Jacobi Conformal Projection of the Triaxial Ellipsoid: New Projection for Mapping of Small Celestial Bodies

Maxim V. Nyrtsov, Maria E. Fleis, Michael M. Borisov and Philip J. Stooke

Abstract In this paper a new technique for recalculating geographic coordinates of a triaxial ellipsoid to elliptical and then to rectangular coordinates of the Jacobi conformal projection is considered. Coordinate lines of the elliptical system and the cartographical grid with the parallels passing through the circular points on the Jacobi projection are shown. This new technique allows us to achieve the conformal mapping of small celestial bodies. A map of asteroid 25143 Itokawa in the Jacobi conformal projection, the first ever published, and a map of asteroid 433 Eros created by the authors in the transverse conformal cylindrical projection of a triaxial ellipsoid are presented for comparison. Asteroids 25143 Itokawa and 433 Eros are near-Earth objects.

Keywords Cartographical projection · Triaxial ellipsoid · Coordinate system

1 Introduction

Carl Jacobi in his lectures (1842–1843) at the University of Königsberg proposed the conformal projection of the triaxial ellipsoid to the plane. The lectures were written down by C.W. Borchardt and published by A. Clebsch in 1866 (Jacobi 1866). They also were translated into many languages particularly in English (Jacobi's lectures 2009). We used a Russian translation made by O.A. Polosukhina

M. V. Nyrtsov (✉)
Moscow State University of Geodesy and Cartography (MIIGAiK), Moscow, Russia
e-mail: nyrtsovmaxim@gmail.com

M. E. Fleis · M. M. Borisov
Institute of Geography, Russian Academy of Sciences, Moscow, Russia

P. J. Stooke
The University of Western Ontario, London, ON, Canada

M. Buchroithner et al. (eds.), *Cartography from Pole to Pole*, Lecture Notes in Geoinformation and Cartography, DOI: 10.1007/978-3-642-32618-9_17,

in 1936 and edited by N.S. Koshlyakov, a specialist in the field of partial differential equations (Jakobi 1936). The interpreters tried to translate the text not only as accurately as possible, but also to keep as much as possible the original features of the language and the character of the descriptions.

This is important because Jacobi's lectures are interesting not only for their results, but also for their depiction of the author's thoughts. For example, we are interested in a map projection obtained as a result of applying the elliptic coordinates to the derivation of the shortest line equation on the triaxial ellipsoid. Many of the materials presented in the lectures were used by mathematicians in development of the theory of surfaces. In the 1970s and 1980s interest in map projections for triaxial ellipsoids began to grow among cartographers in connection with the problem of mapping of small celestial bodies.

The Jacobi conformal projection is considered in Bugaevskiy (1999), but its formulae were found to be inconvenient for practical use because of difficulties in calculating of the integrals. In this paper we consider the technique of recalculating geographic coordinates of a triaxial ellipsoid to elliptical and then to rectangular coordinates of the Jacobi projection and mapping in this projection.

2 Derivation of Jacobi Projection

In his 26th lecture Jacobi introduced the elliptic coordinates λ_1, λ_2, $\lambda_3, \ldots \lambda_n$ for the multidimensional case using the equation

$$\frac{x_1^2}{a_1 + \lambda} + \frac{x_2^2}{a_2 + \lambda} + \cdots + \frac{x_n^2}{a_n + \lambda} = 1. \tag{1}$$

Later they became known as Lamé coordinates. He examines them analytically. For our purpose the important point of this lecture is the relationship between x and y values and the relationship between the squares of their differentials. In the 27th lecture the geometric interpretation of the results of the previous lecture, applied to the plane and three-dimensional space is given. For the three-dimensional case different ranges of λ changes correspond to three systems of confocal quadric surfaces (hyperboloid of one sheet, hyperboloid of two sheets, triaxial ellipsoid). In the three-dimensional space one hyperboloid of one sheet, one hyperboloid of two sheets and one triaxial ellipsoid are going through each point. These surfaces intersect each other at a right angle. According to this, the square of the element of arc of arbitrary curve, expressed in terms of the differentials of elliptic coordinates, doesn't contain the multiplication of differentials of two different values. Geometric interpretation of elliptic coordinates is considered in more detail in Kagan (1947).

The 28th lecture considers an orthogonal system of elliptic coordinates on the surface of the triaxial ellipsoid. Equation

$$\frac{x_1^2}{a_1+\lambda}+\frac{x_2^2}{a_2+\lambda}+\frac{x_3^2}{a_3+\lambda}=1, \tag{2}$$

where $a_1 < a_2 < a_3$ for given values of rectangular coordinates has three real roots: $\lambda_1 > \lambda_2 > \lambda_3$. Root λ_1 corresponds to the ellipsoid, root λ_2 corresponds to a hyperboloid of one sheet, root λ_3 corresponds to a hyperboloid of two sheets.

For the surface of a given ellipsoid the value of λ_1 is constant. After obtaining the formula for the square of the element of arc of an arbitrary curve on the ellipsoid

$$ds^2=\frac{1}{4}\frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}d\lambda_2^2+\frac{1}{4}\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}d\lambda_3^2.$$

Jacobi determines the ratio between the element of arc on the plane (in projection) $d\sigma$ and on the ellipsoid $d\sigma=\frac{2}{\sqrt{\lambda_2-\lambda_3}}ds$. Also Jacobi determines formulae of the conformal projection

$$\begin{aligned} u &= \int d\lambda_2\sqrt{\frac{\lambda_2-\lambda_1}{(a_1+\lambda_2)(a_2+\lambda_2)(a_3+\lambda_2)}} \\ v &= \int d\lambda_3\sqrt{\frac{\lambda_1-\lambda_3}{(a_1+\lambda_3)(a_2+\lambda_3)(a_3+\lambda_3)}}. \end{aligned} \tag{3}$$

Here u and v are rectangular coordinates on the plane.

Note. Values $\lambda_1,\ \lambda_2,\ \lambda_3,\ldots\lambda_n$ are measured in units of x^2, value ds is measured in units of x (units of the triaxial ellipsoid axes), $\frac{2}{\sqrt{\lambda_2-\lambda_3}}$ is measured in units of $\frac{1}{x}$, values $d\sigma,\ u,\ v$ are dimensionless.

3 Coordinate Systems and Coordinate Lines

Later rectangular coordinates on the plane of projection we will sign as x_{proj} and y_{proj}. We have also changed the signing of elliptic coordinates, because the value λ is usually used for longitude. Because a traditionally cartographic grid is used for mapping, we take angular planetocentric coordinates (ϕ—latitude, λ—longitude) as the initial (see Fig. 1).

In a new (more usual) values Eq. (2) becomes

$$\frac{x^2}{a^2-u}+\frac{y^2}{b^2-u}+\frac{z^2}{c^2-u}=1, \tag{4}$$

where

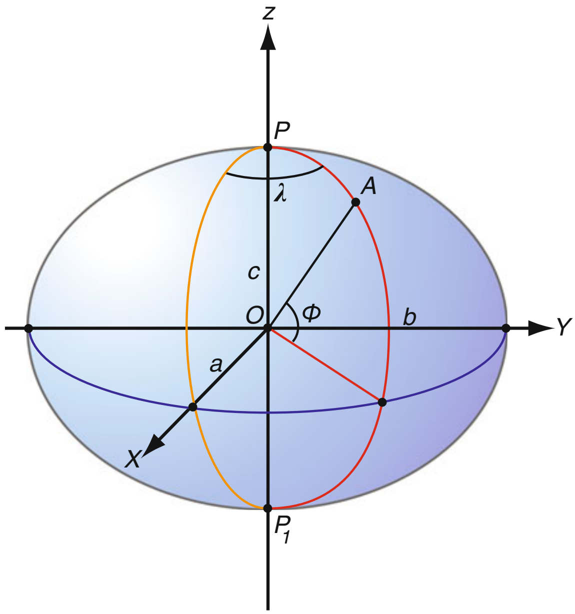

Fig. 1 Traditional coordinate systems of the triaxial ellipsoid

$c^2 = a_1 + \lambda_1, \quad b^2 = a_2 + \lambda_1, \quad a^2 = a_3 + \lambda_1$—squares of semi-axes of the triaxial ellipsoid, $x = x_3, \quad y = x_2, \quad z = x_1$—three-dimensional rectangular coordinates, $u = \lambda_1 - \lambda$.

For given three-dimensional rectangular coordinates elliptic coordinates are getting from Eq. (4) relatively to u

$$x^2(b^2-u)(c^2-u) + y^2(a^2-u)(c^2-u) + z^2(a^2-u)(b^2-u) \\ = (a^2-u)(b^2-u)(c^2-u)$$

$$u^3 + u^2(x^2+y^2+z^2-a^2-b^2-c^2) \\ + u(-x^2b^2 - x^2c^2 - y^2a^2 - y^2c^2 - z^2a^2 - z^2b^2 + a^2b^2 + b^2c^2 + a^2c^2) \\ + x^2b^2c^2 + y^2a^2c^2 + z^2a^2b^2 - a^2b^2c^2 = 0.$$

On the surface of the triaxial ellipsoid
$x^2b^2c^2 + y^2a^2c^2 + z^2a^2b^2 - a^2b^2c^2 = 0$, that is one of the roots $u_3 = 0$.
Solve a quadratic equation with non-zero u

$$u^2 + u(x^2+y^2+z^2-a^2-b^2-c^2) \\ - x^2b^2 - x^2c^2 - y^2a^2 - y^2c^2 - z^2a^2 - z^2b^2 + a^2b^2 + b^2c^2 + a^2c^2 = 0$$

and get two roots

$u_1 = \frac{-p+\sqrt{p^2-4q}}{2}$ and $u_2 = \frac{-p-\sqrt{p^2-4q}}{2}$, where

$$p = x^2 + y^2 + z^2 - a^2 - b^2 - c^2$$
$$q = -x^2b^2 - x^2c^2 - y^2a^2 - y^2c^2 - z^2a^2 - z^2b^2 + a^2b^2 + b^2c^2 + a^2c^2.$$

Sign $u_1 = u,\ u_2 = v$. Integrals (3) become

$$\begin{aligned} x_{proj} &= \int\limits_{b^2}^{u_i} \sqrt{\frac{u}{(c^2-u)(b^2-u)(a^2-u)}}\,du \\ y_{proj} &= \int\limits_{c^2}^{v_i} \sqrt{\frac{-v}{(c^2-v)(b^2-v)(a^2-v)}}\,dv. \end{aligned} \tag{5}$$

The Integral to calculate the horizontal and vertical coordinates in the Jacobi projection is chosen by the fact that the projection axis X_{proj} is directed horizontally to the right, and the Y_{proj} axis vertically upwards.

Intersections of the ellipsoid surface with hyperboloids of one sheet establish on the ellipsoid a system of curves and for each curve a value of u is constant, but v varies from c^2 to b^2. Intersection of the ellipsoid surface with hyperboloids of two sheets derives a system of curves for which v is a constant value, but u varies from b^2 to a^2. Together, these systems define coordinate lines that are orthogonal to each other, i.e., an orthogonal system of curvilinear coordinates. Figure 2 shows the projection (geometric) of the coordinate lines of the elliptic systems to the plane XZ.

Figure 3 shows these coordinate lines on a Jacobi projection to the plane $X_{proj}Y_{proj}$ with a background of a cartographical grid. The spacing of the elliptical coordinate grid is constant on u and v, but the distances between the lines in the projections are not identical. We see this type of grid because of x_{proj} depends of u only and y_{proj} depends of v only but the relation is not linear.

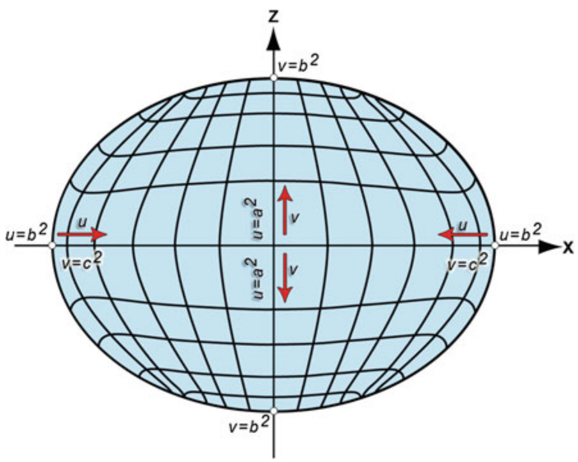

Fig. 2 The projection of the coordinate lines of the elliptic systems to the plane XZ

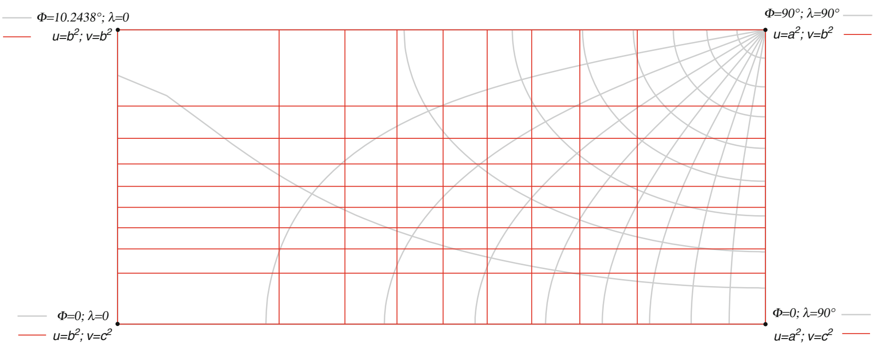

Fig. 3 Coordinate lines of elliptical system on Jacobi projection

Because the squares of three-dimensional rectangular coordinate values are used in the derivation of the formulae we obtain the values of the elliptic coordinates and rectangular coordinates in Jacobi projection respectively for one-eighth of the ellipsoid surface. For the other parts of the surface we flip coordinates in accordance with the sign of the latitude and longitude range.

4 Calculation of the Upper Limits of Integration

The upper limits of integration $u_i,\ v_i$ in (5), we get for each point with the given latitude and longitude step by step.

1. Calculate three-dimensional rectangular coordinates as the function of latitude and longitude (geographical coordinates) See Fig. 1.

$$x = r\cos\phi\cos\lambda,\ y = r\cos\phi\sin\lambda,\ z = r\sin\phi. \tag{6}$$

Here:

$r = \frac{a}{\sqrt{t}},\ t = \cos^2\phi\cos^2\lambda + \frac{\cos^2\phi\sin^2\lambda}{1-e_1^2} + \frac{\sin^2\phi}{1-e^2},\ e_1$—equatorial ellipse eccentricity, e—eccentricity of prime meridian ellipse.

2. Calculate elliptic coordinates as the function of three-dimensional rectangular coordinates, solving the Eq. (4).
3. Integrating (5), we obtain the rectangular coordinates in the projection.
   For the integration we perform a calculation based on Gaussian quadrature rule (GIS Research Centre Site 2012). This quadrature rule is a special case of the integral of the Lagrange interpolation polynomial with a special selection of nodes and weights. Direct calculation of integrals (5) is possible, but does not provide the required accuracy, therefore we express these integrals in terms of

elliptic integrals of the first and third kind, as it is done in (Prudnikov et al. 1986) Sect.1.2.35—formula 8 and Sect.1.2.36—formula 8.

$$x_{proj} = \int_{b^2}^{u_i} \sqrt{\frac{u}{(c^2-u)(b^2-u)(a^2-u)}}du = \frac{2}{\sqrt{(a^2-c^2)b^2}}\left[(b^2-c^2)I_3(\varphi_i,k_2,k_1)+c^2I_1(\varphi_i,k_1)\right] \tag{7}$$

$$\begin{aligned} y_{proj} &= \int_{c^2}^{v_i} \sqrt{\frac{-v}{(c^2-v)(b^2-v)(a^2-v)}}dv = \\ &\frac{2}{\sqrt{(a^2-c^2)b^2}}\left[(b^2-a^2)I_3\left(\frac{\pi}{2},k_2,k_1\right)+a^2I_1\left(\frac{\pi}{2},k_1\right)\right] \\ &-\frac{2}{\sqrt{(a^2-c^2)b^2}}\left[(b^2-a^2)I_3(\varphi_i,k_2,k_1)+a^2I_1(\varphi_i,k)\right]. \end{aligned} \tag{8}$$

The calculation of the vertical coordinate (formula (8) of our paper) as the difference of the integrals due to the fact that in the formula 8 of Sect. 1.2.36 (Prudnikov et al. 1986) upper limit of integration is equal to the maximum value, and the lower one is the current value of the elliptic coordinate.

3.1. Express the integrals (5) to the elliptic integrals of the first and the third kind. Here φ is the variable of integration and k_1, k_2 are constant parameters for the calculation of integrals. φ_i is the upper limit of integration for elliptic integrals.

In integration of (7) $\varphi_i = \arcsin\sqrt{\frac{(a^2-c^2)(u-b^2)}{(a^2-b^2)(u-c^2)}}$, $k_1 = \sqrt{\frac{(a^2-b^2)c^2}{(a^2-c^2)b^2}}$, $k_2 = \frac{a^2-b^2}{a^2-c^2}$, and in integration of (8) $\varphi_i = \arcsin\sqrt{\frac{(a^2-c^2)(b^2-v)}{(b^2-c^2)(a^2-v)}}$, $k_1 = \sqrt{\frac{(b^2-c^2)a^2}{(a^2-c^2)b^2}}$, $k_2 = \frac{b^2-c^2}{a^2-c^2}$.

Elliptic integral of the fist kind:

$$I_1(\varphi_i,k_1) = \int_0^{\varphi_i} \frac{d\varphi}{\sqrt{1-k_1^2\sin^2\varphi}}. \tag{9}$$

Elliptic integral of the third kind:

$$I_3(\varphi_i,k_2,k_1) = \int_0^{\varphi_i} \frac{d\varphi}{\left(1-k_2\sin^2\varphi\right)\sqrt{1-k_1^2\sin^2\varphi}}. \tag{10}$$

3.2. After substituting (9) and (10) to (7) and (8) we integrate given expressions and get coordinates of the Jacobi conformal projection in a range of latitudes from the equator to the North Pole and longitude from the prime meridian to the 90° meridian.
3.3. Flip coordinates in accordance with the actual position of point on the triaxial ellipsoid.
3.4. Multiply coordinates by the size of the major semi-axis in units of the map. In that case, when, $u = b^2$ and $v = b^2$ we get the so-called circular points (Kagan 1947, p. 117) of the triaxial ellipsoid. Taking into account that longitude is equal to zero, we obtain

$$t = \cos^2\phi + \frac{\sin^2\phi}{1-e^2} = \frac{\cos^2\phi - e^2\cos^2\phi + \sin^2\phi}{1-e^2} = \frac{1-e^2\cos^2\phi}{1-e^2},$$

$$x^2 = r^2\cos^2\phi = \frac{a^2\cos^2\phi\,(1-e^2)}{1-e^2\cos^2\phi},\ \textit{i.e.}\ \frac{a^2\cos^2\phi\,(1-e^2)}{1-e^2\cos^2\phi} = \frac{a^2(a^2-b^2)}{a^2-c^2}$$

$$\frac{\cos^2\phi\,(1-e^2)}{1-e^2\cos^2\phi} = \frac{e_1^2}{e^2}$$

$$e^2\cos^2\phi\left(1-e^2\right) = e_1^2 - e^2e_1^2\cos^2\phi$$

$$\cos^2\phi = \frac{e_1^2}{e^2\left(1-e^2+e_1^2\right)}.$$

5 Mapping

The projection exists at all points of the ellipsoid. Coordinate values of the projection are nowhere equal to infinity. However, conformality doesn't exist in circular points, and a gap in the projection is required for the prime meridian and its opposite or between the equator and the circular point, or between the circular point and a pole. Figure 4 shows the cartographical grid of a conformal Jacobi projection for a triaxial ellipsoid with the parameters $a = 267.5$ m, $b = 147$ m, $c = 104.5$ m, for the asteroid 25143 Itokawa, and parallels passing through the circular points. Latitude in degrees of the circular points at such parameters is ±10.2438°. A gap occurs on prime meridian (and 180° meridian) in the northern hemisphere from the equator to the circle point +10.2438° and in the southern hemisphere from the circle point −10.2438° to the equator.

Figure 5 shows a map of Itokawa asteroid in Jacobi conformal projection created by transformation of a photomosaic of the asteroid surface in the azimuthal equidistant projection of a sphere originally created by Philip Stooke. The local-

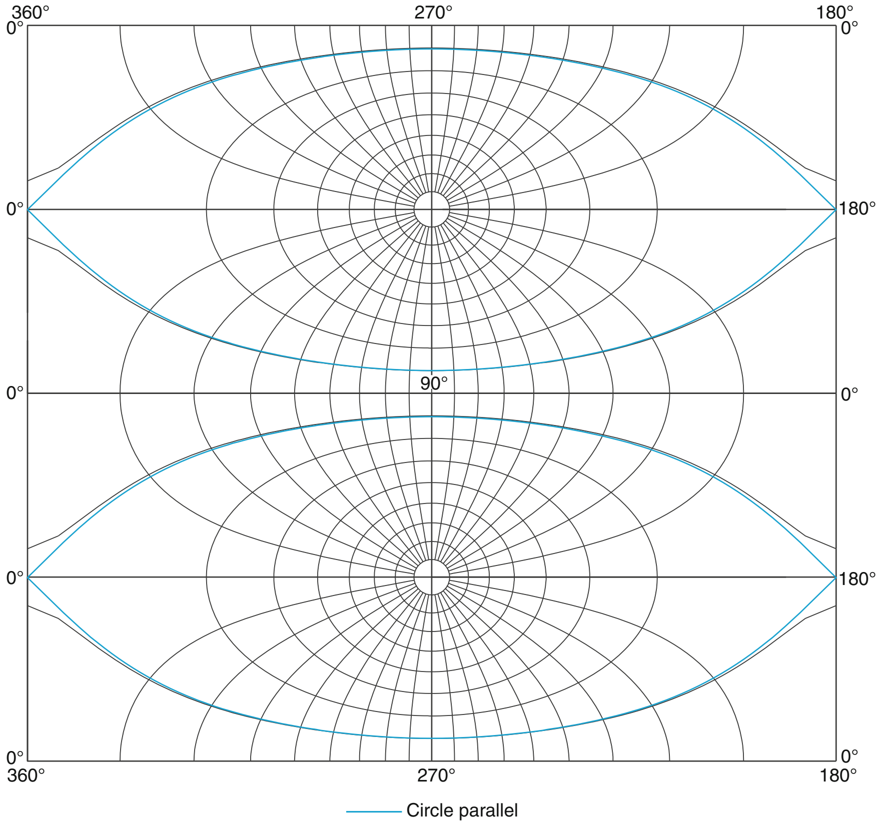

Fig. 4 Cartographic grid in Jacobi conformal projection

affine transformation was used in GeoGraph GIS 2.0 software. The control points for transformation were obtained by the tool for the calculation of the Jacobi projection (GIS Research Centre Site 2012). Unfortunately, due to an incorrect transformation in the neighborhood of circular points the map excluded areas located closer than 20° to the prime meridian and the meridian opposite it. A linear cartographic grid layer and a point craters layer with coordinates taken from (Gazetteer of Planetary Nomenclature—USGS 2012) were imposed on the resulting photomosaic.

Taking into account that the Jacobi projection is based on the relationship between semi-axes sizes a > b > c, in the special case when the polar flattening is equal to equatorial flattening, it is possible to use the cylindrical projection with right angle between meridian and parallel in the transverse orientation. The projection is also present in GIS Research Centre Site (2012). Figure 6 shows a map of asteroid 433 Eros in this projection. The new pole of the triaxial ellipsoid is the point of intersection of the equator and the prime meridian, and the major axis

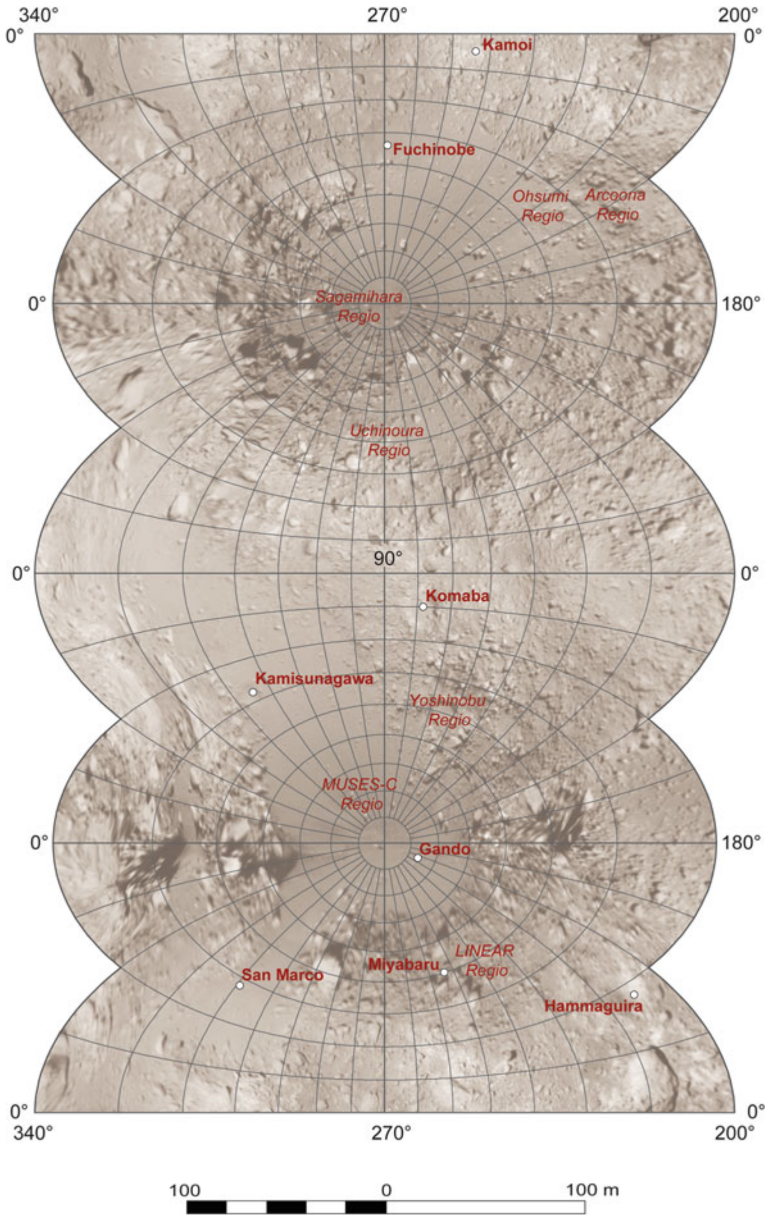

Fig. 5 Map of asteroid 25143 Itokawa in Jacobi conformal projection

becomes polar. In the case when the minor axis of the initial ellipsoid is equal to its polar axis, the resulting projection is conformal, as in fact it is a projection of the ellipsoid of revolution. A map was also created in GeoGraph GIS 2.0 software by transforming a base global photomosaic of the surface of an asteroid from the

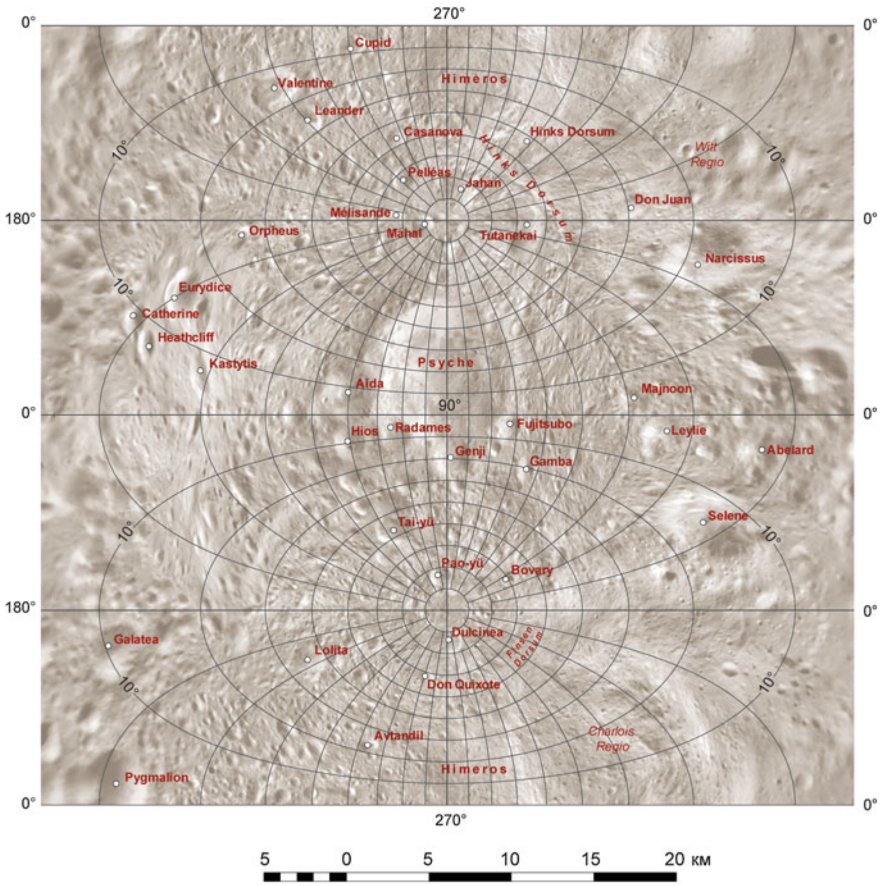

Fig. 6 Map of asteroid 433 Eros in the transverse conformal cylindrical projection of the triaxial ellipsoid

equidistant azimuthal projection of a sphere to transverse cylindrical projection of the triaxial ellipsoid. The transformation was made by special plug-into the module of calculating the projections on the local workplace. Therefore only a small neighborhood of points going to infinity was excluded from a map.

6 Conclusion

Cartographic grids in the transverse conformal cylindrical projection of the triaxial ellipsoid and in the Jacobi conformal projection have some similarities in the area of the pole and the neighborhood of central meridian. As a result, the technology for mapping in these projections also has similar particularities. We use a two step process for transformation of photomosaics: from a simple cylindrical projection into azimuthal projection first and the transformation of obtained photomosaics to conformal projection of the triaxial ellipsoid after. There is also an important feature in common between the transverse conformal cylindrical projection of the triaxial ellipsoid and the Jacobi projection derivation. In both cases, we first go from the latitude/longitude coordinate system to the orthogonal curvilinear coordinates on the surface of the triaxial ellipsoid, and then to the mapping on the projection plane. Only for cylindrical projection the orthogonal coordinates are latitude and longitude referred to the poles and the equator of transverse system, and for the Jacobi projection those are elliptic coordinates.

References

Bugaevskiy LM (1999) Teoriya kartographicheskich proekciy regulyarnych poverchnostey. M. Zlatoust (in Russian)

Clebsch A (2009) (ed) Jacobi's lectures on dynamics, 2nd edn. Hindustan Book Agency, New Delhi

Gazetteer of Planetary Nomenclature—USGS (2012) http://planetarynames.wr.usgs.gov/. Accessed 26 Oct 2012

GIS Research Centre of the Institute of Geography of the Russian Academy of Sciences Cartographical Projections of Triaxial Ellipsoid (2012) http://geocnt.geonet.ru/en/3_axial. Accessed 26 Oct 2012

Jacobi CGJ (1866) Vorlesungen über Dynamik. G. Reimer, Berlin

Jakobi K (1936) Lekcii po dinamike/Perevod s nemeckogo O.A. Polosukhina/Pod redakciey Prof. N.S. Koshlyakova. Glavnaya redakciya obshetekhnicheskoy literatury. Leningrad-Moskva (In Russian)

Kagan VF (1947) Foundations of the theory of surfaces in a tensor setting, Moscow-Leningrad (In Russian)

Prudnikov AP, Brychkov YA, Marichev OI (1986) Integrals and series. Elementary functions, vol 1. Gordon and Breach, New York