# Smooth Version of Johnson's Problem Concerning Derivations of Group Algebras

A. S. Mishchenko
Lomonosov Moscow State University
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# Outline

# Introduction

Consider the Banach algebra $\mathcal{A}$ and a $\mathcal{A}$–bimodule $E$. A linear mapping

$$D : \mathcal{A} \longrightarrow E$$

is called a derivation (or differentiation) if, for any elements $a, b \in \mathcal{A}$, the so-called Leibniz identity (with respect to the two-sided action of the algebra $\mathcal{A}$ on the bimodule $E$)

$$D(ab) = D(a)b + aD(b), \quad a, b \in \mathcal{A}.$$

holds (see Definition 1.8.1 in the Dales paper (2000) [6]).

H. G. Dales *Banach Algebras and Automatic Continuity*, Clarendon Press, Oxford University Press, New York, 2000

# Introduction

Denote the space of all derivations from $\mathcal{A}$ to $E$ by $\mathbf{Der}(\mathcal{A}, E)$ . The inner derivations $\mathbf{Int}\ (\mathcal{A}, E) \subset \mathbf{Der}(\mathcal{A}, E)$ , are defined by the adjoint representations

$$\mathbf{ad}_x(a) \stackrel{def}{=} xa - ax, \quad x \in E, a \in \mathcal{A}.$$

The quotient space $\mathbf{Out}\ (\mathcal{A}, E) = \mathbf{Der}(\mathcal{A}, E)/\mathbf{Int}\ (\mathcal{A}, E)$ is called the space of outer derivations; this space can be interpreted using the one-dimensional Hochschild cohomology of the algebra $\mathcal{A}$ with coefficients in the bimodule $E$ :

$$H^1(\mathcal{A}; E) \approx \mathbf{Out}\ (\mathcal{A}, E),$$

The derivation problem is as follows: is it true that every derivation is inner? (See Dales(2000) [6], (Question 5.6.B, p. 746)); i.e., is it true that

$$H^1(\mathcal{A}; E) \approx \mathbf{Out}\ (\mathcal{A}, E) = 0?$$

A simpler and more natural case occurs when the bimodule $E$ is isomorphic to the algebra $\mathcal{A}$, which is certainly a bimodule over the algebra $\mathcal{A}$ itself.

The passage to more general bimodules enabled V. Losert to solve Johnson's problem by finding inner derivations using elements of an algebra larger than $\mathcal{A}$ rather than elements of the original algebra $\mathcal{A}$.
Namely, the derivation problem is stated as follows: are all derivations inner? This problem was considered for the group algebras $\mathcal{A} = C[G]$ of some group $G$ rather than for all algebras.

In this very setting, Losert ([5]) proved that

$$\textbf{Out}\,(L^1(G), M(G)) = 0,$$

where $M(G)$ stands for the algebra of all bounded measures on $G$ with the multiplication operation defined by the convolution of measures.

In the present talk, we are interested in a dense subalgebra $\mathcal{A} = C[G] \subset \overline{\mathcal{A}}$ of the Banach algebra $\mathcal{A} = L^1(G)$ only rather than in the whole algebra $\mathcal{A} = L^1(G)$. The subalgebra $\mathcal{A} = C[G]$ consists of a kind of smooth elements of the algebra $\mathcal{A} = L^1(G)$.

## Statement of the problem

Consider the group algebra $\mathcal{A} = C[G]$. We assume that the group $G$ is a finitely generated discrete group. There is a natural problem to describe all derivations of $\mathcal{A}$. The inner derivations of $\mathcal{A} = C[G]$ form an ideal $\mathbf{Int}\ (\mathcal{A}) \subseteq \mathbf{Der}(\mathcal{A})$ in the algebra $\mathbf{Der}(\mathcal{A})$ of all derivations.

## Statement of the problem

To every group $G$ we assign the groupoid of the adjoint action of the group $G$ , $\mathcal{G}$ , and show that every derivation of the algebra $\mathcal{A} = C[G]$ is uniquely defined by an additive function on $\mathcal{G}$ which satisfies some natural finiteness conditions for the support.
For the case in which the group $G$ is finitely presented and its presentation is of the form $G = F < X, R >$ , one can transfer the presentation using the generators and defining relations to the groupoid $\mathcal{G}$ : $\mathcal{G} = \mathcal{F} < \mathcal{X}, \mathcal{R} >$ . This presentation enables us to construct the Cayley complex $\mathcal{C}a(\mathcal{G})$ of the groupoid $\mathcal{G}$ as a two-dimensional complex whose vertices are the objects of the groupoid $\mathcal{G}$ , the edges are the system of generating morphisms, and the two-dimensional cells are formed by the system of defining relations.

## Statement of the problem

Thus, the problem is to prove that the algebra of outer derivations $\mathbf{Out}\ (\mathbf{A}) = \mathbf{Der}(\mathcal{A})/\mathbf{Int}\ (\mathcal{A})$ of the algebra $\mathcal{A}$ is isomorphic to the one-dimensional cohomology of the Cayley complex $\mathcal{K}(\mathcal{G})$ of the groupoid $\mathcal{G}$ with finite supports:

$$\mathbf{Out}\ (C[G]) \approx H^1_f(\mathcal{K}(\mathcal{G}); \mathbf{R}).$$

# Outline

Consider the group algebra $\mathcal{A} = C[G]$. We assume that $G$ is a finitely presented discrete group. An arbitrary element $u \in \mathcal{A}$ is a finite linear combination $u = \sum\limits_{g \in G} \lambda^g \cdot g$. Consider an arbitrary linear operator on the group algebra $\mathcal{A} = C[G]$ , $X : \mathcal{A} \longrightarrow \mathcal{A}$. The linear operator $X$ has the following matrix form:

$$X(g) = \sum_h x_g^h \cdot h \in \mathcal{A}.$$

The matrix $X = \|x_g^h\|_{g,h \in G}$ must satisfy the natural finiteness condition:

(**F**1) For every subscript $g \in G$ , the set of the superscripts $h \in G$ for which $x_g^h$ is nonzero is finite.

If a matrix $X = \|x_h^g\|_{g,h\in G}$ satisfies condition $(\mathbf{F}1)$, then it well defines a linear operator $X : \mathcal{A} \longrightarrow \mathcal{A}$ . All this justifies that both the operator $X$ and its matrix $X = \|x_g^h\|_{g,h\in G}$ are denoted by the same symbol $X$.

Consider now a so-called differentiation (derivation) in the algebra $\mathcal{A}$ , i.e., an operator $X$ for which the following condition holds:

$(\mathbf{F}2)$ $X(u \cdot v) = X(u) \cdot v + u \cdot X(v), u, v \in \mathcal{A}.$

The set of all derivations of the algebra $\mathcal{A}$ is denoted by $\mathbf{Der}(\mathcal{A})$ and forms a Lie algebra with respect to the commutator of operators.

There is a natural problem to describe all derivations of the algebra $\mathcal{A}$. To this end, it is necessary to satisfy two conditions, (**F**1) and (**F**2). It is more or less simple to verify each of the conditions separately. The simultaneous validity of these conditions is one of the tasks of this talk.
All inner derivations satisfy automatically both the conditions (**F**1) and (**F**2).

Denote by $\mathcal{G}$ the groupoid associated with the adjoint action of the group $G$. The groupoid $\mathcal{G}$ consists of the objects $\mathbf{Obj}(\mathcal{G}) = G$ and the morphisms

$$\mathbf{Mor}(a, b) = \{g \in G : ga = bg \text{ or } b = \mathbf{ad}(g)(a)\}, \quad a, b \in \mathbf{Obj}(\mathcal{G}).$$

It is convenient to denote the elements of the set of all morphisms $\mathbf{Mor}(\mathcal{G})$ in the form of columns

$$\xi = \left(\frac{a \longrightarrow b}{g}\right) \in \mathbf{Mor}(a, b), \quad b = gag^{-1} = \mathbf{ad}(g)(a).$$

Note that the groupoid $\mathcal{G}$ is decomposed into the disjoint union of its subgroupoids $\mathcal{G}_{\langle g\rangle}$ that are indexed by the conjugacy classes $\langle g\rangle$ of the group $G$ :

$$\mathcal{G} = \coprod_{\langle g\rangle \in \langle G\rangle} \mathcal{G}_{\langle g\rangle},$$

where $\langle G\rangle$ stands for the set of conjugacy classes of the group $G$. The subgroupoid $\mathcal{G}_{\langle g\rangle}$ consists of the objects $\mathbf{Obj}(\mathcal{G}_{\langle g\rangle}) = \langle g\rangle$ and the morphisms $\mathbf{Mor}(\mathcal{G}_{\langle g\rangle}) = \coprod_{a,b\in\langle g\rangle} \mathbf{Mor}(a,b)$ .

A linear operator $X : \mathcal{A} \longrightarrow \mathcal{A}$ is described by the matrix $X = \|x_g^h\|_{g,h \in G}$ satisfying condition $(\mathbf{F}1)$. The same matrix $X$ defines a function on the groupoid $\mathcal{G}$ :

$$T^X : \mathbf{Mor}(\mathcal{G}) \longrightarrow R,$$

associated with $X$ , which is defined by the formula:

$$T^X(\xi) = T^X\left(\frac{a \longrightarrow b}{g}\right) = x_g^{ga=bg}.$$

Condition (**F**1) can be reformulated in terms of the function $T$ on the morphisms $\mathbf{Mor}(\mathcal{G})$ of the groupoid $\mathcal{G}$ :

(**T**1) for every element $g \in G$ , the set of morphisms of the form $\xi = \left(\frac{a \longrightarrow b}{g}\right)$ for which $T^X(\xi) \neq 0$, is finite.

Then the condition (**T**1) imposed on the function $T$ can equivalently be reformulated as follows:

**Proposition**

*A function* $T : \mathbf{Mor}(\mathcal{G}) \longrightarrow \mathbf{C}$ , *is defined by a linear operator*

$$X : \mathcal{A} \longrightarrow \mathcal{A}, \quad T = T^X,$$

*if and only if, for any element* $g \in G$, *the restriction* $(T)\big|_g$ *is a finitely supported function.*

In this case we say that $T$ is locally finitely supported.

Denote the set of locally finitely supported functions on the groupoid $\mathcal{G}$ by $C_f(\mathcal{G})$ .

**Theorem**

*The homomorphism*

$$T : \mathbf{Hom}\,(\mathcal{A},\mathcal{A}) \longrightarrow C_f(\mathcal{G})$$

*is an isomorphism.*

# Outline

# Derivations of the group algebra as characters on the groupoid

The algebra of derivations $\mathbf{Der}(\mathcal{A})$ treated as linear operators is a subspace of $\mathbf{Hom}\,(\mathcal{A}, \mathcal{A})$ . Thus, the correspondence $T$ takes the algebra of derivations $\mathbf{Der}(\mathcal{A})$ to some subspace $\mathbf{T}_f(\mathcal{G}) \subset C_f(\mathcal{G})$ .

Consider two morphisms $\xi = \left(\frac{a \longrightarrow b}{g_1}\right)$ and $\eta = \left(\frac{b \longrightarrow c}{g_2}\right)$ , which thus admit the composition

$$\eta * \xi = \left(\frac{a \longrightarrow c}{g_2 g_1}\right).$$

# Derivations of the group algebra as characters on the groupoid

**Theorem**

*An operator* $X : \mathcal{A} \longrightarrow \mathcal{A}$ *is a differentiation (i.e., a derivation) if and only if the function* $T^X$ *(on the groupoid* $\mathcal{G}$ *) associated with the operator* $X$ *satisfies the additivity condition*

**(T2)**

$$T^X(\eta * \xi) = T^X(\eta) + T^X(\xi)$$

*for every pair of morphisms* $\xi$ *and* $\eta$ *admitting the composition* $\eta * \xi$ .

# Derivations of the group algebra as characters on the groupoid

Every function $T : \mathbf{Mor}(\mathcal{G}) \longrightarrow R$ on the groupoid $\mathcal{G}$ satisfying the additivity condition (T2) is called a *character*. Denote the set of all characters on the groupoid $\mathcal{G}$ by $\mathbf{T}(\mathcal{G})$. Denote the space of all locally finitely supported characters of the groupoid $\mathcal{G}$ by $\mathbf{T}_f(\mathcal{G}) \subset \mathbf{T}(\mathcal{G})$.
Thus, the correspondence $T$ defines a mapping from the algebra of derivations $\mathbf{Der}(\mathcal{A})$ to the space $\mathbf{T}_f(\mathcal{G})$ of locally finitely supported characters on the groupoid $\mathcal{G}$ :

**Theorem**

*The mapping*

$$T : \mathbf{Der}(\mathcal{A}) \longrightarrow \mathbf{T}_f(\mathcal{G}),$$

*is an isomorphism.*

# Outline

## Cayley complex of a groupoid

Here we intend to apply the so-called geometric methods of combinatorial group theory to study the problem of describing the derivations of the group algebra of a finitely presentable discrete group. Following, for example, the book of R. Lyndon and P. Schupp (1980, [24]), one can assign to every discrete finitely presentable group the so-called Cayley graph and its two-dimensional generalization, the Cayley complex, which consists of the elements of the group as vertices, of the system of generators as edges, and of the system of defining relations as two-dimensional cells. The topological properties of the Cayley complex are responsible for certain algebraic properties of the group $G$ itself.

# Cayley complex of a groupoid

The geometric construction of the Cayley complex for a finitely presentable group $G$ can be generalized to the case of groupoids; in particular, to the case of the groupoid $\mathcal{G}$ of the adjoint action of the group $G$ . Since the derivations of the group algebra $\mathbf{Der}(C[G])$ can be described as characters on the groupoid $\mathcal{G}$, it follows that the topological properties of the Cayley complex $\mathcal{C}a(\mathcal{G})$ of the groupoid $\mathcal{G}$ enable us to describe some properties of derivations.

Consider a finitely presentable group $G$ ,

$$G = F < X, R >,$$

where $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set of generators and $R = \{r_1, r_2, \ldots, r_m\}$ is a finite set of defining relations.

The finite set of generators $X = \{x_1, x_2, \ldots, x_n\}$ and the finite set of defining relations $R = \{r_1, r_2, \ldots, r_m\}$ are transferred to the generators and relations of the groupoid $\mathcal{G}$ , which we denote by $\mathcal{X}$ and $\mathcal{R}$ . Thus, the set of morphisms $\mathbf{Mor}(\mathcal{G})$ can be denoted by $\mathcal{F} < \mathcal{X}, \mathcal{R} >$,

$$\mathbf{Mor}(\mathcal{G}) = \mathcal{F} < \mathcal{X}, \mathcal{R} > .$$

Let us define $\mathcal{X}$ as the set of all morphisms of the form

$$\mathcal{X} = \left\{ \xi = \left( \frac{a \longrightarrow b}{x} \right) : x \in X, a \in \mathbf{Obj}(\mathcal{G}), b = a^x \right\}.$$

Let $\mathcal{Y} = \mathcal{X} \sqcup \mathcal{X}^{-1}$; consider $\mathcal{Y}$ as an alphabet,

$$\mathcal{Y} = \left\{ \xi = \left( \frac{a \longrightarrow b}{y} \right) : y \in Y = X \sqcup X^{-1}, a \in \mathbf{Obj}(\mathcal{G}), b = a^y \right\}.$$

The set $S(\mathcal{Y})$ is the set of all admissible words $s$ in the alphabet $\mathcal{Y}$, i.e., words formed by the letters of the alphabet $\mathcal{Y}$, $s = \xi_1 \xi_2 \xi_3 \cdots \xi_l$ such that

$$\xi_i = \left( \frac{a_i \longrightarrow a_{i+1}}{y_i} \right), \quad \xi_i \in \mathcal{Y}, \quad 1 \leq i \leq l.$$

Every admissible word $s \in S(\mathcal{Y})$ defines a morphism $\xi(s) \in \mathbf{Mor}(\mathcal{G})$ by the formula

$$\xi(s) = \xi_1 * \xi_2 * \xi_3 * \cdots * \xi_l.$$

Define first the system of relations $\mathcal{R}$ generated by the set $R$ of defining relations for the group $G$. Every relation $r_i \in R$ is written out in the form of a word

$$r_i = y_{i1} y_{i2} y_{i3} \cdots y_{il_i}, \quad y_{ij} \in Y.$$

The relations $r_i$ generate the system of admissible words $\rho_{i,a}, a \in \mathbf{Obj}(\mathcal{G})$, of the form

$$\rho_{i,a} = \left(\frac{a_1 \longrightarrow a_2}{y_{i1}}\right)\left(\frac{a_2 \longrightarrow a_3}{y_{i2}}\right)\left(\frac{a_3 \longrightarrow a_4}{y_{i3}}\right)\cdots\left(\frac{a_{l_i} \longrightarrow a_1}{y_{il_i}}\right),$$

$$a = a_1, \quad a_{j+1} = a_j^{y_{ij}}, \quad 1 \leq j \leq l_i, \quad a_{l_i+1} = a_1,$$

which serve as the defining relations of the groupoid $\mathcal{G}$. Denote the set of all admissible words of the form $\rho_{i,a}$ by $\mathcal{R}$,

$$\mathcal{R} = \{\rho_{i,a} : 1 \leq i \leq l_i, \quad a \in \mathbf{Obj}(\mathcal{G})\},$$

$\mathcal{R} \subset S(\mathcal{Y})$.

Thus, two admissible words $s$ and $s'$ define the same morphism, i.e.,

$$\xi(s) = \xi(s') \in \mathbf{Mor}(\mathcal{G}),$$

if and only if the words are equivalent, $s \sim s'$ , i.e., when there is a finite sequence of operations of two types:
1) the operation of reduction,
2) the operation of admissible insertion.

Before constructing the Cayley complex of the groupoid $\mathcal{G}$ by analogy with the Cayley complex of the group $G$ itself, recall the construction of the Cayley complex of $G$ from its presentation in the form of finitely many generators $X$ and finitely many defining relations $R$, $\mathcal{F}(X)/R$.

By the definition in the book by Lyndon and Schupp (1980, [24], p. 174), for the group $G$ , the Cayley complex $\mathcal{C}a(G)$ consists of vertices, edges, and two-dimensional cells. The set of vertices $\mathcal{C}a_0(G)$ is the set of all elements the group $G$.
The set of edges is formed by the morphisms of the form
$\xi = \left(\frac{a \longrightarrow ag}{g}\right)$ , $g \in X \sqcup X^{-1}$, i.e., $\xi \in \mathcal{X} \sqcup \mathcal{X}^{-1} = \mathcal{Y}$.

Thus, the edges $\xi \in \mathcal{C}a_1(G)$ are defined by the set $\mathcal{X}$ of generators of the groupoid $r\mathcal{G}$ .
The set of two-dimensional cells, $\mathcal{C}a_2(G)$ , is defined using sequences of morphisms defined by words $\rho \in \mathcal{R} \sqcup \mathcal{R}^{-1} \subset S(\mathcal{X} \sqcup \mathcal{X}^{-1}) = S(\mathcal{Y})$ .
The two-dimensional cells are the planar orientable polygons $\sigma(\rho)$ defined by the words $\rho \in \mathcal{R} \sqcup \mathcal{R}^{-1}$ that determine the boundaries of the polygons $\sigma(\rho)$ as closed cycles formed by the edges of the word $\rho$.
The two-dimensional cells $\sigma(\rho)$ are pasted to the 1-skeleton of the complex $\mathcal{C}a(G)$ by the natural identification of the edges of the boundary of the cell $\sigma(\rho)$ with the corresponding edges of the complex $\mathcal{C}a(G)$, preserving the orientation.

The only difference between the groupoid $\mathcal{G}$ and the group $G$ is that the former is defined by another action of the group $G$ , namely, the adjoint action: $\mathbf{Ad}_g(a) = gag^{-1}$ , $g, a \in G$ .
Therefore, the Cayley complex of the groupoid $\mathcal{G}$ is constructed by analogy with the Cayley complex of the group $G$ .
Namely, the vertices, i.e., the zero-dimensional cells $\mathcal{C}a_0(\mathcal{G})$ of the complex $\mathcal{C}a(\mathcal{G})$, are the objects, $a \in \mathbf{Obj}(\mathcal{G}) \approx G.$

The one-dimensional edges, i.e., the oriented cells of dimension 1, $\mathcal{K}_1(\mathcal{G})$ , joining vertices $a$ and $b$ , are the morphisms $\xi \in \mathbf{Mor}(a, b)$ of the form

$$\xi = \left( \frac{a \longrightarrow b}{y} \right), \quad y \in Y = X \sqcup X^{-1}, \quad a \in \mathbf{Obj}(\mathcal{G}), \quad b = yay^{-1} \in \mathbf{Obj}$$

The set of edges described above is denoted by $\mathcal{X}$ ; let $\mathcal{Y} = \mathcal{X} \sqcup \mathcal{X}^{-1}$ . These edges form a system of generators of the groupoid $\mathcal{G}$ , i.e., every morphism $\eta \in \mathbf{Mor}(a, c)$ can be represented as an admissible composition of generators,

$$\eta = \xi_1 * \xi_2 * \cdots * \xi_k, \quad \xi_i \in \mathcal{Y} \quad 1 \leq i \leq k.$$

Finally, the two-dimensional cells $\mathcal{K}_2(\mathcal{G})$ are the planar orientable polygons $\sigma(\rho)$ given by words $\rho \in \mathcal{R} \sqcup \mathcal{R}^{-1}$ that define the boundaries of the polygons $\sigma(\rho)$ as closed cycles composed of the edges of the words $\rho$. The two-dimensional cells $\sigma(\rho)$ are pasted to the 1-skeleton of the complex $\mathcal{C}a(\mathcal{G})$ by the natural identification of the edges of the boundary of a cell $\sigma(\rho)$ to the corresponding edge of the complex $\mathcal{C}a(\mathcal{G})$ , preserving the orientation.

## Groups of chains of the Cayley complex of the groupoid $\mathcal{G}$

The two-dimensional Cayley complex $\mathcal{C}a(\mathcal{G})$ generates the cochain complex

$$C^0(\mathcal{C}a(\mathcal{G})) \xrightarrow{d_0} C^1(\mathcal{C}a(\mathcal{G})) \xrightarrow{d_1} C^2(\mathcal{C}a(\mathcal{G})).$$

This cochain complex has a natural subcomplex of finitely supported cochains, because every cell of dimension 0 or 1 satisfies the condition that the set of cells that abut on cells of lesser dimension is finite.

# Groups of chains of the Cayley complex of the groupoid $\mathcal{G}$

This, taken together, gives the commutative diagram

$$\begin{tikzcd}
C^0(\mathcal{C}a(\mathcal{G})) \arrow[r, "d_0"] & C^1(\mathcal{C}a(\mathcal{G})) \arrow[r, "d_1"] & C^2(\mathcal{C}a(\mathcal{G})) \\
C^0_f(\mathcal{C}a(\mathcal{G})) \arrow[u, "\cup"] \arrow[r, "d_0^f"] & C^1_f(\mathcal{C}a(\mathcal{G})) \arrow[u, "\cup"] \arrow[r, "d_1^f"] & C^2_f(\mathcal{C}a(\mathcal{G})). \arrow[u, "\cup"]
\end{tikzcd}$$

We identify the one-dimensional finitely supported cochains $C^1_f(\mathcal{C}a(\mathcal{G}))$ with the derivations $\mathbf{Der}(C[G])$ by the composition of the mappings

$$H : \mathbf{Der}(C[G]) \xrightarrow{T} \mathbf{T}_f(\mathcal{G}) \xrightarrow{\varphi^*} C^1_f(\mathcal{C}a(\mathcal{G})).$$

# Groups of chains of the Cayley complex of the groupoid $\mathcal{G}$

We have the commutative diagram:

$$\begin{tikzcd}
C^0(\mathcal{C}a(\mathcal{G})) \arrow[r, "d_0"] & C^1(\mathcal{C}a(\mathcal{G})) \arrow[r, "d_1"] & C^2(\mathcal{C}a(\mathcal{G})) \\
C^0_f(\mathcal{C}a(\mathcal{G})) \arrow[r, "d_0^f"] \arrow[u, "\cup"] & C^1_f(\mathcal{C}a(\mathcal{G})) \arrow[r, "d_1^f"] \arrow[u, "\cup"] & C^2_f(\mathcal{C}a(\mathcal{G})). \arrow[u, "\cup"] \\
\mathbf{Int}\ (C[G]) \arrow[r] \arrow[u, "H"] & \mathbf{Der}(C[G]) \arrow[r] \arrow[u, "H"] & 0 \arrow[u]
\end{tikzcd}$$

## Groups of chains of the Cayley complex of the groupoid $\mathcal{G}$

**Theorem**

*The homomorphism* $H$ *is a monomorphism onto the kernel of the differential* $d_1$ *:*

$$\mathbf{Im}\,(H) = \mathbf{Ker}\,(d_1^f) \subset C^1(\mathcal{C}a(\mathcal{G})).$$

*The image of the algebra of inner derivations* $H(\mathbf{Int}\,(C[G])) \subset C^1(\mathcal{C}a(\mathcal{G}))$ *is equal to the image of the differential* $d_0^f$ *:*

$$H(\mathbf{Int}\,(C[G])) = \mathbf{Im}\,(d_0^f) \subset C^1(\mathcal{C}a(\mathcal{G})).$$

## Groups of chains of the Cayley complex of the groupoid $\mathcal{G}$

### Corollary

*The homomorphism* $H$ *induces an isomorphism of the algebra of outer derivations* $\mathbf{Out}\,(C[G])$ *onto the group of the one-dimensional cohomology with finite supports of the Cayley complex of the groupoid* $\mathcal{G}$ *of the adjoint action of the group* $G$ *:*

$$H : \mathbf{Out}\,(C[G]) \xrightarrow{\approx} H^1_f(\mathcal{C}a(\mathcal{G}); \mathbf{R}).$$

# Description of the Cayley complex of the groupoid $\mathcal{G}$

Let $K(X,R)$ be the two dimensional complex associated with the representation $(X,R)$ of the group $G = F\langle X,R\rangle$ . Then the universal covering $\widetilde{K(X,R)}$ is homeomorphic to the Cayley complex of the group $G$ :

$$\mathcal{C}a(G) \approx \widetilde{K(X,R)}.$$

Then the Cayley complex $\mathcal{C}a(\mathcal{G}_{\langle a\rangle})$ is homeomorphic to the covering $\widetilde{K(X,R)}^{C_G(a)}$ that corresponds to the subgroup $C_G(a) \subset \pi_1(K(X,R)) \approx G$ where $C_G(a)$ is the centralizer of the conjugated class $\langle a\rangle \in \langle G\rangle$ :

$$\mathcal{C}a(\mathcal{G}_{\langle a\rangle}) \approx \widetilde{K(X,R)}^{C_G(a)}.$$

## Description of the Cayley complex of the groupoid $\mathcal{G}$

In particular one has

**Theorem**

$$H_f^*(\mathcal{C}a(\mathcal{G});\mathbf{R}) \approx \bigoplus_{\langle a\rangle\in\langle G\rangle} H_f^*(\mathcal{C}a(\mathcal{G}_{\langle a\rangle});\mathbf{R}).$$

# Description of the Cayley complex of the groupoid $\mathcal{G}$

For outer derivation one has

**Corollary**

*The homomorphism $H$ induces the isomorphism of the algebra of outer derivations $\mathbf{Out}\ (C[G])$ onto the direct sum of one dimensional finite cohomologies of the Cayley of groupoids $\mathcal{G}_{\langle a\rangle}$ of adjoint action of the group $G$ :*

$$H : \mathbf{Out}\ (C[G]) \xrightarrow{\approx} H^1_f(\mathcal{C}a(\mathcal{G}); \mathbf{R}) \approx \bigoplus_{\langle a\rangle \in \langle G\rangle} H^*_f(\mathcal{C}a(\mathcal{G}_{\langle a\rangle}); \mathbf{R}).$$

# Thank you!

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