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Asymptotically Optimum Arrangements for a Special Class of Normed Spaces

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Abstract—A nonclassical queuing-theory problem with calls arising in a space is considered. Stations must be placed to minimize the service time for arising calls. The service time is an increasing function that depends on the distance between a call and a station. The time spent to overcome the same distance frequently depends on the direction of motion. In this case, a metric that considers the nonequivalence of coordinates of the space in order must be chosen to construct an adequate mathematical model. Optimum arrangements of stations can be found for problems of this kind only in exceptional situations. However, an asymptotical solution to the problem can be found that is acceptable from a practical viewpoint. An algorithm is given for constructing asymptotically optimum arrangements.

Keywords: service stations, zone of influence, affine transformation, optimality criterion, asymptotically optimum arrangements.

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1. INTRODUCTION

The interest in the problem of arranging service stations is mainly due to the practical need to distribute restricted resources efficiently. For example, this problem was investigated in [1–4] to find the best arrangements in a variety of regions.

In practice, arrangement problems arise in finding the optimum arrangement of, e.g., car-care centers on motorways, and selecting locations for maintenance stations and first-aid points. The call service time is an increasing function that depends on the distance between a call and a station. In real practical models, the time spent to overcome the same distance frequently depends on the direction of motion. To construct an adequate mathematical model, we must select a metric that considers the nonequivalence of the coordinates of the space.

In this work, we provide the theoretical reasoning and arrangement algorithm for service stations, constructed according to this reasoning and ensuring the efficient location of the stations.

Let ρ be a norm defining the distance between spatial points, and the algorithm for constructing asymptotically optimum arrangements be known for the said norm. A class of Minkowski metrics, $\mathcal{M}$, can be constructed according to this norm, such that the role of the unit sphere is played by a geometric body obtained using affine map h of the unit sphere in the space with norm ρ. Let δ be a metric belonging to specified class $\mathcal{M}$. The idea of constructing asymptotically optimum arrangements in the space with metric δ (note that this metric is a norm) is described below.

In the space with norm ρ, optimum arrangement x^* is constructed to minimize a criterion on this arrangement such that the said criterion is a function of the distance from incoming call ξ to the closest station. In the space with norm δ, the arrangement obtained using map h of arrangement x^* has the

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same optimality, since $\rho(\xi, x_i) = \delta(h(\xi), h(x_i))$, where ξ is a random call from the zone of influence of service station x_i.

In this work, we generalize the result obtained in [5] for the mean service time criterion. Note that the problem of finding the optimum locations of stations in the space is closely related to the so-called problem of the densest packing of balls from the combinatorial geometry (see [6]). For example, such problems were considered in [7] to find the uniform arrangement of points on a sphere.

Problems of optimum resource distribution are actively investigated abroad as well. Optimization in transport problems is of special interest (see [8–10]).

2. STATEMENT OF THE PROBLEM

Let us consider exact formulations. We assume that the distance between points R^N is defined by norm δ, calls arise from random points ξ with distribution density f, and number n of service stations is given. A call arrives at the closest station. The service time is an increasing function that depends on the distance between the call and the station. The stations must be positioned to minimize the service time for arising calls.

Definition 1. An arrangement of n stations in space is set $x = \{x_1, \dots, x_n\}$ of spatial points where these stations are located.

A service station and the spatial point of its location are denoted by the same symbol.

Definition 2. The zone of influence of station x_i is set C_i of points of space R^N such that the said station is the one closest to all those points; i,e.,

$$C_i = \{v \in R^N : |v - x_i| \leqslant |v - x_j|,\ j = 1, 2, \dots, n\}.$$

Arrangements of stations $x = \{x_1, \dots, x_n\}$ are estimated using the criterion

$$\varphi_\delta(x) = \mathrm{E} \min_{1 \leqslant i \leqslant n} \delta^s(\xi, x_i), \quad s > 0.$$

Definition 3. We say that arrangement $x^* = \{x_1^*, \dots, x_n^*\}$ is optimum among arrangements $x = \{x_1, \dots, x_n\}$ with respect to criterion φ if

$$\varphi(x^*) = \min_x \varphi(x).$$

Definition 4. Let F be a criterion for spatial arrangements. We say that arrangement $x = \{x_1, \dots, x_n\}$ is asymptotically optimal if

$$\lim_{n \to \infty} \frac{F(x)}{F(x^*)} = 1,$$

where $x^* = \{x_1^*, \dots, x_n^*\}$ is the optimum arrangement with respect to criterion F.

Our aim is to find the asymptotically optimum arrangement of service stations.

3. PROPERTIES OF ASYMPTOTICALLY OPTIMUM ARRANGEMENTS OF SERVICE STATIONS

Let us change the coordinate values of R^N by means of affine transformation h^{-1}. Distribution density p of random variable $\eta = h^{-1}(\xi)$ is then presented in the form $p(u) = f(v)\triangle$, where $\triangle$ is the modulus of the determinant of affine transformation h, $v = h(u)$. The corresponding probabilistic model is thus constructed: in space R^N, calls at random points η arise with distribution density p, the distance between points of the space is determined using norm ρ, and the optimality criterion is given by the expression

$$\varphi_\rho(x) = \mathrm{E} \min_{1 \leqslant i \leqslant n} \rho^s(\eta, x_i).$$

In this space, we construct asymptotically optimum arrangement $z = \{z_1, \dots, z_n\}$. Let x denote the arrangement obtained with mapping h of arrangement z:

$$x = \{h(z_1), \dots, h(z_n)\}.$$