{H}$ of the form (4).]

One can readily show that if $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_d$ are linearly independent rows, then the rows

$$\begin{aligned} &C_1,\ C_1A,\ \dots,\ C_1A^{\varrho_1-1}, \\ &C_2,\ C_2A,\ \dots,\ C_2A^{\varrho_2-1},\ \dots, \\ &C_d,\ C_dA,\ \dots,\ C_dA^{\varrho_d-1} \end{aligned} \quad (5)$$

are linearly independent as well. Indeed, consider the linear combination of the above-mentioned rows equal to zero,

$$\sum_{i=1}^{d}\sum_{j=1}^{\varrho_i} \alpha_{ij}C_iA^{j-1} = 0. \quad (6)$$

We multiply this relation by the matrix B on the right with regard to the relation $C_iA^{j-1}B = 0$ for $j < \varrho_i$,

$$\sum_{i=1}^{d} \alpha_{i\varrho_i}C_iA^{\varrho_i-1}B = \sum_{i=1}^{d} \alpha_{i\varrho_i}\mathcal{H}_i = 0.$$

This, together with the linear independence of the rows $\mathcal{H}_i$, implies that $\alpha_{i\varrho_i} = 0$, $i = 1, \dots, d$. In a similar way, by multiplying relation (6) by $AB, A^2B, \dots, A^{\max_i \varrho_i - 1}B$, we obtain $\alpha_{ij} = 0$, $i = 1, \dots, d$, $j = 1, \dots, \varrho_i$. Note that the set of rows (5) contains exactly σ_0 rows.

Take $n - \sigma_0$ rows $V_1, V_2, \dots, V_{n-\sigma_0}$ such that the vector system

$$\{\{C_iA^j\}_{j=0}^{\varrho_i-1}\}_{i=1}^{d} \cup \{V_k\}_{k=1}^{n-\sigma_0}$$

is linearly independent, and

$$V_kB_j = 0, \qquad j = 1, \dots, d, \qquad k = 1, \dots, n - \sigma_0,$$

where B_j is the jth column of the matrix B. In other words, we require that the columns V_k^T belong to the kernel of the matrix ${B^*}^T = (B_1, B_2, \ldots, B_d)^T$. This is possible, because, of the vectors (5), only $\sigma_0 - d$ vectors belong to the kernel of the matrix ${B^*}^T$ (because $C_i A^{\varrho_i - 1} B \neq 0$ and $C_i A^{j-1} B = 0$, $j < \varrho_i$, $i = 1, \ldots, d$, by virtue of the definition of the degenerate relative degree), and the dimension of its kernel is equal to $n - d$ [because it is a matrix of full rank in accordance with the original assumptions about system (1)].

is considered the following transformation of coordinates:

$$
\begin{aligned}
&z_j^{(i)}(t) = C_i A^{j-1} x(t), \qquad i = 1, \ldots, d, \qquad j = 1, \ldots, \varrho_i,\\
&z_j^{(0)}(t) = V_j x(t), \qquad j = 1, \ldots, n - \sigma_0,
\end{aligned}
$$

or $z = Mx$, where

$$
\begin{aligned}
M &= (C_1^T, \ldots, (C_1 A^{\varrho_1 - 1})^T, \ldots, C_d^T, \ldots, (C_d A^{\varrho_d - 1})^T, V_1^T, \ldots, V_{n-\sigma_0}^T)^T,\\
z &= (z_1^{(1)}, z_2^{(1)}, \ldots, z_{\varrho_1}^{(1)}, \ldots, z_1^{(d)}, z_2^{(d)}, \ldots, z_{\varrho_d}^{(d)}, z_1^{(0)}, z_2^{(0)}, \ldots, z_{n-\sigma_0}^{(0)})^T;
\end{aligned}
$$

in addition,

$$
\dot z_j^{(i)}(t) = C_i A^{j-1} \dot x(t) = C_i A^j x(t) = z_{j+1}^{(i)}(t), \qquad i = 1, \ldots, d, \qquad j = 1, \ldots, \varrho_i - 1.
$$

Therefore, the transformed matrix $\hat A = MAM^{-1}$ acquires the form

$$
\hat A = \begin{pmatrix}
\hat A_{11} & \hat A_{12} & \ldots & \hat A_{1d} & \hat A_1\\
\hat A_{21} & \hat A_{22} & \ldots & \hat A_{2d} & \hat A_2\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
\hat A_{d1} & \hat A_{d2} & \ldots & \hat A_{dd} & \hat A_d\\
\hat{\bar A}_1 & \hat{\bar A}_2 & \ldots & \hat{\bar A}_d & \hat{\bar A}
\end{pmatrix},
$$

where

$$
\hat A_{ii} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0\\
0 & 0 & 1 & \ldots & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
0 & 0 & 0 & \ldots & 1\\
a_1^{(ii)} & a_2^{(ii)} & a_3^{(ii)} & \ldots & a_{\varrho_i}^{(ii)}
\end{pmatrix} \in \mathbb{R}^{\varrho_i \times \varrho_i}, \quad
\hat A_{ij} = \begin{pmatrix}
0 & \ldots & 0\\
\ldots & \ldots & \ldots\\
0 & \ldots & 0\\
a_1^{(ij)} & \ldots & a_{\varrho_j}^{(ij)}
\end{pmatrix} \in \mathbb{R}^{\varrho_i \times \varrho_j} \quad (j \neq i),
$$

$$
\hat A_j = \begin{pmatrix}
0 & \ldots & 0\\
\ldots & \ldots & \ldots\\
0 & \ldots & 0\\
a_1^{(j)} & \ldots & a_{n-\sigma_0}^{(j)}
\end{pmatrix} \in \mathbb{R}^{\varrho_j \times (n-\sigma_0)}, \quad
\hat{\bar A}_j \in \mathbb{R}^{(n-\sigma_0) \times \varrho_j}, \quad
\hat{\bar A} \in \mathbb{R}^{(n-\sigma_0) \times (n-\sigma_0)}, \quad i, j = 1, \ldots, l.
$$

Under this change of variables, the matrix B acquires the form

$$\hat{B} = MB = \begin{pmatrix} C_1 \\ \dots \\ C_1 A^{\varrho_1 - 1} \\ \dots \\ C_l \\ \dots \\ C_l A^{\varrho_l - 1} \\ V_1 \\ \dots \\ V_{n-\sigma_0} \end{pmatrix} B = \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} & \dots & \hat{B}_{1l} \\ \hat{B}_{21} & \hat{B}_{22} & \dots & \hat{B}_{2l} \\ \dots & \dots & \dots & \dots \\ \hat{B}_{d1} & \hat{B}_{d2} & \dots & \hat{B}_{dl} \\ \hat{\hat{B}}_1 & \hat{\hat{B}}_2 & \dots & \hat{\hat{B}}_l \end{pmatrix},$$

where

$$\hat{B}_{ii} = (0, 0, \dots, 0, 1)^T \in \mathbb{R}^{\varrho_i}, \quad i = 1, \dots, d, \quad \hat{B}_{ij} = 0 \in \mathbb{R}^{\varrho_i}, \quad i \neq j,$$
$$\hat{\hat{B}}_i \in \mathbb{R}^{n-\sigma_0}, \quad \hat{\hat{B}}_i = 0 \quad \text{for} \quad i \leq d,$$

because $C_i A^{q-1} B = 0$, $q < \varrho_i$, $C_i A^{\varrho_i - 1} B = \mathcal{H}_i$, and $V_k B_j = 0$, $k = 1, \dots, n - \sigma_0$, $j = 1, \dots, d$. Note that if $d < l$, then $n - \sigma_0 > 0$. (Otherwise, the last $l - d$ columns of the matrix $\hat{B}$ are zero, which is impossible, because $\operatorname{rank} B = l$.) The matrix $\hat{B}^T$ has the form (the block structure is defined in it)

$$\hat{B}^T = \left(\begin{array}{|cccc|c|cccc|ccc|} 0 & \dots & 0 & 1 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & 0 & * & \dots & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \end{array} \\ \begin{array}{|cccc|c|cccc|ccc|} 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & 0 & * & \dots & * \end{array}\right).$$

Since the outputs $y_i(t)$ of the original system are equal to $y_i(t) = C_i x(t)$, it follows from the relation $z_1^{(i)}(t) = C_i x(t)$ that the output matrix of the transformed system has the form

$$\hat{C} = \left(\begin{array}{|cccc|c|cccc|ccc|} 1 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ * & * & \dots & * & \dots & * & * & \dots & * & * & \dots & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \end{array} \\ \begin{array}{|cccc|c|cccc|ccc|} * & * & \dots & * & \dots & * & * & \dots & * & * & \dots & * \end{array}\right)$$

$$= \begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} & \dots & \hat{C}_{1d} & \hat{\hat{C}}_1 \\ \dots & \dots & \dots & \dots & \dots \\ \hat{C}_{d1} & \hat{C}_{d2} & \dots & \hat{C}_{dd} & \hat{\hat{C}}_d \\ \hat{C}_{d+1,1} & \hat{C}_{d+1,2} & \dots & \hat{C}_{d+1,d} & \hat{\hat{C}}_{d+1} \\ \dots & \dots & \dots & \dots & \dots \\ \hat{C}_{l1} & \hat{C}_{l2} & \dots & \hat{C}_{ld} & \hat{\hat{C}}_l \end{pmatrix}.$$

The sizes of blocks of this matrix are the same as in the matrix $\hat{B}^T$.

Thus, any system that has a nonzero degenerate relative degree vector can be reduced (by changes of inputs, outputs, and state variables) to the form

$$\begin{aligned}
\dot{z}^{(0)}(t) &= \sum_{j=1}^{d} \hat{\hat{A}}_j z^{(j)}(t) + \hat{\hat{A}} z^{(0)}(t) + \hat{\hat{B}} \bar{u}(t),\\
\dot{z}_j^{(i)}(t) &= z_{j+1}^{(i)}(t), \qquad i = 1, \ldots, d, \qquad j = 1, \ldots, \varrho_i - 1,\\
\dot{z}_{\varrho_i}^{(i)}(t) &= \sum_{j=1}^{d} \sum_{k=1}^{\varrho_j} a_k^{(ij)} z_k^{(j)}(t) + \sum_{k=1}^{n-\sigma_0} a_k^{(i)} z_k^{(0)}(t) + \tilde{u}_i(t), \qquad i = 1, \ldots, d,\\
\tilde{y}_i(t) &= z_1^{(i)}(t), \qquad i = 1, \ldots, d, \qquad \bar{y}(t) = \sum_{j=1}^{d} \hat{C}^{(j)} z^{(j)}(t) + \hat{\hat{C}} z^{(0)}(t).
\end{aligned} \tag{7}$$

Here we have used the notation

$$\tilde{u}(t) = (u_1, \ldots, u_d)^T, \quad \bar{u}(t) = (u_{d+1}, \ldots, u_l)^T, \quad \tilde{y}(t) = (y_1, \ldots, y_d)^T, \quad \bar{y}(t) = (y_{d+1}, \ldots, y_l)^T,$$

$$\hat{\hat{C}} = \begin{pmatrix} \hat{\hat{C}}_{d+1} \\ \ldots \\ \hat{\hat{C}}_l \end{pmatrix}, \qquad \hat{C}^{(i)} = \begin{pmatrix} \hat{C}_{d+1,i} \\ \ldots \\ \hat{C}_{li} \end{pmatrix},$$

$$z^{(0)} = (z_1^{(0)}, \ldots, z_{n-\sigma_0}^{(0)})^T, \quad z^{(i)} = (z_1^{(i)}, \ldots, z_{\varrho_i}^{(i)}), \quad i = 1, \ldots, d, \quad \hat{\hat{B}} = (\hat{\hat{B}}_{d+1}, \hat{\hat{B}}_{d+2}, \ldots, \hat{\hat{B}}_l).$$

We have thereby proved the following assertion.

Theorem 1. *Any system* (1) *that has a nonzero degenerate relative degree vector can be reduced by nonsingular transformations of inputs, outputs, and state coordinates to the form* (7). *In addition, if* $d = \operatorname{rank} \mathcal{H} < l$, *then* $n_0 > 0$, *where* n_0 *is the dimension of the vector* $z^{(0)}$.

From system (7), one can obtain an estimate for the dimension of the zero dynamics of the system. Indeed, if $\tilde{y}(t) \equiv 0$, then $z_j^{(i)}(t) \equiv 0$, $i = 1, \ldots, d$, $j = 1, \ldots, \varrho_i$, since $z_j^{(i)}(t) = \dfrac{d^{j-1}\tilde{y}_i(t)}{dt^{j-1}}$, $j = 1, \ldots, \varrho_i$. Therefore, if the trajectory $z(t)$ belongs to the zero dynamics, then only components $z^{(0)}(t)$ of the state vector can be nonzero.

From the condition $\bar{y}(t) \equiv 0$, we find that $z^{(0)}(t)$ satisfies the equation $\hat{\hat{C}} z^{(0)}(t) = 0$ for arbitrary t. Therefore, the trajectories $z(t)$ that belong to the zero dynamics of the system (and only these trajectories) satisfy the system of equations

$$z_j^{(i)}(t) = 0, \quad i = 1, \ldots, d, \quad j = 1, \ldots, \varrho_i, \quad \dot{z}^{(0)}(t) = \hat{\hat{A}} z^{(0)}(t) + \hat{\hat{B}} \bar{u}(t), \quad \hat{\hat{C}} z^{(0)}(t) = 0, \tag{8}$$

which implies that the dimension of the zero dynamics does not exceed the dimension of the matrix $\hat{\hat{A}}$, i.e., $n - \sigma_0$.

Example 1. Consider the system $\{A, B, C\}$ with the matrices:

$$A = \begin{pmatrix} 1 & 1 & 0 & -2 & 0 & -3 \\ 1 & 1 & 4 & -6 & 4 & -1 \\ -1 & -1 & -3 & 7 & -3 & 2 \\ 0 & -1 & -3 & 6 & -4 & 1 \\ 1 & 0 & 0 & 1 & -2 & 0 \\ 1 & 1 & 3 & -5 & 3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can readily see that $\beta(s) \equiv 0$ for this system. We find its degenerate relative degree vector

$$\begin{aligned} &C_1B = 0, \qquad C_2B = 0, \qquad C_3B = (-1,0,0), \\ &C_1AB = 0, \qquad C_2AB = (0,1,0), \qquad C_1A^2B = (1,0,0), \end{aligned} \tag{9}$$

whence it follows that $\varrho = (3,2,1)$. Note that in this case the vector ϱ is also a leading incomplete relative degree vector (arranged in decreasing order). Obviously, the matrix $\mathcal{H}(\varrho)$ corresponding to this vector is singular (this implies that the system has relative degree); moreover, $d = \operatorname{rank} \mathcal{H}(\varrho) = 2$.

It follows from system (9) that

$$\max_{\substack{j_1,j_2:\\ \mathcal{H}_{j_1},\mathcal{H}_{j_2}\ rmlin.indep.}} \sigma(j_1,j_2) = 5;$$

i.e., $\sigma_0 = 5$, a moreover, the maximum is attained for $j_1 = 1$ and $j_2 = 2$. (In this case, $\varrho_1 = 3$ and $\varrho_2 = 2$.) Therefore, the dimension of the zero dynamics of the system does not exceed $n - \sigma_0 = 1$.

Consider the rows

$$\begin{aligned} &C_1 = (1,0,1,-1,0,0), \qquad C_2 = (0,0,0,1,-1,1), \\ &C_1A = (0,1,0,-1,1,-2), \qquad C_2A = (0,0,0,0,1,0), \qquad C_1A^2 = (0,0,1,-1,0,0). \end{aligned}$$

In accordance with above-performed argument, these rows are linearly independent. Note that if this system of rows is supplemented with the row $V_1 = (0,0,0,0,0,1)$ such that $V_1B_1 = 0$ and $V_1B_2 = 0$, then the row system remains linearly independent.

Let us use the change of coordinates $z = Mx$, where

$$M = \begin{pmatrix} C_1 \\ C_1A \\ C_1A^2 \\ C_2 \\ C_2A \\ V_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

After the transformation, the matrices of the system acquire the form (the block structure is singled out in them)

$$\hat{A} = MAM^{-1} \left(\begin{array}{|ccc|cc|c|} \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & -1 & -1 \\ \hline 1 & 1 & 2 & -1 & 1 & 2 \\ \hline \end{array}\right), \quad \hat{B} = MB = \left(\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array}\right),$$

$$\hat{C} = CM^{-1} = \left(\begin{array}{|ccc|cc|c|} \hline 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 \\ \hline \end{array}\right);$$

i.e.,

$$\begin{aligned}
&\dot z_1^{(1)}(t) = z_2^{(1)}(t), \qquad \dot z_2^{(1)}(t) = z_3^{(1)}(t),\\
&\dot z_3^{(1)}(t) = -z_1^{(1)}(t) + z_3^{(1)}(t) + z_1^{(2)}(t) + 2z_2^{(2)}(t) + \tilde u_1(t),\\
&\dot z_1^{(2)}(t) = z_2^{(2)}(t), \qquad \dot z_2^{(2)}(t) = z_1^{(1)}(t) - z_3^{(1)}(t) + z_1^{(2)}(t) - z_2^{(2)}(t) - z_1^{(0)}(t) + \tilde u_2(t),\\
&\dot z_1^{(0)}(t) = z_1^{(1)}(t) + z_2^{(1)}(t) + 2z_3^{(1)}(t) - z_1^{(2)}(t) + z_2^{(2)}(t) + 2z_1^{(0)}(t) + \bar u_1(t),\\
&\quad \tilde y_1(t) = z_1^{(1)}(t), \qquad \tilde y_2(t) = z_1^{(2)}(t), \qquad \bar y_1(t) = z_1^{(1)}(t) - z_3^{(1)}(t).
\end{aligned} \tag{10}$$

If $y(t) = (y_1(t), y_2(t), y_3(t)) \equiv 0$, then from system (10), we obtain $z_j^{(i)}(t) \equiv 0$, $i = 1, 2$, $j = 1, \ldots, \varrho_i$. It follows that if a trajectory $z(t)$ belongs to the zero dynamics, then the first five components of the vector $z(t)$ are zero for any t, and the latter satisfies the condition

$$z_1^{(0)}(t) = 2z_1^{(0)}(t) + \bar u_1(t).$$

Hence it follows that the dimension of the zero dynamics of the system is equal to unity, which corresponds to the estimate. In this case the zero dynamics is controllable by the output $\bar u_1(t)$; i.e., the spectrum can be assigned arbitrarily. This corresponds to the case in which $\beta(s) = 0$, $s \in \mathbb{C}$.

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