Remark on a finite axiomatization of finite intermediate propositional logics

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ABSTRACT. A simple method of axiomatizing every finite intermediate propositional logic by a finite set of axioms with the minimal number of variables is proposed. The method is based on Jankov's characteristic formulas.

It is well-known that every finite intermediate propositional logic (i.e. the logic of a finite Heyting algebra or, equivalently, of a finite Kripke frame **A**) is finitely axiomatizable. This was first proved by De Jongh (unpublished; announced in [7]). Later several different procedures of axiomatizing finite logics were suggested (e.g. in [3, 6, 8]). However, all these axiomatizing techniques were rather non-optimal. Bellissima [2, §2] described the minimal number of variables $d(\mathbf{A})$ sufficient for axiomatizing the finite logic $Th(\mathbf{A})$ of a frame **A**. His method is based on the construction of free Kripke models representing finitely generated free Heyting algebras [1]. Here we give another simple proof of Bellissima's result and present another axiomatics of finite logics $Th(\mathbf{A})$ (in $d(\mathbf{A})$ variables), based on Jankov's characteristic formulas [4, 5]. The author was always sure that this method almost immediately follows from Jankov's consrtuctions, i.e. it essentially belongs to Jankov, and must be well-known[1]. Nevertheless, it was never published, so we present it here.

To make the paper self-contained, first we recall basic notions and results related to Jankov's formulas.

1 Intermediate logics and Kripke frames

Propositional formulas are built from variables $P = \{p_\alpha : \alpha \in \omega\}$ and constants $T, \bot$ using connectives $\wedge, \vee, \rightarrow$ (as usual, $\neg\varphi = (\varphi \rightarrow \bot)$ and $(\varphi \leftrightarrow \psi) = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$). An *α-formula* is a formula in variables $P_\alpha = \{p_\beta : \beta < \alpha\}$ (for $\alpha \in \omega$). An intermediate (propositional) logic L is called *α-axiomatizable* if $L = (Int + \Gamma)$ for a set of α-formulas Γ (where Int is intuitionistic propositional calculus).

[1] This turns out to be wrong: V.A.Jankov has told me recently that he did not know this way of axiomatizing finite logics and never thought about this problem.

A *Kripke frame* $\mathbf{A}$ is a non-empty partially ordered set. A *cone* (*s.g.* in [2]) is $(x\uparrow) = \{y \in \mathbf{A} : x \leq y\}$. A subset $a \subseteq \mathbf{A}$ is called *increasing* if $a = \cup(x\uparrow : x \in a)$. All increasing subsets of $\mathbf{A}$ constitute a Heyting algebra $O(\mathbf{A})$. A *Kripke model* is a pair $\mathcal{U} = (\mathbf{A}, \rho)$ where ρ is a valuation of variables in the Heyting algebra $O(\mathbf{A})$. Then $Th(x,\mathcal{U}) = \{\varphi : x \in \rho(\varphi)\}$ (for $x \in \mathbf{A}$) and $Th(\mathcal{U}) = \cap(Th(x,\mathcal{U}) : x \in \mathbf{A})$. Finally, $Th(\mathbf{A}) = \cap(Th(\mathbf{A},\rho) : \rho))$ is the *logic* of a frame $\mathbf{A}$ (i.e. of its Heyting algebra $O(\mathbf{A})$). The corresponding sets of α-formulas (for $\alpha \in \omega$) are denoted $Th_\alpha(x,\mathcal{U}), Th_\alpha(\mathcal{U}), Th_\alpha(\mathbf{A})$ respectively. A model $\mathcal{U} = (\mathbf{A}, \rho)$ is called *α-distinuishable* if

$$\forall x, y \in \mathbf{A}[(x \neq y) \Rightarrow (Th_\alpha(x,\mathcal{U}) \neq Th_\alpha(y,\mathcal{U}))].$$

An intermediate logic L is called *finite* if $L = Th(\mathbf{C})$ for a finite frame $\mathbf{C}$.

Let $\pi_0 = \bot$ and $\pi_{\alpha+1} = p_\alpha \vee (p_\alpha \rightarrow \pi_\alpha)$ for $\alpha \in \omega$ (hence π_α is an α-formula). It is well-known that $\pi_\alpha \in Th(\mathbf{A})$ iff $h(\mathbf{A}) \leq \alpha$, where $h(\mathbf{A})$ is the *height* of a frame $\mathbf{A}$, i.e. the greatest length of chains in $\mathbf{A}$ (recall that $h(\mathbf{A}) = sup(1 + \lambda(x) : x \in \mathbf{A})$ in terminology from [1]). Let $\mathcal{F}$ be the set of all finite cones, i.e. finite frames $\mathbf{D}$ with the least element $0_\mathbf{D}$. In other words, $\mathbf{D} \in \mathcal{F}$ iff $O(\mathbf{D})$ is a finite strongly compact Heyting algebra (i.e. it contains the greatest element $\tau = (\mathbf{D}\backslash\{0_\mathbf{D}\})$). Let $\delta(\mathbf{D})$ (for $\mathbf{D} \in \mathcal{F}$) be the least α such that there exists an α-distinguishable model $\mathcal{U} = (\mathbf{D}, \rho)$ (or, equivalently, such that the Heyting algebra $O(\mathbf{D})$ is generated by an α-element set). Obviously, $\delta(\mathbf{D}) < card(\mathbf{D})$ (since $\{(x\uparrow) : x \neq 0_\mathbf{D}\}$ generates $O(\mathbf{D})$, cf. Lemma 2.0 in [2]) and $\delta(\mathbf{D}) = \alpha$ for the $(\alpha + 1)$-element chain $\mathbf{D}$.

A *p-morphism* is a function F from a frame $\mathbf{A}$ onto a frame $\mathbf{B}$ such that: (1) $x \leq_\mathbf{A} y \Rightarrow F(x) \leq_\mathbf{B} F(y)$; (2) $F(x) \leq_\mathbf{B} z \Rightarrow \exists y \in (x\uparrow)(F(y) = z)$.

Then F^{-1} is an embedding of $O(\mathbf{B})$ into $O(\mathbf{A})$. The following lemma seems to be well-known:

Lemma 1. Let $\mathbf{A}$ be a cone (possibly infinite) and $\mathbf{D} \in \mathcal{F}$. Then every embedding f of $O(\mathbf{D})$ into $O(\mathbf{A})$ can be represented as F^{-1} for a p-morphism $F : \mathbf{A} \rightarrow \mathbf{D}$.

Proof: $F(x)$ (for $x \in \mathbf{A}$) is defined as the greatest element y of $\mathbf{D}$ such that $x \in f(y\uparrow)$. ■

Let X be a class of frames, then IX and QX are the classes of isomorphic copies and of p-morphic images for frames from X (respectively) and $SX = \cup(O(\mathbf{A}) : \mathbf{A} \in X)$ (here $\mathbf{B} \in O(\mathbf{A})$ is a frame with an ordering restricted from $\mathbf{A}$). Obviously, $Th(\mathbf{A}) \subseteq Th(\mathbf{B})$ if $\mathbf{B} \in QS\{\mathbf{A}\}$, and $\mathcal{F} \cap QS\{\mathbf{C}\}$ is the class of all p-morphic images of cones in $\mathbf{C}$ (for a finite frame $\mathbf{C}$). Recall that $SQX = QSX$ (cf. Remark 1.4 in [2]).

2 Jankov's formulas

The *Jankov's* characteristic *formula* of a frame $\mathbf{D} \in \mathcal{F}$ is:

$$\chi_D = (\wedge\Delta(\mathbf{D}) \rightarrow p_\tau)$$

where $\Delta(\mathbf{D})$ is the following set of formulas (in variables $p_a : a \in O(\mathbf{D})$) :

$$\begin{array}{rcl} \Delta(\mathbf{D}) & = & \{(p_\emptyset \leftrightarrow \bot)\} \cup \\ & & \cup\{(p_{(a\cap b)} \leftrightarrow (p_a \wedge p_b)) \ : \ a, b \in O(\mathbf{D})\} \cup \\ & & \cup\{(p_{(a\cup b)} \leftrightarrow (p_a \vee p_b)) \ : \ a, b \in O(\mathbf{D})\} \cup \\ & & \cup\{(p_{(a\Rightarrow b)} \leftrightarrow (p_a \rightarrow p_b)) \ : \ a, b \in O(\mathbf{D})\} \end{array}$$

(here $\cap, \cup, \Rightarrow$ are the operations of the Heyting algebra $O(\mathbf{D})$, $\emptyset$ is its least element, and $\tau = (\mathbf{D}\backslash\{0_\mathbf{D}\})$).

Let χ'_D be a $\delta(\mathbf{D})$-formula in variables $p_b : b \in G$, where G is a $\delta(\mathbf{D})$-element set generating $O(\mathbf{D})$: namely, we replace every p_a (for $a \in O(\mathbf{D})$) by a representation of a through G (thus χ'_D is a substitution instance of χ_D). Obviously, $\chi_D \notin Th(\mathbf{D})$ (and $\chi'_D \notin Th(\mathbf{D})$ too). Jankov [4, 5] proved the following properties of characteristic formulas.

Lemma 2. (1) $(Int + \varphi) \vdash \chi_D$ iff $\varphi \notin Th(\mathbf{D})$, for a formula φ.
(2) $L \subseteq Th(\mathbf{D})$ iff $\chi_D \notin L$, for an intermediate logic L.

Proof: if $\varphi(a_1, \ldots, a_n) \leq \tau$ in $O(\mathbf{D})$ then $\Delta(\mathbf{D}) \vdash [\varphi(p_{a_1}, \ldots, p_{a_n}) \rightarrow p_\tau]$. ■

Corollary. $(Int + \chi_D) = (Int + \chi'_D)$.

Remark. $(Int + \pi_\alpha) = (Int + \chi_{\mathbf{C}_{\alpha+1}})$, where $\mathbf{C}_{\alpha+1}$ is the $(\alpha + 1)$-element chain.

Lemma 3. $\chi_D \notin Th(\mathbf{A})$ iff $\mathbf{D} \in QS\{\mathbf{A}\}$, for an arbitrary frame $\mathbf{A}$.

Proof: if there exist a model $\mathcal{U} = (\mathbf{A}, \rho)$ and $x \in \mathbf{A}$ such that $x \models \Delta(\mathbf{D})$ and $x \not\models p_\tau$ then $f(a) = \{y \geq x : p_a \in Th(y, \mathcal{U})\}$ is an embedding of $O(\mathbf{D})$ into $O(x\uparrow)$. ■

Jankov [4] defined the following relations on $\mathcal{F}$:

$(\mathbf{C} \leq \mathbf{D})$ iff $(\mathbf{C} \in QS\{\mathbf{D}\})$ iff $(\chi_C \notin Th(\mathbf{D}))$
iff $(Th(\mathbf{D}) \subseteq Th(\mathbf{C}))$ iff $((Int + \chi_C) \vdash \chi_D)$;

$(\mathbf{C} < \mathbf{D})$ iff $(\mathbf{C} \leq \mathbf{D})\&\neg(\mathbf{D} \leq \mathbf{C})$ iff $(Th(\mathbf{D}) \subset Th(\mathbf{C}))$.

It is clear that (cf., e.g., [4]):

(i) $\leq$ is a pre-ordering on $\mathcal{F}$ (i.e. it is reflexive and transitive);

(ii) $(\mathbf{C} \leq \mathbf{D})\&(\mathbf{D} \leq \mathbf{C})$ iff $(\mathbf{C} \in I\{\mathbf{D}\})$;

(iii) $(\mathbf{C} \leq \mathbf{D}) \Rightarrow (card(\mathbf{C}) \leq card(\mathbf{D}))$,
$(\mathbf{C} < \mathbf{D}) \Rightarrow (card(\mathbf{C}) < card(\mathbf{D}))$;

(iv) $\{\mathbf{C} \in \mathcal{F} : \mathbf{C} \leq \mathbf{D}\}$ is finite, for any $\mathbf{D} \in \mathcal{F}$.

3 An axiomatization of finite logics

Let $\mathbf{A}$ be a finite frame, and let

$$\begin{array}{rcl} Z^*(\mathbf{A}) & = & (\mathcal{F}\backslash QS\{\mathbf{A}\}) = \{\mathbf{D} \in \mathcal{F} : \chi_D \in Th(\mathbf{A})\} = \{\mathbf{D} \in \mathcal{F} : Th(\mathbf{A}) \not\subseteq Th(\mathbf{D})\}; \\ Z(\mathbf{A}) & = & \{\mathbf{D} \in Z^*(\mathbf{A}) : (QS\{\mathbf{D}\}\backslash I\{\mathbf{D}\}) \subseteq QS\{\mathbf{A}\}\} \\ & & \text{(the set of all } \leq\text{-minimal elements of } Z^*(\mathbf{A}) \text{);} \\ d(\mathbf{A}) & = & max\{\delta(\mathbf{D}) : \mathbf{D} \in Z(\mathbf{A})\}. \end{array}$$

The set $Z(\mathbf{A})$ is finite, since $\forall \mathbf{D} \in Z(\mathbf{A})[card(\mathbf{D}) \leq (1 + card(\mathbf{A}))]$ (cf. the proof of Lemma 2.0 in [2]: namely, by Theorem 1.3 in [2], $\forall \mathbf{D} \exists \mathbf{C} \in Q\{\mathbf{D}\}(card(\mathbf{C}) = card(\mathbf{D}) - 1)$). Hence $d(\mathbf{A}) \leq card(\mathbf{A})$. Note also that $d(\mathbf{A}) \geq h(\mathbf{A})$ (since the $(h(\mathbf{A}) + 1)$- element chain $\mathbf{C}_{h(\mathbf{A})+1}$ belongs to $Z(\mathbf{A})$). Bellissima proved the following result (Theorem 2.2 in [2]), using free Kripke models with a finite number of variables [1].

Theorem. Let $L = Th(\mathbf{A})$ be a finite logic. Then L is α-axiomatizable iff $\alpha \geq d(\mathbf{A})$ (moreover, there exists a finite $d(\mathbf{A})$-axiomatization of L).

Here we give another proof of this theorem basing on Jankov's formulas.

($\Rightarrow$). Suppose that $\alpha < d(\mathbf{A})$. Take a frame $\mathbf{D} \in Z(\mathbf{A})$ such that $\delta(\mathbf{D}) > \alpha$. Then $\chi_D \in Th(\mathbf{A})$. On the other hand, the following Lemma shows that $\chi_D \notin (Int + Th_\alpha(\mathbf{A}))$ (and hence, $(Int + Th_\alpha(\mathbf{A})) \neq Th(\mathbf{A})$).

Lemma 4. $Th_\alpha(\mathbf{A}) \subseteq Th(\mathbf{D})$ if $(Q\{\mathbf{D}\} \backslash I\{\mathbf{D}\}) \subseteq QS\{\mathbf{A}\}$ and $\alpha < \delta(\mathbf{D})$.

Proof. Let φ be an α-formula, $\varphi \notin Th(\mathcal{U})$ for some model $\mathcal{U} = (\mathbf{D}, \rho)$. Then $\{\rho(p) : \beta < \alpha\}$ generates a proper subalgebra in $O(\mathbf{D})$, and hence $\varphi \notin Th(\mathbf{C})$ for some $\mathbf{C} \in (Q\{\mathbf{D}\} \backslash I\{\mathbf{D}\}) \subseteq QS\{\mathbf{A}\}$. Thus $\varphi \notin Th(\mathbf{A})$. ∎

($\Leftarrow$).

Lemma 5. $Th(\mathbf{A}) = (Int + \Gamma)$ where $\Gamma = \{\chi'_D : \mathbf{D} \in Z(\mathbf{A})\}$ is a set of $d(\mathbf{A})$-formulas.

Proof. Note that $L = (Int + \Gamma) \supseteq \Gamma' = \{\chi_C : \mathbf{C} \in Z^*(\mathbf{A})\}$ (since $\forall \mathbf{C} \in Z^*(\mathbf{A}) \exists \mathbf{D} \in Z(\mathbf{A})(\mathbf{D} \leq \mathbf{C})$). Also $\pi_{h(\mathbf{A})} \in L$ since $\mathbf{C}_{h(\mathbf{A})+1} \in Z(\mathbf{A})$ (cf. Remark after Lemma 2). It is known that every extension of $(Int + \pi_\alpha)$ (for $\alpha < \omega$) has the finite model property (since all its finitely generated Heyting algebras are finite). Now, if $L \not\vdash \varphi$ then $\varphi \notin Th(\mathbf{C})$ for some finite frame $\mathbf{C}$ (from $\mathcal{F}$) validating L. Then $\mathbf{C} \in QS\{\mathbf{A}\}$ (since $\Gamma' \subseteq Th(\mathbf{C})$), $Th(\mathbf{A}) \subseteq Th(\mathbf{C})$ and $\varphi \notin Th(\mathbf{A})$. ∎

We did not compare which axiomatics is simpler: the axiomatics from our Lemma 5, or that given by Theorem 2.2 in [2]. Note however, that one can get our axioms almost immediately from the calculation of $\alpha = d(\mathbf{A})$: namely, one has to construct all frames $\mathbf{D}$ from $Z(\mathbf{A})$ and to find minimal generating subsets of their algebras $O(\mathbf{D})$. On the other hand, to obtain axioms from [2] you have besides to construct $h(\mathbf{A})$ slices of the free model in α variables and to find formulas ψ_w for its points w (see Theorem 2.7 from [1]).

4 Concluding remarks

Now we will compare our constructions with Bellissima's [2]; some terminology and notations from [2] will be used without additional explanations.

Note that Theorem 2.2 from [2] is stated in a slightly different form. Namely, the following set of frames is considered (for a finite frame $\mathbf{A}$):

$$X(\mathbf{A}) = \{\mathbf{D} \in Z^*(\mathbf{A}) : (Q\{\mathbf{D}\} \backslash I\{\mathbf{D}\}) \subseteq QS\{\mathbf{A}\}\ \}.$$

Obviously, $Z(\mathbf{A}) \subseteq X(\mathbf{A}) \subset Z^*(\mathbf{A})$ (and $X(\mathbf{A})$ is finite, as well as $Z(\mathbf{A})$: see Lemma 2.0 in [2]). Here the first inclusion can be proper.

Example (see Fig.1):
$\mathbf{D} \in (X(\mathbf{A}) \backslash Z(\mathbf{A}))$ since $(Q\{\mathbf{D}\} \backslash I\{\mathbf{D}\}) = Q\{\mathbf{D}_1\} \cup Q\{\mathbf{D}_2\} \subseteq Q\{\mathbf{A}\}$ (use Theorem 1.3 from [2]) and $(x\uparrow) \in (S\{\mathbf{D}\} \backslash QS\{\mathbf{A}\})$.

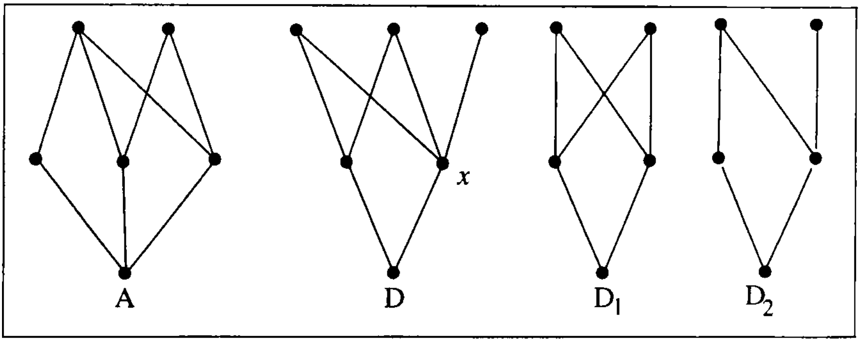

Figure 1

Now, Theorem 2.2 in [2] is stated with $d'(\mathbf{A}) = max\{\delta(\mathbf{D}) : \mathbf{D} \in X(\mathbf{A})\}$ instead of our d($\mathbf{A}$) (clearly, $d(\mathbf{A}) \leq d'(\mathbf{A})$). The proof of *only if* part in [2, p.409] seems to contain a minor gap and guarantees only that $\alpha \geq d(\mathbf{A})$ (but not $\alpha \geq d'(\mathbf{A})$) for an α-axiomatization of $Th(\mathbf{A})$ (namely, $M_\alpha(\mathbf{A}) = M_\alpha(\mathbf{C})$ follows from Lemma 2.1 in [2] only if $(SQ\{\mathbf{B}\} \backslash I\{\mathbf{B}\}) \subseteq QS\{\mathbf{A}\}$). On the other hand, our Lemma 4 guarantees that $\alpha \geq d'(\mathbf{A})$. Therefore we can conclude that $d'(\mathbf{A}) = d(\mathbf{A})$, and the formulation of Theorem 2.2 from [2] is equivalent to ours.

Note also that methods from [2] allow us to prove that $d'(\mathbf{A}) = d(\mathbf{A})$ directly. Namely, it is easily seen that:

> if $\mathbf{D} \in X(\mathbf{A})$ then $(x\uparrow) \in Z(\mathbf{A})$ for some $x \in \mathbf{D}$ such that every degenerate or duplicate pair $\{u, v\}$ in $\mathbf{D}$ belongs to $(x\uparrow)$

(for the definitions of degenerate and duplicate pairs see [2, §1, p.407]). Now, $\delta(\mathbf{D}) = \delta(x\uparrow)$ for such x, since any α-distinguishable model $\mathcal{U} = (x\uparrow\ ,\ \rho)$ gives rise to an α-distinguishable model $\mathcal{U}' = (\mathbf{D}\ ,\ \rho')$ (use Remark 0.1(iii) from [2, p.404]).

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