Year 1999 Olympiad

Level A

Problem 1. Compare the fractions $x = \frac{111110}{111111}$, $y = \frac{222221}{222223}$, and $z = \frac{333331}{333334}$, and arrange them in ascending order.

Problem 2. Show how to cut any quadrilateral into three trapezoids. (A parallelogram is also considered a trapezoid.)

Problem 3. Find four pairwise distinct positive integers a, b, c, and d for which the numbers $a^2 + 2cd + b^2$ and $c^2 + 2ab + d^2$ are perfect squares.

Problem 4. Annie has 500 dollars on her bank account. The bank allows only two kinds of operations: withdrawing \$300 or adding \$198. What is the largest sum that Annie can take from her account if she has no other money?

Problem 5. In a right triangle ABC, the midpoint of the hypotenuse AC is labeled O. Points M and N are chosen on the legs AB and BC so that $\angle MON = 90°$. Prove that $AM^2 + CN^2 = MN^2$.

Problem 6. Each participant in a chess tournament played each other twice: once as white and once as black. The final scores of all the players were the same. (A win is worth one point, a tie half a point, and a loss zero points.) Prove that at least two competitors won the same number of games playing white. page 41

Level B

Problem 1. Two numbers are written on a blackboard in a laboratory. Every day at noon a researcher erases both numbers and writes their arithmetic and harmonic means instead. The numbers written on the blackboard on the morning of the first day were 1 and 2. Find the product of the numbers written on the blackboard at the end of the 1999th day. (The arithmetic and harmonic means of two numbers a and b are

$$\frac{a+b}{2} \quad \text{and} \quad \frac{2}{\frac{1}{a}+\frac{1}{b}},$$

respectively.)

Problem 2. Two players play the following game: the first writes one letter A or B per turn, from left to right, starting from nothing; the second player, after each play by the first, can either pass or interchange any two letters already written. When both players have had 1999 turns each, the game is over.

Can the second player ensure that the final string is a palindrome no matter what the first player does? (A palindrome is a string that reads the same from left to right or from right to left.)

Problem 3. The diagonals of a parallelogram $ABCD$ meet at a point O. The circle passing through A, O, and B is tangent to the line BC. Prove that the circle passing through the points B, O, and C is tangent to the line CD.

Problem 4. Find all positive integers k such that the number

$$\underbrace{1\dots1\overbrace{2\dots2}^{k}}_{2000}-\underbrace{2\dots2}_{1001}$$

is a perfect square.

Problem 5. The incircle of a triangle ABC, for which $AB > BC$, touches the sides AB and AC at P and Q, respectively. The midline parallel to AB is labeled RS, and it intersects PQ at T. Prove that T lies on the bisector of the angle B of the triangle.

page 42

Problem 6. A sports competition has n scored events and 2^n participants. The first event eliminates the bottom half the participants, according to the scores obtained; the second eliminates half of the remaining ones (a quarter of the total), and so on, until only an overall winner is left.

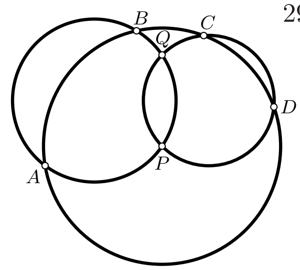
Suppose a ranking of expected performance is made for each event separately, prior to the start of the competition. A contestant is called a *potential winner* if, for some ordering of the events and assuming the expected ranking holds true, that contestant will win the tournament.

(a) Prove that it is possible for 2^{n-1} contestants to be potential winners.
(b) Prove that it is not possible for more than $2^n - n$ contestants to be potential winners.
(c) $\star$ Prove that it is possible for exactly $2^n - n$ contestants to be potential winners.

Level C

Problem 1. It is known that $(a + b + c)c < 0$. Prove that $b^2 > 4ac$.

Problem 2. Two circles meet at points P and Q. The third circle, centered at P, meets the first one at points A and B, and the second one at points



C and D (see figure). Prove that the angles AQD and BQC are equal.

Remark. To avoid having to consider numerous cases, at the Olympiad the problem was proposed only for the configuration shown in the figure. Nonetheless, the statement remains true for other cases.

Problem 3*. Find all the pairs of positive integers x, y such that $x^3 + y$ and $y^3 + x$ are divisible by $x^2 + y^2$.

Problem 4*. A disk is divided by $2n$ radii into $2n$ congruent sectors, n of them blue and n red. Two sectors, blue and red, are chosen at will. Starting with the chosen blue sector, the numbers from 1 to n are written in the blue sectors counterclockwise. Similarly, starting with the chosen red sector, the numbers from 1 to n are written in the red sectors clockwise. Prove that that there is a half-disk that contains all the numbers from 1 to n. page 43

Problem 5. A grasshopper jumps along the interval $[0, 1]$. From a point x it can jump either to $x/\sqrt{3}$ or to $x/\sqrt{3} + (1 - 1/\sqrt{3})$. A point a is chosen on the interval $[0, 1]$. Prove that, starting from any point, the grasshopper can, after a number of jumps, reach a point at a distance of no more than $\frac{1}{100}$ from a.

Problem 6*. The numbers $1, \ldots, 1999$ are written around a circle in some order; then the sum of the products of all sets of 10 consecutive numbers is computed. Find the arrangement for which this sum is the greatest.

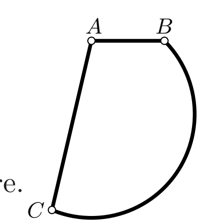
Level D

Remark. At the Olympiad, problems 5–7 were scored and the best two numbers were added to the scores of problems 1–4. *

Problem 1. Let a, b, c be the sides of a triangle. Prove the inequality

$$\frac{a^2 + 2bc}{b^2 + c^2} + \frac{b^2 + 2ac}{c^2 + a^2} + \frac{c^2 + 2ab}{a^2 + b^2} > 3.$$

Problem 2. A plane convex figure is bounded by two line segments, AB and AC, and an arc of circle BC (see figure).



(a) Construct a line that bisects the perimeter of the figure.
(b) Construct a line that bisects the area of the figure.

Problem 3*. The faces of a regular octahedron are colored white or black. Any two faces that share a common edge are of different colors. Prove that the sum of distances from any point inside the octahedron to the planes of the white faces is equal to the sum of its distances to the planes of the black faces.

Problem 4. A square meadow contains a round clearing. A grasshopper is hopping around in the meadow. Before each jump it chooses a vertex of the square and jumps exactly halfway toward it. Can the grasshopper always arrange to land in the clearing after a number of jumps, no matter where it starts?

Problem 5. A graph is a set of points, called nodes, some of which are connected by lines, called edges; each edge connects exactly two nodes. A coloring of the nodes is said to be regular if no two nodes of the same color are connected by an edge. A certain graph is regularly colored in k colors and cannot be regularly colored in fewer colors. Prove that there is a path in this graph that visits nodes of all k colors, each of them once. page 44

Problem 6*. Solve the equation $(1 + n^k)^l = 1 + n^m$ in positive integers, where $l > 1$.

Problem 7. Prove that the first digits of the numbers of the form 2^{2^n} form a nonperiodic sequence.

Year 1999 Olympiad

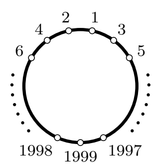
Level A

1. $x < z < y$. **3.** For example, $a = 1$, $b = 6$, $c = 2$, $d = 3$. **4.** \$498.

Level B

1. 2. **2.** Yes. **4.** $k = 2$.

Level C

3. $x = 1$, $y = 1$. **6.** 

is the only solution, apart from rotations and reflections. *

Level D

4. Yes, it can. **6.** Unique solution: $n = 2$, $k = 1$, $l = 2$, $m = 3$. *

Year 1999 Olympiad

Level A

1. Consider the numbers $1 - x$, $1 - y$, and $1 - z$.
3. If $ab = cd$, then $a^2 + 2cd + b^2$ and $c^2 + 2ab + d^2$ are perfect squares.
4. Both 300 and 198 are divisible by 6.
5. Consider a point symmetric to N with respect to O.
6. Otherwise there is a participant who won all the games as white and one who won no games as white.

Level B

1. The product of the numbers on the blackboard does not change.
2. Try to arrange it so that the after 1001 moves the last three letters form a palindrome. *
3. Use the theorem about the angle between a tangent and a chord.
4. A perfect square ends in an even number of zeros; the perfect square closest to n^2 is $n^2 - 2n + 1$.
5. Prove that $ST = BS$.

Level C

1. If $f(x) = ax^2 + bx + c$, then $a + b + c = f(1)$ and $c = f(0)$.
3. $x(x^2 + y^2) - (x^3 + y) = y(xy - 1)$.
4. Consider two equal numbers with the smallest distance between them.
5. Both maps defined by the grasshopper's jump contract the interval $[0, 1]$ by a factor of $\sqrt{3}$.

Level D

1. Apply the triangle inequality.
2. Consider the midpoint of the arc BC.
3. The planes containing white faces bound a regular tetrahedron.
4. Divide the square into 4^n small squares and prove by induction that the grasshopper can hit any of them.

* removed rest of hint

page 87

5. If, in a regular coloring, we reassign color 1 to the nodes of color 2 not connected with nodes of color 1, the entire coloring remains regular.
6. Use the binomial formula.
7. Consider the half-open intervals $[0, \lg 2), \dots, [\lg 9, 0)$ on a circle of length 1. The first digit of 2^k is determined by the interval that contains the fractional part of $\lg 2^k = k \cdot \lg 2$.

Year 1999 Olympiad

Level A

Problem 1. Consider the numbers

$$1-x=\frac{1}{111111},\quad 1-y=\frac{2}{222223},\quad 1-z=\frac{3}{333334},$$

and their reciprocals

$$\frac{1}{1-x}=111111,\quad \frac{1}{1-y}=111111\tfrac{1}{2},\quad \frac{1}{1-z}=111111\tfrac{1}{3}.$$

Since $1/(1-x)<1/(1-z)<1/(1-y)$, and all three of these numbers are positive, we obtain $1-x>1-z>1-y$. Thus, $x<z<y$. *

Problem 2. The solution is shown in the figures on the next page. However, to make it rigorous, we must do some work. (At the Olympiad, even solutions without rigorous explanations were accepted by the graders.)

page 233

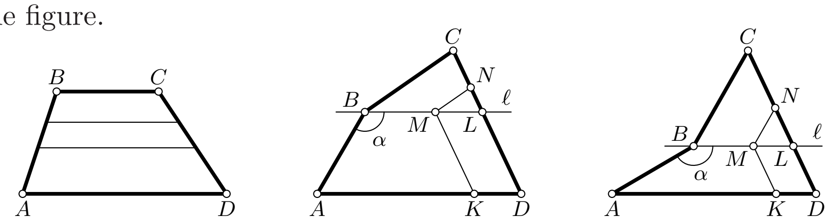
I don't understand why the original considered the nonconvex case separately.

Let $ABCD$ be an arbitrary quadrilateral. It has two neighboring internal angles whose sum is at least 180°, because

$$(\angle A+\angle B)+(\angle C+\angle D)=360^\circ.$$

(This is true even for nonconvex quadrilaterals.) Hence one of the sums in parentheses is at least 180°. We may assume, by relabeling if necessary, that $\angle A+\angle B\geq 180^\circ$.

If this sum *equals* 180°, then $BC \parallel AD$ and $ABCD$ is a trapezoid. Any two cuts parallel to the bases then solve the problem. See the left diagram in the figure.



Next, suppose $\angle A+\angle B>180^\circ$. Consider the line ℓ passing through the point B parallel to AD, as in the middle and right diagrams above. (The rightmost diagram illustrates the case of a nonconvex quadrilateral.)

Let α be the angle formed by ℓ and the side AB, and situated opposite angle A relative to the parallel lines ℓ and AD. Then $\alpha = 180° - \angle A < \angle B$, where the equality holds because interior opposite angles are supplementary. Therefore, the line ℓ goes inside the quadrilateral, and so, upon going out, it intersects the side CD at a certain point L.

Now draw a line parallel to CD and intersecting the segments AD and BL. Suppose this line intersects the side AD at K, and the segment BL at M. If we now draw the line through M parallel to BC until it meets the segment CL at a point N, we will have cut our quadrilateral into trapezoids $AKMB$, $KDNM$, and $CNMB$. removed "this ... done" * *

Problem 3. It suffices to choose the numbers so that the product of the first two is equal to that of the last two, $ab = cd$; indeed, in this case, we have page 234

$$a^2 + 2cd + b^2 = a^2 + 2ab + b^2 = (a+b)^2,$$
$$c^2 + 2ab + d^2 = c^2 + 2cd + d^2 = (c+d)^2.$$

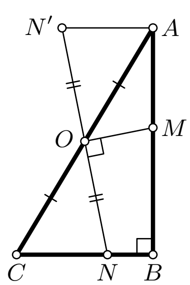
Thus, we only need to find a number n that can be represented as the product of two different factors in two different ways: $n = ab = cd$. For example, we can take the number $6 = 1 \cdot 6 = 2 \cdot 3$. * removed "see also Fact 10".

Problem 4. Since 300 and 198 are divisible by 6, Annie can withdraw only multiples of 6 dollars (see Fact 5). The largest multiple of 6 not exceeding 500 is 498.

Let's see how to withdraw 498 dollars. After the following operations: $500-300=200$, $200+198=398$, $398-300=98$, $98+198=296$, $296+198=494$, the sum on the account decreases by 6 dollars.

Having repeated this procedure 16 times, Annie will have withdrawn 96 dollars. Then she can take out 300, deposit 198, and take out 300 again to end up with 498 dollars.

Problem 5. Denote by N' the point symmetric to N with respect to O. The triangles ONC and $ON'A$ are congruent by the SAS property. In addition, $N'AM$ is a right angle, because



$$\angle N'AM = \angle N'AO + \angle MAO = \angle ACB + \angle BAC = 90°.$$

Then, by the Pythagorean Theorem,

$$AM^2 + CN^2 = AM^2 + AN'^2 = MN'^2.$$

Therefore, it remains to prove that $MN' = MN$. But this equality follows from the fact that the right triangles $N'OM$ and NOM are congruent by the congruence of their legs.

Problem 6. Suppose there were n participants in the tournament. The total number of games played was therefore $n(n-1)$, and for each one of them, one point was awarded in total. Thus the final scores of the players, being all the same, must equal $n-1$ points. page 235

Each participant played white in $n-1$ games, and the number of wins for a given player among his or her games played as white must be one of the numbers $0, \dots, n-1$. Suppose that the statement of the problem is not true: each player won a different number of games while playing white. These different numbers then must take all the n available values 0, 1, …, $n-1$. Let A be the player who won $n-1$ games as white, and B the one who won no games as white.

Who won the game between them in which A played black? A scored $n-1$ points playing white, and so won no games playing black. In particular, B won when playing white against A. On the other hand, we know that B won no games playing white, so we have a contradiction.

Level B

Problem 1. On the 1999th day the product of the numbers on the blackboard will still be the same as on the first day! This product does not change from day to day:

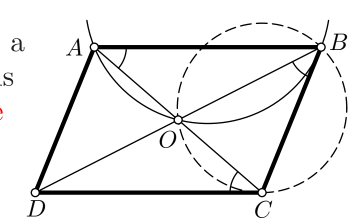
$$\frac{a+b}{2} \cdot \frac{2}{1/a+1/b} = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab.$$

Problem 2. Here is the strategy of the second player. This player skips the first 1000 moves. Her k-th move after that is made so the last $2k+1$ letters form a palindrome. Let's prove by induction on k (see Fact 24) that this is always possible.

For $k = 0$, this is obvious. Suppose that after $1000+(k-1)$ moves by the second player the last $2k-1$ letters form a palindrome. If the next letter added by the first player, which is in position $1000+k$, coincides with the one in position $1000-k$, the second player has nothing to do. *

If the $(1000+k)$-th and $(1000-k)$-th letters are different, one of them also differs from the letter in the 1000th place. In this case, the second player exchanges it with the 1000th letter. This does not destroy the ealier $(2k-1)$-letter palindrome, because the middle letter has no match. Now the last $2k+1$ letters form a palindrome. page 236

After 1999 moves — that is, when $k = 999$ — the entire word becomes a palindrome.



Problem 3. The angle between a tangent and a chord drawn from the point of contact equals half the measure of the corresponding arc (see Fact 15); hence $\angle CBO = \angle BAC$. At the same time we have $\angle BAC = \angle ACD$, because alternate interior angles between two parallels are equal. Thus $\angle CBO = \angle OCD$. Applying the converse of the theorem about the angle between a tangent and a chord, we see that the line CD is tangent to the circle passing through the points B, O, and C.

* moved from below

* for layout

Remark. The parallelogram considered in the problem is such that the ratio of the lengths of the diagonal BD and the side CD is equal to $\sqrt{2}$. This property can be derived from the similarity of the triangles BCD and COD. Another consequence of this similarity is the fact that the midpoints of its sides are vertices of a parallelogram similar to the given one.

Problem 4. Set $n = 1000$. Consider two cases (see Fact 11):

(a) $k > n$. Then

$$\underbrace{1\ldots1\overbrace{2\ldots2}^{k}}_{2n}-\underbrace{2\ldots2}_{n+1} \;=\; \underbrace{1\ldots1}_{2n-k}\;\underbrace{2\ldots2}_{k-(n+1)}\;\underbrace{0\ldots0}_{n+1}.$$

This number ends in $n+1 = 1001$ zeros. But the number of zeros at the end of the decimal representation of a perfect square must always be even! Thus this number is not a perfect square.

(b) $k \le n$. Then page 237

$$\underbrace{1\ldots1\overbrace{2\ldots2}^{k}}_{2n}-\underbrace{2\ldots2}_{n+1}$$
$$= \underbrace{1\ldots1}_{2n-k}\underbrace{0\ldots0}_{k}-\underbrace{2\ldots2}_{n+1-k}\underbrace{0\ldots0}_{k} = 10^k(\underbrace{1\ldots1}_{2n-k}-\underbrace{2\ldots2}_{n+1-k}). \quad (1)$$

This number ends in k zeros. As we explained above, this number can be a perfect square only if k is even. Set $l = k/2$.

Clearly, the number in (1) is a perfect square if and only if

$$A = \underbrace{1\ldots1}_{2n-2l}-\underbrace{2\ldots2}_{n+1-2l}$$

is a perfect square.

Notice that

$$A = \frac{1}{9}\cdot\underbrace{9\ldots9}_{2n-2l}-\frac{2}{9}\cdot\underbrace{9\ldots9}_{n+1-2l} = \frac{1}{9}\big(10^{2n-2l}-1-2(10^{n+1-2l}-1)\big);$$

see Fact 11. Let $B = 9A$. Then A is a perfect square if and only if B is. Write B as

$$B = 10^{2n-2l} - 2\cdot10^{n+1-2l} + 1 = (10^{n-l})^2 - 2\cdot10^{n-l}\cdot10^{1-l} + 1.$$

For $l = 1$, the right-hand side of this equality coincides with the square of a difference:

$$B = (10^{n-1})^2 - 2\cdot10^{n-1} + 1 = (10^{n-1}-1)^2.$$

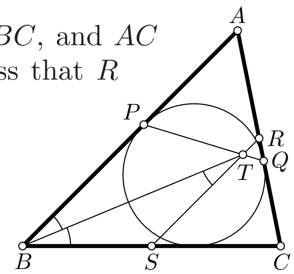
Now suppose that $l > 1$. We remark that if $X = Y^2$ is a perfect square (with $Y > 0$), the perfect square closest to X is $(Y-1)^2 = Y^2 - 2Y + 1$. That is, if a number Z satisfies

$$Y^2 - 2Y + 1 < Z < Y^2,$$

then Z is not a perfect square. We apply this to $Y = 10^{n-l}$ and $Z = B$. Clearly, $Z < Y^2$. In addition,

$$Z = (10^{n-l})^2 - 2 \cdot 10^{n-l} \cdot 10^{1-l} + 1 > (10^{n-l})^2 - 2 \cdot 10^{n-l} + 1.$$

By the remark, this number cannot be a perfect square; therefore, only $l = 1$ satisfies our requirement, implying that $k = 2$.



Problem 5. Denote the lengths of the sides AB, BC, and AC by c, a, and b, respectively. Assume for definiteness that R lies on AC and S on BC. Then (see the remark) page 238

$$RQ = |RC - QC| = \left|\frac{b}{2} - \frac{a+b-c}{2}\right| = \frac{c-a}{2}.$$

Since $\triangle TRQ$ is similar to the isosceles triangle PAQ, we have $RQ = RT$. Therefore,

$$ST = RS - RT = RS - RQ = \frac{c}{2} - \frac{c-a}{2} = \frac{a}{2} = BS.$$

It follows that TSB is an isosceles triangle, and $\angle SBT = \angle STB = \angle TBA$, which means that BT is the bisector of the angle ABC.

Remark. In any triangle, the distance from a vertex to the tangency point of the incircle with either side incident on that vertex is equal to $p - a$, where p is the semiperimeter and a is the length of the opposite side.

Proof. Let the incircle touch the sides AB, BC, and AC at C', A', and B', respectively. The lengths of the two tangents to a circle drawn from the same point are equal, so the triangle's perimeter can be written as

$$2AB' + 2BA' + 2A'C = 2AB' + 2BC.$$

Our statement follows readily. □

Problem 6. Each part of this problem is solved by induction (see Fact 24), but the complexity of the reasoning increases sharply. To streamline the exposition, we define an $(k, 2^n)$-*ranking* to be a set of rankings of 2^n players in k events: in other words, an array of k columns, each corresponding to one event and ranking the 2^n players according to how well they're expected to perform in it. *
*

The typical situation, then, is this: given an $(n, 2^n)$-ranking and some player, can the events be scheduled in such an order that the player wins the competition — assuming, of course, that the $(n, 2^n)$-ranking is accurate?

Part (a) uses induction on n. The base of induction, with $n = 1$, is obvious: the winner of the single event is one-half of the two players.

Suppose we have solved the problem for an n-event competition, so we have an $(n, 2^n)$-ranking C_n admitting 2^{n-1} potential winners. After adding a new event, we must describe an $(n{+}1, 2^{n+1})$-ranking C_{n+1} admitting 2^n potential winners. We do this as follows.

First divide the 2^{n+1} players into two equal groups A and A'. The defining conditions for C_{n+1} are: page 239

(1) In both groups, the ranking with respect to the old events is given by the $(n, 2^n)$-ranking C_n.
(2) Any member of A' ranks higher in the new event than any member of A, whereas in the old events any member of A ranks higher than any member of A'.
(3) The ranking of members of A with respect to the new event is the same as the ranking for some fixed old event, which we denote by α. (The ∗
ranking of members of A' in the new event is arbitrary.)

Now, if we start the competition with the new event, only members of A' will remain. By the induction hypothesis, half this group are potential winners.

If we start the competition with event α, only members of A will remain. We claim that half of them are potential winners as well. By the induction hypothesis, for half the members of A we can find a schedule of events ∗ multiple times
allowing them to become winners. But this includes the event α, which has already taken place! To fix this, we replace event α in the schedule by the new event and use the fact that the ranking within A is the same for the new event as for α.

Thus, half of the players in A, as well as half of those in A', are potential winners. This completes the induction step.

(b) In this case we fix n and an $(n, 2^n)$-ranking. To prove there are n players who are not potential winners, we will find inductively for each event a different player who gets beaten either before or in this event, irrespective of the schedule of events. ∗∗

For the induction we must imagine the n events ordered in a certain way, once and for all. We will call this their *alphabetical order*. This has nothing to do with the scheduling order.

Base of the induction. For the alphabetically first event α, we ∗
simply take the player rated lowest in α. This person cannot stay past this ∗
event, regardless of when it takes place.

Induction step. Suppose that we have selected a set $A_k = \{a_1, \dots, a_k\}$ of players such that a_i leaves after the (alphabetically) i-th event or earlier.

Of the remaining players, let a_{k+1} be the lowest-rated in the alphabetically $(k+1)$-st event. We prove that a_{k+1} leaves right after this event or earlier, irrespective of the schedule of events.

Suppose the alphabetically $(k+1)$-st event is the r-th in the schedule, and that w players from the set A_k have left after the first $r-1$ events. In the r-th event, 2^{n-r} players must leave. Therefore, a_{k+1} can only go on to page 240
the next event if $2^{n-r} \le k-w$ (since only $k-w$ of the remaining players ∗
can be weaker than a_{k+1} in $(k+1)$-th event). But after the alphabetically $(k+1)$-st event, at least $k-w$ of the k-th alphabetically first events are still to be held. Therefore, $k-w \le n-r < 2^{n-r}$, where we have used that $2^l > l$ ∗
for all l. This contradiction concludes the induction step.

(c) To perform the induction in this case we need an auxiliary construction, also inductive. It involves a ranking of 2^n players in $n+1$ events, that is, a $(n+1, 2^n)$-ranking. We borrow the notion of alphabetical order from (b). $\ast$

Lemma 1. *There exists an* $(n+1, 2^n)$*-ranking* E_n *such that every player but one is a potential winner in some* n*-event subcompetition that includes the alphabetically last event. The exception is the player ranked lowest in this event.*

Before proving this statement, we explain how it is used to solve part (c) of the problem. We use induction on n. The base of induction, for $n = 1$, is just as in part (a).

For the induction step, suppose that we have an $(n, 2^n)$-ranking D_n such that $2^n - n$ players are potential winners. We add a new (alphabetically last) event and describe an $(n+1, 2^{n+1})$-ranking that continues the induction.

The construction is similar to the one used in item (a). We again divide the 2^{n+1} players into two groups of 2^n players, A and A'. The defining $\ast$ conditions for D_{n+1} are:

(1) The members of A' are ranked with respect to the old events as in the $(n, 2^n)$-ranking D_n.
(2) Any member of A' ranks higher in the new event than any member of A, whereas in the old events any member of A ranks higher than any member of A'.
(3) The members of A are ranked according to E_n, the $(n+1, 2^n)$-ranking provided by the lemma.

If we start our $(n+1)$-athlon with the new event, only members of A' page 241 will remain. By the induction hypothesis, $2^n - n$ players from this group are potential winners.

Next we show that a member of A that is potential winners in the ranking E_n is also a potential winner in the ranking D_{n+1}. Given such a player, Lemma 1 says that there is a schedule of n events, including the new one, which makes this player win. Let α be the event omitted in this schedule. If we hold α first, it eliminates members of A', leaving those of A; now we play out the schedule of n events that ensures the win of this player in the ranking E_n.

Thus, there are $2^n - n$ potential winners in group A' and $2^n - 1$ potential winners in A. This gives $2^n - n + 2^n - 1 = 2^{n+1} - (n+1)$ potential winners in the ranking E_{n+1}, as needed.

The proof of the lemma also involves induction. In fact we need a slightly stronger statement (the change is highlighted): $\ast$

Lemma 2. *There exists an* $(n+1, 2^n)$*-ranking* E_n *such that every player but one is a potential winner in some* n*-event subcompetition that includes the alphabetically last event. The exception is the player ranked lowest in this event;* ***this player, called the sure loser, nonetheless can progress to the final.*** $\ast$

Base of the induction. We take for E_1 the $(2,2)$-ranking that says that the winner in the alphabetically first event, denoted by α, is the loser in the second. * *

Induction step. We construct an $(n+2, 2^{n+1})$-ranking E_{n+1}, assuming the existence of E_n, where we have removed the alphabetically last event. * Divide the 2^{n+1} players into two equal groups, B and B'. The defining conditions for E_{n+1} are:

(1) For $2 \le j \le n+2$, the members of B are ranked with respect to the alphabetically j-th event in the same order as they are with respect to the $(j-1)$-st event in E_n. The same is true for members of B'. * *

(2) Any member of B' ranks higher in event α than any member of B, whereas in the other events any member of B ranks higher than any member of B'. * several times

(3) In B, the sure loser according to E_n is ranked first in α.

If we start with α, we have 2^n-1 potential winners from B', and we can organize the competition so that the sure loser from B' reaches the final, by the induction hypothesis. page 242

If we do not hold event α at all, then all players from B' leave after the first event in the schedule, and then n events are held. According to the induction hypothesis, by choosing the first event and the sequence of subsequent events, we can guarantee that any of 2^n-1 members of B wins. *

We still have to explain how to arrange a win for the sure loser from B. To this end, we *start with the event that is the last in the schedule of events that lets the sure loser reach the final*. After this round only members of group B remain. Then we proceed in the sequence that leads the sure loser to the final, and we hold event α last. The sure loser then wins the competition. * *

Level C

Problem 1. *First solution.* If $a = 0$, then $b \ne 0$ (otherwise $c^2 < 0$). Then $b^2 > 0 = 4ac$ and we're done.

If $a \ne 0$, we consider the quadratic trinomial $f(x) = ax^2 + bx + c$. We have $f(1) = a + b + c$ and $f(0) = c$. Hence, by assumption,

$$f(1)f(0) = (a+b+c)c < 0.$$

It follows that one of the numbers $f(1)$ and $f(0)$ is negative and the other is positive. Therefore, the parabola $y = f(x)$ intersects the x-axis, which means that the discriminant of this quadratic polynomial is positive: $b^2 - 4ac > 0$.

Second solution. We can avoid the consideration of two cases. Consider the quadratic polynomial $g(x) = x^2 + bx + ac$. It follows from the assumption that $g(c) = c^2 + bc + ac = (a+b+c)c < 0$. Since the branches of the parabola $y = g(x)$ are directed upward and $g(c) < 0$, this parabola meets the * removed "The... c"

x-axis at two points, i.e., it has two distinct roots. Hence the discriminant of this quadratic polynomial is positive: $b^2 - 4ac > 0$.

Problem 2. The triangles APB and DPC are isosceles, since AP, BP, CP, and DP are radii of the third circle. Denote the angles at their bases by $\angle ABP = \angle BAP = \alpha$ and $\angle DCP = \angle CDP = \beta$. The quadrilaterals $ABQP$ and $DCQP$ are cyclic; hence the angles AQP and ABP both equal α, while $\angle DQP$ and $\angle DCP$ equal β (see figure).

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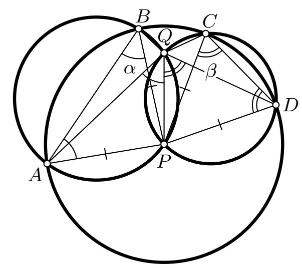

We have

$$\angle AQD = \angle AQP + \angle DQP = \alpha + \beta.$$

Further, $\angle BQP = \pi - \angle BAP = \pi - \alpha$, similarly, $\angle CQP = \pi - \beta$. Thus,

$$\angle BQC = 2\pi - \angle BQP - \angle CQP = \alpha + \beta.$$

Problem 3. We first prove that x and y are coprime. Suppose otherwise; that is, let x and y be divisible by some prime number p. Let $a \geq 1$ and $b \geq 1$ be the exponents of p in the prime factorizations of x and y, respectively. Without loss of generality we can assume that $a \geq b$. Then the exponent of the highest power of p that divides $x^3 + y$ is equal to b (since x^3 is divisible by p^{3a} and hence by p^{b+1}, whereas y is divisible by p^b and not by p^{b+1}). On the other hand, $x^2 + y^2$ is divisible by p^{2b}. It follows that $x^3 + y$ is not divisible by $x^2 + y^2$. This contradiction shows that x and y are coprime.

Further, $x(x^2 + y^2) - (x^3 + y) = y(xy - 1)$ must be divisible by $x^2 + y^2$. Notice that y and $x^2 + y^2$ cannot have a common factor greater than 1 (because x and y are coprime); therefore, $xy - 1$ is divisible by $x^2 + y^2$ (see Fact 9). But this is impossible whenever $xy - 1 > 0$, because $x^2 + y^2 \geq 2xy > xy - 1$.

Problem 4. The numbers will be called red and blue according to the colors of sectors in which they are written. All the numbers are paired up with their equals; we take a pair of equal numbers that are closest, meaning that the number of sectors in the smaller arc ω lying strictly between the two numbers is minimal.

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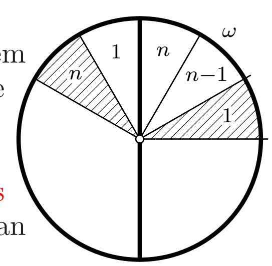
Only the cyclic order of the numbers from 1 to n matters in the problem; a cyclic shift of the numbering through the same amount of sectors for both red and blue numbers replaces the problem by an equivalent one. Therefore, we can assume that the chosen pair consists of 1s.

We can also assume that the arc ω between them runs counterclockwise from the red sector to the blue one; see the figure. Further, all the sectors in ω, if there are any, have the same color. Indeed, if both colors are represented, the red and the blue n sectors are both in ω; but then this pair is closer together than the pair of 1s, contrary to our assumption.



Suppose that all the numbers on this arc (if there are any) are blue; the case of red numbers is similar. Draw the diameter separating the blue 1 from its clockwise neighbor (this is either blue n or red 1); we will show that this is the diameter we need. Indeed, consider the half-disk containing the blue 1. The blue numbers in this half-disk read up, counterclockwise, from 1 through some positive integer l. Now imagine reading the red numbers, also counterclockwise. Since there are no red numbers in the arc ω, the first number in our half-disk is n. It follows that the red numbers we read are the numbers n, $n-1$, ..., $n-m$, where m is also a positive integer. *

Thus, there are l blue and $m+1$ red numbers in this half-disk. Since all in all there are n numbers in the half-disk, we have $l+(m+1)=n$; that is, $n-m=l+1$. Therefore, the numbers in the half-disk are the blue numbers from 1 to l and the red numbers from n to $l+1$. Together these amount to all the numbers from 1 to n, each taken once.

Problem 5. Let

$$f\colon [0,1] \to [0,1], \quad f(x) = x/\sqrt{3} \quad \text{and}$$
$$g\colon [0,1] \to [0,1], \quad g(x) = x/\sqrt{3} + (1 - 1/\sqrt{3})$$

be the functions describing the jumps of the grasshopper. The range of f is the interval $[0, 1/\sqrt{3}]$, while the range of g is $[1-1/\sqrt{3}, 1]$. Each of these intervals is of length $1/\sqrt{3}$ and together they cover all of $[0,1]$. page 245

Let n be a positive integer. Consider all possible compositions of functions of the form

$$h_1(h_2(\ldots(h_n(x))\ldots))\colon [0,1] \to [0,1],$$

where each function h_i is either f or g. It can be readily seen that the range of any of these functions is an interval of length $(1/\sqrt{3})^n$. We'll prove by induction on n (see Fact 24) that these intervals cover all of $[0,1]$. For $n=1$ this statement has already been verified. Suppose that the ranges of all possible functions $h_1(h_2(\ldots(h_{k-1}(x))\ldots))$ cover $[0,1]$. But the range of a particular $h_1(h_2(\ldots(h_{k-1}(x))\ldots))$ is covered by the ranges of the functions $h_1(h_2(\ldots(h_{k-1}(f(x)))\ldots))$ and $h_1(h_2(\ldots(h_{k-1}(g(x)))\ldots))$. This proves the statement.

Now suppose that a point a is chosen on the interval $[0,1]$. Consider the interval $(a-0,01,\ a+0,01)$. We show that the grasshopper can hit it. Choose n large enough to satisfy the inequality $(1/\sqrt{3})^n < 0,01$ (for instance, $n=10$). As we have proved, it is possible to find a function $h_1(h_2(\ldots(h_n(x))\ldots))$ whose range contains a. Then the entire range of this function — an interval of length $(1/\sqrt{3})^n$ — lies inside $(a-0,01,\ a+0,01)$. This means that starting anywhere in $[0,1]$ the grasshopper will hit the interval $(a-0,01,\ a+0,01)$ after making the jumps corresponding to the functions $h_n, h_{n-1}, \ldots, h_1$.

Remarks. (a) Compare with Problems 99114 and 99117. REF 99114
REF 99117

(b) This problem, as well as Problem 99114, are based on properties of contracting mappings. A differentiable mapping is said to be *contracting* if its derivative has absolute value everywhere less than some fixed number $\delta < 1$. REF 99114

Let $f\colon [0,1] \to [0,1]$ and $g\colon [0,1] \to [0,1]$ be two contracting mappings, and let x be a point on the interval $[0,1]$. It is natural to ask: *What can we say about the position of this point after it undergoes a large number of the consecutive mappings f and g?* More exactly, consider an infinite sequence of mappings f and g, and let x_n be the image of x after the first n mappings. page 246

Since the mappings are contracting, $\lim_{n\to\infty} x_n$ does not depend on x. However, it depends on the order in which the mappings f and g are applied.

If the ranges of the mappings f and g cover the entire interval $[0,1]$, then x_n can approach any point in the interval $[0,1]$ (in fact, this is just the statement of our problem). Otherwise, the set of possible limits is like the Cantor set. For instance, the functions $f(x) = \frac{1}{3}x$ and $g(x) = \frac{1}{3}(x+2)$ generate exactly the usual Cantor set.

Problem 6. Lemma. *Suppose that* 1999 *distinct positive numbers $a_1, a_2, \ldots, a_{1999}$, are arranged around a circle, and that $a_1 > a_{1998}$. For all $i = 2, 3, \ldots, 999$, we perform the following operation: the numbers a_i and a_{1999-i} are swapped if $a_i < a_{1999-i}$, and stay in place otherwise. If at least one pair is swapped, then the sum of all products of ten consecutive numbers increases.* *

Proof. Consider two groups of 10 numbers in a row, arranged symmetrically: $a_i, \ldots, a_{i+9}$ and $a_{1999-i}, \ldots, a_{1990-i}$. Take the sums of the products of the numbers in either group. We first show that this sum can never decrease.

Consider the product z of the numbers appearing in both groups. (If there are no such numbers, we set $z = 1$.) Also let x and x' be the products of the numbers that belong only to the first or only to the second group, respectively, and that don't get swapped; while y and y' are the products of the numbers that belong only to the first or only to the second group and do get swapped. (Again, if no numbers satisfy a specified condition, the corresponding product is defined as 1.)

The sum of products of numbers in the two groups before the operation is $s_1 = zxy + zx'y'$; after the operation, it is $s_2 = zxy' + zx'y$. We have $s_1 - s_2 = z(x - x')(y - y')$. It is easily seen that $x' \le x$ and $y' \ge y$. Hence $s_1 - s_2 \le 0$. page 247

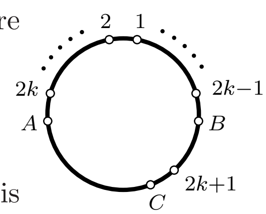
Now we consider all the numbers again. We show that if some of the numbers are swapped when the operation of the lemma is performed, the difference $s_1 - s_2$ is strictly negative for at least one of the symmetric pairs of groups of 10 numbers. This will imply the lemma.

Clearly, if at least one pair of numbers (in a given group) was swapped, then $y' > y$. And if at least one pair of numbers stayed in place, then $x' < x$, since all the numbers are different. So it suffices to show that we can find two symmetric groups of ten numbers each for which at least one pair was swapped and at least one pair was not. But this is obvious, because by assumption some pair was swapped and the pair (a_1, a_{1998}) was not. $\square$ *

To apply the lemma to the problem, we assume that the numbers from 1 through 1999 are placed at the vertices of an 1999-gon, and that the desired condition is satisfied: the sum of all products of numbers taken 10 in a row is maximal.

Draw the diameter through some number k. We claim that, for all pairs symmetric about this diameter, the smaller number lies in one semicircle and the larger one in the other. Indeed, denote by $a_1, \dots, a_{1999}$ the numbers around the circle, starting from the greater of the neighbors of k and ending with k. Then $a_1 > a_{1998}$, and we can apply the lemma: since the arrangement is optimal, we cannot increase our sum of products by swapping symmetric numbers, hence all numbers on one side of the diameter are greater than the corresponding numbers on the other side.

Ignoring rotations and reflections, there is only one arrangement of numbers satisfying this property for all diameters. We show this by induction * (see Fact 24). First, 2 must be next to 1; otherwise we can find a diameter separating 2 from 1 and such that the two numbers are not symmetric about this diameter. Denote by A and B the numbers symmetric to 1 and 2 about this diameter. Then $A > 1$ and $2 < B$, contradicting the claim in * the previous paragraph.



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Suppose that we have proved that $1, 2, \dots, 2k$, where $1 \le k \le 998$, must be arranged as in the answer, that is, in the order $2k$, $2k-2$, $\dots$, 2, 1, 3, $\dots$, $2k-1$ (say clockwise, for definiteness). Denote by A and B the numbers following $2k$ counterclockwise and $2k-1$ clockwise; see figure. Suppose that the number $2k+1$ is distinct from A and B. Then let C be the clockwise neighbor of $2k+1$. The number C is distinct from $1, 2, \dots, 2k$. The numbers C and $2k-1$, as well as $2k+1$ and B are symmetric about the diameter, but $C > 2k-1$ and $2k+1 < B$. This is a contradiction, which means that either $A = 2k+1$ or $B = 2k+1$. But the assumption $A = 2k+1$ immediately leads to a contradiction: it suffices to consider the diameter which is the symmetry axis of the numbers $2k$ and $2k-1$. Hence $B = 2k+1$.

In a similar way we show that $A = 2k+2$, completing the induction step.

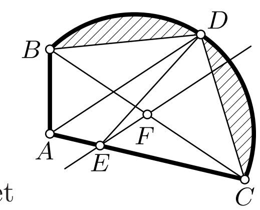
Level D

Problem 1. By the triangle inequality, $a > |b-c|$. Squaring both sides we * get $a^2 > (b-c)^2$. Hence

$$a^2 + 2bc > b^2 + c^2.$$

The right-hand side here is positive, so we can divide both sides by it. After the division, we see that the first term in the inequality we are proving is greater than 1. The same is true for the other two terms. Therefore, their sum is greater than 3.

Problem 2. (a) It is not difficult to construct the midpoint of the broken line BAC, that is, the point that divides the broken line into two broken lines of equal length. Clearly, the line joining this point to the midpoint of the arc is the desired one.

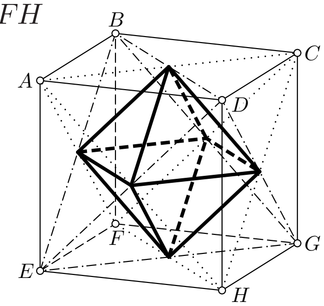


(b) Let D be the midpoint of the given arc. The shaded caps shown in the figure are clearly congruent. Therefore, it suffices to draw the line through D bisecting the area of the quadrilateral $ABDC$.

Let F be the midpoint of the diagonal BC, and let l be the line drawn through F parallel to the diagonal AD. For definiteness, assume that the line l intersects the segment AC (the case in which l intersects the segment AB is considered similarly). If E is the intersection point, then DE is the line to be constructed. page 249

Indeed, the sum of the areas of triangles ABF and BDF is half the area of $ABDC$. But this sum is also equal to the area of quadrilateral $ABDE$, because the triangles AED and AFD have a common side AD and equal altitudes dropped on this side, and so have equal area.

Problem 3. *First solution.* The planes containing four faces of the same color bound congruent regular tetrahedra. To grasp why, imagine a cube $ABCDEFGH$ and the two regular tetrahedra $ACFH$ and $BDEG$. The intersection of these tetrahedra is a regular octahedron. Indeed, the vertices of this intersection are the centers of the cube's faces, and the centers of a cube's faces are vertices of a regular octahedron. (See the figure.)



The black faces of the given octahedron lie on one tetrahedron, and the white faces on the other.

The result in the problem follows from the fact that the sum of distances from a point in a regular tetrahedron to its faces is constant and equals three times the volume of the tetrahedron divided by its face area. To prove this last statement, let A, B, C, and D be the vertices of the tetrahedron. Denote by h_A, h_B, h_C, and h_D the distances from a point O inside the tetrahedron to the planes BCD, ACD, ABD, and ABC, respectively, and by S the area of any face of the tetrahedron. Then the volumes of the tetrahedra $BCDO$, $ACDO$, $ABDO$, and $ABCO$ are equal to $\frac{1}{3}Sh_A$, $\frac{1}{3}Sh_B$, $\frac{1}{3}Sh_C$, and $\frac{1}{3}Sh_D$, respectively. Therefore, the volume of the tetrahedron $ABCD$ is equal to

* removed “altitude”

$$\frac{S}{3}(h_A + h_B + h_C + h_D),$$

which is equivalent to our statement.

Second solution. (Above grade level.) Consider the directed distances from a point to the face planes. More exactly, if the point and the octahedron are on the same side of the plane, then the distance is taken with the plus sign page 250

(so the directed distance for these points is just the ordinary one); otherwise, the sign is reversed.

For brevity, the distance to the plane containing a face will be called the distance to that face. We prove a claim stronger than that of the problem:

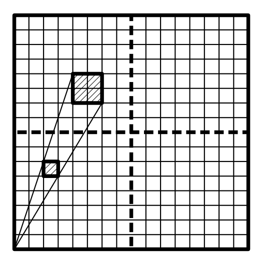
The sum of the directed distances from any point to the black faces is equal to the sum of the directed distances to the white faces, even for points lying outside the octahedron.

The directed distance to a plane is a linear functional in space (see Fact 25 on page 208). Hence the sum of distances to the faces of a certain color is also a linear functional. Denote the sum of distances from a point X to white faces by $l_w(X)$ and a similar sum for the black faces by $l_b(X)$. We want to prove that $l_b - l_w = 0$. If this were not so, the set of zeros of the linear functional $l_b - l_w$ is a plane. But it is readily seen that it vanishes at all the vertices of the octahedron. Hence $l_b - l_w$ is identically zero.

Problem 4. Suppose the square has side length 1. Divide each side into 2^n equal segments and draw the lines parallel to the sides through all partition points. They cut the square into small squares with side length 2^{-n}. If n is sufficiently large, then one of these small squares will be completely inside the hole (for instance, for n such that 2^{-n} is less than half the radius of the hole, we can take the small square containing the center of the hole).

Therefore, it will suffice to prove that for any n the grasshopper can hit any of the 2^{2n} small squares. page 251

We will prove this by induction (see Fact 24). For $n = 0$ the claim is trivial. Let us describe the induction step from n to $n + 1$. Consider a $2^{-n-1} \times 2^{-n-1}$ square Q.



Cut the initial square into four squares with side length $\frac{1}{2}$. Without loss of generality we can assume that Q lies in the bottom left square, whose outer corner we denote by A (see figure). The dilation with center A and ratio 2 maps the chosen small square Q onto a square Q' with side length 2^{-n}. Clearly, this is one of the squares obtained by cutting the initial square into 2^{2n} squares with side length 2^{-n}.

By the induction hypothesis, the grasshopper can get into Q'. Now, if it jumps half the distance to the vertex A, it will hit the target square Q.

Remark. Compare with Problems 99105 and 99117. REF 99105 REF 99117

Problem 5. We number the colors of the nodes from 1 to k. Then we take the nodes of color 2 not adjacent with nodes of color 1 and reassign them color 1. The new coloring is regular as well; therefore, it involves k colors. This means that some of the nodes of color 2 were not recolored; hence they are adjacent with nodes of color 1.

Next we take the nodes of color 3 not adjacent with nodes of color 2 (that were not recolored at the first step), and we reassign them color 2. We continue in this fashion with all the colors; at the last step we reassign color $k-1$ to certain nodes of color k.

After that, consider any node of color k. It has not been recolored; therefore, it is adjacent with a node of color $k-1$. This node has not been recolored either: otherwise, its initial color would be k and it would be adjacent to a node of the same color, which is impossible by the assumption of regularity. Since the last node preserved its initial color after the recoloring, it is adjacent with a node of color $k-2$, and so on. This process eventually yields a path of length k visiting the nodes of all the k colors that were not changed.

removed "See also"

Remark. We have, in fact, proved a stronger statement: there exists a path that visits nodes of all k colors, each color once, *in a given order.*

Problem 6. *First solution.* Let p be a prime factor of l. Since $n^m = (1+n^k)^l - 1$, the number n^m is divisible by $(1+n^k)^p - 1$ (see Fact 8). But, page 252 by the binomial theorem (see Remark below), we have

$$(1+n^k)^p - 1 = n^k\, p + n^{2k}\, \frac{p(p-1)}{2} + n^{3k}\, r,$$

where r is a nonnegative integer. Dividing both sides by n^k, we see that n^m is divisible by

$$p + n^k\, \frac{p(p-1)}{2} + n^{2k}\, r.$$

If n is divisible by p, then this expression is coprime with n, and n^m cannot be divisible by it. Hence p is a divisor of n (see Fact 9). Then *

$$1 + n^k\, \frac{p-1}{2} + \frac{n^{2k}}{p}\, r$$

is a positive integer greater than 1. If $k > 1$ or p is odd, then the second term is divisible by n (the third term is always divisible by n); therefore, the sum is coprime with n, and so is not a divisor of n^m. This contradiction shows that $k = 1$ and $p = 2$. Therefore, 2 is the only prime factor of the number l and l can be written as 2^s. *

Using the binomial theorem again, we have

$$n^m = (1+n^k)^l - 1 = (1+n)^l - 1 = ln + \frac{l(l-1)}{2}n^2 + \cdots + n^l.$$

After the first, all terms on the right-hand side are divisible by n^2. Since $m > 1$, it follows that l is divisible by n. Therefore, n, as well as l, is a power of two. *

Since l is even, $(1+n)^l - 1$ is divisible by $(1+n)^2 - 1 = n(n+2)$. Since n^m is a power of two, the number $n+2$ is also a power of two. Since n and $n+2$ are powers of 2, we see that $n = 2$. *

If $l \geq 4$, then $(1+n)^l - 1$ is divisible by $(1+n)^4 - 1 = 80$, and cannot be a power of two. Therefore, $l = 2$, whence $m = 3$.

Outline of second solution. Since $1+n^m$ is divisible by $1+n^k$, m is divisible by k for $n \neq 1$ (see Fact 8). Therefore, replacing n^k by n and m/k by m, * we reduce the problem to the case $k = 1$. Suppose that n is divisible by p^t, but not by p^{t+1}, where p is a prime number ($t > 0$). Let p^s be the greatest power of p that divides l.

Now use the binomial theorem (see Remark below) to write page 253

$$ln + C_l^2 n^2 + \cdots + n^l = n^m.$$

Assume that $p \neq 2$ or $t > 1$. One can show that the right-hand side and all summands except the first are divisible by p^{t+s+1}. The contradiction obtained shows that $n = 2$. * and removed earlier paragraph

Remark. The formulas for $(a+b)^2$ and $(a+b)^3$ are well known. A similar formula for $(a+b)^n$, for any n, is the subject of the *binomial theorem*. To write this formula, we'll need the notion of combinations. The number of *combinations of n elements taken k at a time* is the number of ways of picking k things out of a set of n different things, without regard to order. This number is denoted by ${}_nC_k$ or $\binom{n}{k}$, and is sometimes read "n choose k". We stress that the order in which the k elements are picked does not matter; in other words, ${}_nC_k$ is the number of k-element subsets in an n-element set.

This number is given by

$${}_nC_k = \frac{n!}{k!\,(n-k)!} = \frac{n\,(n-1)(n-2)\ldots(n-k+1)}{k\,(k-1)\cdots 2\cdot 1}.$$

The *binomial theorem* (or *binomial formula*) now says that

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} {}_nC_k a^k b^{n-k}\\ &= a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \cdots + {}_nC_{n-2}a^2b^{n-2} + {}_nC_{n-1}ab^{n-1} + b^n.\end{aligned}$$

For this reason the ${}_nC_k$ are also called *binomial coefficients*.

Problem 7. If we think of a circle of length 1 as the interval $[0, 1]$ with endpoints identified (compare solution to Problem 56.10.4), then the fractional part f_m of the number $\log_{10}(2^m) = m\log_{10} 2$ can be viewed as a point on this circle. Consider the points REF 56.10.4

$$0, \log_{10} 2, \ldots, \log_{10} 9$$

on the circle and the nine half-open intervals into which they divide the circle. Denote these intervals by $I_1 = [0, \log_{10} 2)$, ..., $I_9 = [\log_{10} 9, 0)$.

The first digit of the number 2^m is equal to s if and only if f_m belongs to the interval I_s. For instance, if 2^m begins with 7, then

$$7 \cdot 10^l \leq 2^m < 8 \cdot 10^l$$

for a certain positive integer l. The fractional part of $m\log_{10} 2$ is equal to $m\log_{10} 2 - l$ and lies between $\log_{10} 7$ and $\log_{10} 8$. page 254

Suppose that the first digits of 2^{2^n} repeat with period k, after some preperiod of length n_0. Then the fractional parts of $2^n\log_{10} 2$ and $2^{n+k}\log_{10} 2$ hit the same interval I_s for any $n > n_0$.

It can be readily seen that the longest of the intervals is the first one, and its length is $\log_{10} 2 < \frac{1}{3}$.

Step 1. Mark the fractional parts of two positive numbers A and B on the circle. Suppose these fractional parts are distinct and are not antipodal points of the circle. Let x be the shorter of the two arcs into which these points divide the circle. Then the length of one of the arcs joining the fractional parts of the numbers $2A$ and $2B$ is equal to $2x$. (Explain why.)

Step 2. Now suppose that the fractional parts of the numbers A and B lie in the same interval I_s, and consider the pairs $2A$ and $2B$, $4A$ and $4B$, ∗ etc. It follows from Step 1 that the length of the shorter arc joining the fractional parts of a pair is doubled until it becomes greater than or equal to $\frac{1}{2}$. Therefore, at a certain step, one of the arcs joining the fractional parts of a pair will become greater than $\frac{1}{3}$, but less than $\frac{2}{3}$. Then these fractional parts belong to different interval on the circle.

Step 3. Consider the numbers $A = 2^{n_0} \log_{10} 2$ and $B = 2^{n_0+k} \log_{10} 2$. These numbers considered as points on the circle are different and are not antipodal, since $\log_{10} 2$ is irrational (see below). Therefore, we can apply Step 1 to these numbers, which yields a contradiction with the periodicity assumption.

It remains to prove that $\log_{10} 2$ is an irrational number. If $\log_{10} 2 = p/q$, then $2^p = 10^q$; this is obviously impossible, by prime factorization.

Remarks. (a) We have in fact proved that if α is not a rational power of ten, the sequence of the first digits of α^{2^n} is nonperiodic.

On the other hand, if we look instead at the sequence α^{10^n}, it is possible page 255 to find α that is not a rational power of ten and the sequence of first digits is periodic. For instance, let

$$\log_{10} \alpha = 0.101001000100001\ldots$$

(the number of zeros between consecutive ones increases). This decimal is not periodic; hence α is not a rational power of ten (see Fact 13). But for any n we ∗ have $\{10^n \log_{10} \alpha\} < 0.11 < \log_{10} 2$. Therefore, for all n, the first digit of the number α^{10^n} is 1! ∗

(b) Problems 99105 and 99114 are about contracting mappings of an interval into REF 99105 itself (see the remarks to those probelms). The problem we have just considered REF 99114 is a problem about *expanding* mappings.