ALGORITHMS FOR NUMERICAL EVALUATION OF THE COMPOUND POISSON DISTRIBUTION

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1. The class of compound (generalized) Poisson distributions (CPD) is fairly representative: Neyman type A, B, C distributions, negative binomial distribution, Hermite distribution, etc. [1]. These distributions describe a wide range of phenomena. However, the application of CPD models often runs into the obstacle of numerical evaluation of the corresponding probability distribution functions. The purpose of this study is to suggest some techniques for numerical evaluation of compound Poisson distributions and to provide examples of their use.

The compound Poisson distribution is introduced in the following way [2, 3]. Given are a random variable (r.v.) ν that follows a Poisson distribution with the parameter λ and a collection $\xi_1, \xi_2, \ldots$ of identically distributed r.v.s that are independent of one another and of the number ν. The distribution of the r.v. $\xi = \xi_1 + \xi_2 + \ldots \xi_\nu$ is called a compound (generalized) Poisson distribution. The characteristic function (c.f.) of the random variable ξ is

$$\varphi(t) = \exp\{\lambda(\psi(t)-1)\}, \tag{1}$$

where $\psi(t)$ is the c.f. of the distribution of the components.

In what follows, we consider CPD only in the class of nonnegative integer r.v.s. Therefore, along with (1), we use the probability generating function (p.g.f.)

$$\Phi(z) = \exp\{\lambda(\Psi(z)-1)\}, \quad |z| \leqslant 1, \tag{2}$$

where $\Psi(z) = \psi(-i \ln z)$ is the p.g.f. of the random variable ξ_j.

Consider the following problem: given λ and $\Psi(z)$, compute N values of the probability distribution function $p_n = P\{\xi = n\}$ at the points $n = 0, 1, \ldots, N-1$. The value of N is assumed given from some reasonable practical considerations. The following stopping rules may be used to end the computation: 1) $p_{N-1} < \delta$; 2) $N \geq N_0, p_{N-1} < \delta$; 3) $1 - \sum_{n=0}^{N-1} p_n < \varepsilon$, where N_0, δ, and ε are given.

2. Noting that $\Phi(z) = \sum_{n=0}^{\infty} p_n z^n$, it is natural to apply one of the methods for inversion of the p.g.f. The existing analytical techniques [4] usually are not adapted for numerical evaluation of p_n. Our problem can be solved by applying direct and inverse Fourier transforms. Indeed, from the representation of the c.f.

$$\varphi(t) = Ee^{it\xi} = \sum_{n=0}^{\infty} p_n e^{int} = \exp\{\lambda(\psi(t)-1)\} \tag{1'}$$

it follows that $\varphi(t)$ is 2π-periodic and p_n, $n = 0, 1, \ldots$, are Fourier coefficients. Therefore

$$p_n = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t) e^{-int}\, dt = \frac{1}{2\pi i}\int_{|z|=1} \Phi(z) z^{-n-1}\, dz. \tag{3}$$

Approximating the first integral by the sum

$$\hat{p}_n = \frac{1}{N}\sum_{m=0}^{N-1} \varphi(2\pi m/N) \exp\{-2\pi i n m/N\}, \quad n = \overline{0, N-1}, \tag{4}$$

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we see that $\hat{p}_n$ is the discrete Fourier transform of the complex periodic sequence $\{\varphi(2\pi m/N)\}$. For numerical implementation of (4), we can apply the fast Fourier transform (FFT) algorithm [8]. The rounding and approximation errors of this method of evaluation of p_n were analyzed in [5, 6]. By (1′), the evaluation of $\varphi(2\pi m/N)$ reduces to evaluation of $\psi(2\pi m/N)$, $m = 0, 1, \ldots, N-1$. It is possible that $\psi(t)$ is defined not in functional form but in series form

$$\psi(t) = \sum_{r=0}^{k} f_r e^{irt},$$

where $f_0, f_1, \ldots, f_k$ are given, $\sum_{r=0}^{k} f_r = 1$, and for any $0 \le l < k$, $\sum_{r=0}^{l} f_r < 1$. Then for $N \ge k+1$, $\{\psi(2\pi m/N)\}$ is evaluated using the inverse Fourier transform

$$\psi(2\pi m/N) = \sum_{r=0}^{N-1} f_r \exp\{2\pi irm/N\}, \quad m = \overline{0, N-1},$$

which also can be implemented by the FFT algorithm.

3. If the probability distribution f_r, $r = 0, 1, \ldots, k$, corresponding to the p.g.f. $\Psi(z)$ is given, the problem reduces to evaluating p_n for known λ and f_r. It can be reduced for $f_0 \ne 0$ to evaluation of p_n from given values

$$\tilde{\lambda} = \lambda \sum_{r=1}^{k} f_r, \quad \tilde{f}_0 = 0, \quad \tilde{f}_r = \lambda f_r/\tilde{\lambda}, \quad r = \overline{1, k} \quad \Big(\sum_{r=1}^{k} \tilde{f}_r = 1\Big). \tag{5}$$

Indeed, the p.g.f. (2) is rewritten in the form

$$\Phi(z) = \exp\Big\{\lambda\Big(\sum_{r=0}^{k} f_r z^r - 1\Big)\Big\} = \exp\Big\{\lambda \sum_{r=1}^{k} f_r (z^r - 1)\Big\} = \exp\Big\{\tilde{\lambda} \sum_{r=1}^{k} \tilde{f}_r (z^r - 1)\Big\} = \exp\Big\{\tilde{\lambda}\Big(\sum_{r=0}^{k} \tilde{f}_r z^r - 1\Big)\Big\}. \tag{6}$$

Denoting $a_r = \tilde{\lambda}\tilde{f}_r$, $r = 1, \ldots, k$, and using (5), we obtain for the p.g.f. (2)

$$\Phi(z) = \exp\Big\{\sum_{r=1}^{k} a_r (z^r - 1)\Big\}. \tag{7}$$

The problem of evaluating p_n for CPD thus can be restated as follow: given the sequence $\{a_r\}$, evaluate the distribution function p_n corresponding to the p.g.f. (7). In addition to the c.f. inversion method described above, this problem can be solved by methods based both on explicit representations of the distribution function p_n and on recursion. Different explicit forms of p_n are determined by different problem formulations leading to the CPD (2), (7).

4. Given the CPD definition from (5)-(7), the distribution function p_n, $n = 0, 1, \ldots$, corresponds to the r.v. $\xi = \gamma_1 + \gamma_2 + \ldots + \gamma_\mu$, where $\{\gamma_j\}$ is a sequence of independent identically distributed random variables with distribution functions $P\{\gamma_j = r\} = \tilde{f}_r$, $r = 0, 1, \ldots$, and μ is a r.v. independent of γ_j and Poisson-distributed with the parameter $\tilde{\lambda}$. Here

$$\tilde{\lambda} = \sum_{r=1}^{k} a_r, \quad \tilde{f}_r = a_r/\tilde{\lambda}, \quad r = \overline{1, k}, \quad \tilde{f}_r = 0, \quad r = 0, \ k+1, \ k+2, \ \ldots \ . \tag{8}$$

Then by the complete probability formula the distribution function of ξ is [7]

$$p_n = \sum_{r=0}^{\infty} P\{\mu = r\} P\{\gamma_1 + \gamma_2 + \ldots + \gamma_r = n\} = \sum_{r=0}^{\infty} e^{-\tilde{\lambda}} \tilde{\lambda}^r / r! \, \tilde{f}_n^{r*}, \tag{9}$$

where $\{\tilde{f}_n^{r*}\} = \{\tilde{f}_n\}^{r*}$ is the r-fold composition of the sequence $\{\tilde{f}_n\}$ defined recursively by

$$\{\tilde{f}_n\}^{r*} = \{\tilde{f}_n\}^{(r-1)*} * \{\tilde{f}_n\}, \quad \tilde{f}_n^{r*} = \sum_{j=0}^{n} \tilde{f}_j^{(r-1)*} \tilde{f}_{n-j}, \quad r = 1, 2, \ldots,$$
$$\{\tilde{f}_n\}^{0*} = (1, 0, 0, \ldots), \quad \{\tilde{f}_n\}^{1*} = \{\tilde{f}_n\}. \tag{10}$$

By (8), the distribution function p_n (9) is written in the form

$$p_n = p_0 \sum_{r=1}^{n} a_n^{r*}/r!, \quad n = 1, 2, \ldots, \quad p_0 = \exp\Big\{-\sum_{r=1}^{k} a_r\Big\}. \tag{11}$$

because $\{\bar{\lambda}\bar{f}_n^{\,r*}\} = \{a_n\}^{r*}$, $a_n = \bar{\lambda}\bar{f}_n$, $n = 0, 1, \ldots$

The evaluation of p_n from (11) is restricted by the technical capabilities of the computer due to the presence of factorials in the formula. Thus, it becomes impossible for $n > 21$ on the BÉSM-6 computer and for $n > 59$ on ES-1045. However, the computational process based on (11) can be organized differently. Indeed, denoting $A_n^{(r)} = a_n^{r*}/r!$, we replace (11) with the algorithm

$$\{A_n^{(1)}\} = \{a_n\}, \quad p_0 = \exp\Big\{-\sum_{n=1}^{k} A_n^{(1)}\Big\},$$
$$\{A_n^{(r)}\} = (\{A_n^{(r-1)}\} * \{A_n^{(1)}\})/r, \quad r = \overline{2, n}, \quad p_n = p_0 \sum_{r=1}^{n} A_n^{(r)}, \quad n = 1, 2, \ldots. \tag{12}$$

5. Another explicit form of the distribution function is obtained from the following formulation of the problem for CPD (7). Let ξ_r, $r = 1, \ldots, k$, be independent Poisson r.v.s with the respective parameters a_r. Then the r.v. $\xi = \zeta_1 + 2\zeta_2 + \ldots + k\zeta_k$ with the p.g.f.

$$\Phi(z) = Ez^{\sum_{r=1}^{k} r\zeta_r} = \prod_{r=1}^{k} Ez^{r\zeta_r} = \prod_{r=1}^{k} \Phi_{\zeta_r}(z^r) = \prod_{r=1}^{k} \exp\{a_r(z^r - 1)\},$$

which obviously coincides with (7) is distributed according to a CPD.

Let us determine the explicit form of the distribution function of ξ which follows directly from its representation.

Consider the events $\mathcal{A}_{i_r}^{(r)} = \{\zeta_r = i_r\}$, $r = 1, \ldots, k$, $i_1, i_2, \ldots, i_k$ are nonnegative integers. Let $\mathcal{B}_i = \mathcal{B}_{i_1,\ldots,i_k} = \bigcap_{r=1}^{k} \mathcal{A}_{i_r}^{(r)}$, $i \in I$ be a family of events and let $I = \{i = (i_1, \ldots, i_k)\colon 0 \le i_r$ are integers, $r = 1, \ldots, k\}$. It is clear from the condition that $P(\mathcal{B}_i) = \prod_{r=1}^{k} e^{-a_r} a_r^{i_r}/i_r!$.

Isolate from $\mathcal{B}_i$, $i \in I$, the family of mutually exclusive events $\mathcal{B}_i$, $i \in I_n$, where

$$I_n = \Big\{i = (i_1, \ldots, i_k) : I_1^{(k)} = \sum_{r=1}^{k} r i_r = n\Big\} \subset I, \quad n = 0, 1, \ldots.$$

For $k > n > 0$ we obviously have

$$I_n = \{i = (i_1, \ldots, i_n, 0, \ldots, 0) : I_1^{(n)} = n\}, \quad I_0 = \{i = (0, \ldots, 0)\}.$$

Then, denoting $m = \min\{k, n\}$, we can write for any $k = 1, 2, \ldots$, $n = 0, 1, \ldots$,

$$I_n = \Big\{i = (i_1, \ldots, i_k) : I_1^{(m)} = n, \sum_{l=m+1}^{k} i_l = 0\Big\}, \quad n = 0, 1, \ldots.$$

The event $\{\xi = n\} = \{\zeta_1 + 2\zeta_2 + \ldots + k\zeta_k = n\}$ occurs if at least one of the mutually exclusive events of the family $\mathcal{B}_i$, $i \in I_n$, occurs, i.e.,

$$P\{\xi = n\} = P\Big(\bigcup_{i \in I_n} \mathcal{B}_i\Big) = \sum_{i \in I_n} P(\mathcal{B}_i) = \sum_{I_1^{(m)} = n} \prod_{r=1}^{k} e^{-a_r} a_r^{i_r}/i_r! = \exp\Big\{-\sum_{r=1}^{k} a_r\Big\} \sum_{I_1^{(m)} = n} \prod_{r=1}^{m} a_r^{i_r}/i_r!, \quad n = 0, 1, \ldots. \tag{13}$$

The evaluation of p_n from (13) is difficult because it requires finding all the nonnegative integer solutions of the equation $I_1^{(m)} = n$ over which summation is performed. Note that this set of solutions is described by the integer relationships

$$0 \leqslant i_m \leqslant [n/m], \quad 0 \leqslant i_{m-1} \leqslant [(n - m i_m)/(m-1)],$$

$$0 \leqslant i_{m-2} \leqslant [(n-I_{m-1}^{(m)})/(m-2)],$$

$$\ldots,\ 0 \leqslant i_r \leqslant [(n-I_{r+1}^{(m)})/r],\ \ldots,\ 0 \leqslant i_2 \leqslant [(n-I_3^{(m)})/2],\ i_1 = n - I_2^{(m)}$$

where $I_r^{(m)} = \sum_{l=r}^{m} l i_l$ and $[\cdot]$ is the whole part. Then p_n may be written as the m-fold sum

$$p_n = \exp\left\{-\sum_{r=1}^{k} a_r\right\} \sum_{i_r=0,\ r=\overline{m,2},\ i_1=n-I_2^{(m)}}^{[(n-I_{r+1}^{(m)})/r]} \prod_{r=1}^{m} a_r^{i_r}/i_r!,\quad n=0,\ 1,\ \ldots,$$

or for $k \geq 2$ as the $(m-1)$-fold sum $\left(p_0 = \exp\left\{-\sum_{r=1}^{k} a_r\right\}\right)$

$$p_n = p_0 \sum_{i_r=0,\ r=\overline{m,2}}^{[(n-I_{r+1}^{(m)})/r]} a_1^{n-I_2^{(m)}}/(n-I_2^{(m)})! \prod_{r=2}^{m} a_r^{i_r}/i_r!,\quad n=1,\ 2,\ \ldots. \tag{14}$$

For (14), as for (11), denoting $A_r^{(l)} = a_r^l/l!$, we obtain the following algorithm for the evaluation of p_n:

$$A_r^{(0)} = 1,\ A_r^{(i_r)} = a_r A_r^{(i_r-1)}/i_r,\ i_r = \overline{1,[n/r]},\ r = \overline{1,m},\ p_0 = e^{-\sum_{r=1}^{k} A_r^{(1)}}$$

$$p_n = p_0 \sum_{i_r=0,\ r=\overline{m,2}}^{[(n-I_{r+1}^{(m)})/r]} A_1^{(n-I_2^{(m)})} \prod_{r=2}^{m} A_r^{(i_r)},\quad n=1,\ 2,\ \ldots. \tag{15}$$

6. Note that the r.v.s $\eta_r = r\zeta_r$, $r = 1, \ldots, k$, are independent and follow the "r-fold" Poisson distribution

$$p_n^{(r)} = P\{\eta_r = n\} = \begin{cases} e^{-a_r} a_r^{n/r}/(n/r)!, & n \equiv 0 \pmod r, \\ 0, & n \not\equiv 0 \pmod r \end{cases}$$

with the p.g.f.

$$\Phi_{\eta_r}(z) = \Phi_{\zeta_r}(z^r) = \exp\{a_r(z^r - 1)\}.$$

Then the probability distribution $\{p_n\}$ of the random variable $\xi = \eta_1 + \eta_2 + \ldots + \eta_k$ may be written, as in (9), in terms of a $(k-1)$ composition of the distributions $\{p_n^{(r)}\}$:

$$\{p_n\} = \{p_n^{(1)}\} * \{p_n^{(2)}\} * \ldots * \{p_n^{(k)}\}.$$

By the obvious recursion for nonzero probabilities $p_n^{(r)}$

$$p_n^{(r)} = r a_r p_{n-r}^{(r)}/n,\quad n = rl,\quad l = 1,\ 2,\ \ldots,\ p_0^{(r)} = e^{-a_r},$$

the algorithm for evaluation of p_n is based on the equalities

$$p_0^{(r)} = e^{-a_r},\ r = \overline{1,k},\ p_n^{(r)} = \begin{cases} r a_r p_{n-r}^{(r)}/n, & n \equiv 0 \pmod r),\ n \neq 0, \\ 0, & n \not\equiv 0 \pmod r), \end{cases}$$
$$\{p_n\} = \{p_n^{(1)}\} * \{p_n^{(2)}\} * \ldots * \{p_n^{(k)}\}. \tag{16}$$

The algorithm (16) is applicable only when k is finite. Let us improve it in the following way.

7. Denoting $b_n^{(r)} = e^{a_r} p_n^{(r)}$, $r = 1, \ldots, k$, $\{b_n^{(0)}\} = (1, 0, \ldots)$, we have

$$\{p_n\} = \{e^{-a_1} b_n^{(1)}\} * \ldots * \{e^{-a_k} b_n^{(k)}\} = e^{-\sum_{r=1}^{k} a_r} \{b_n^{(1)}\} * \ldots * \{b_n^{(k)}\}.$$

Consider the case $n < k$. Since

$$\{b_n^{(r)}\} = (1, \underbrace{0, \ldots, 0}_{r-1}, a_r, \underbrace{0, \ldots, 0}_{r-1}, a_{r/2}^2, 0, \ldots),$$

TABLE 1

n	p_n	[9]	(19)	(20)	Double precision
0	0,00673	8	79	79	79469990855
1	0,0336	90	897	897	897349954273
2	0,08422	4	43	43	43374885683
3	0,14037	4	38	39	38958142805
4	0,17546	7	73	73	73697678507
5	0,17546	7	73	73	73697678507
6	0,14622	3	27	27	28081398756
7	0,10444	5	47	47	48629570540
8	0,06527	8	79	79	80393481587
9	0,03626	6	55	55	55774156437
10	0,01813	8	28	28	27887078219
11	0,00824	2	22	22	21766853736
12	0,00343	4	42	42	42402855723
13	0,00132	1	09	09	08616482970
14	0,00047	2	17	17	17363029632
15	0,00015	7	72	72	72454343211
Total	0,99993	6	01	02	09917581441

i.e., the first $(n+1)$ components of all sequences $\{b_n^{(r)}\}$, $r = n+1, \ldots, k$, are 1, 0, ..., 0, then $\{p_n\} = p_0\{b_n^{(1)}\} * \ldots * \{b_n^{(n)}\}$. Thus, for any $k = 1, 2, \ldots$, $n = 0, 1, \ldots$, we replace (16) with the algorithm

$$p_0 = \exp\Big\{-\sum_{r=1}^{k} a_r\Big\}, \quad b_0^{(r)} = 1, \quad r = \overline{1, m},$$
$$b_n^{(r)} = \begin{cases} r a_r b_{n-r/n}^{(r)}, & n \equiv 0 \pmod r, \quad n \neq 0, \\ 0, & n \not\equiv 0 \pmod r, \end{cases} \tag{17}$$
$$\{p_n\} = p_0\{b_n^{(1)}\} * \{b_n^{(2)}\} * \ldots * \{b_n^{(m)}\}.$$

8. Finally, let us consider an algorithm based on the recursive property of the CPD:

$$p_0 = \exp\Big\{-\sum_{r=1}^{k} a_r\Big\}, \quad p_n = \frac{1}{n}\sum_{r=1}^{m} r a_r p_{n-r}, \quad n = 1, 2, \ldots . \tag{18}$$

Indeed, from the inversion formula for the p.g.f. (12) and Leibniz formula we obtain the chain of equalities

$$p_n = \Phi^{(n)}(0)/n! = \lambda\,(\Phi(z)\,\Psi'(z))^{(n-1)}/n!\,|_{z=0} = \frac{\lambda}{n!}\sum_{l=0}^{n-1} C_{n-1}^{l}\,\Phi^{(n-1-l)}(0)\,\Psi^{(l+1)}(0) =$$

$$= \frac{\lambda}{n}\sum_{r=1}^{n}\frac{\Phi^{(n-r)}(0)}{(n-r)!}\,\frac{\Psi^{(r)}(0)}{(r-1)!} = \frac{\lambda}{n}\sum_{r=1}^{\min(k,n)} r f_r p_{n-r},$$

which coincides with (18).

9. The algorithms described above for the evaluation of the compound Poisson distribution p_n, $n = 0, 1, \ldots, N-1$, can be classified in three distinct groups. Algorithm (4) performs computations simultaneously for all the required N points. The algorithms of the second group (12), (15)-(17) evaluate p_n at one point n using only p_0 (with the exception of algorithm (16)). The recursive algorithm (18) successively evaluates $p_0, p_1, \ldots, p_{N-1}$ and is the simplest to implement.

The explicit dependence of p_n on p_0 in algorithms (12), (15), (17), (18) obviously makes them inappropriate for computer implementation when p_0 is the machine zero, e.g., when $\sum_{r=1}^{k} a_r > 44.361$ (for BÉSM-6) or $\sum_{r=1}^{k} a_r > 180.218$ (for ES-1045). For these algorithms, the evaluation of p_0 usually requires summing the numerical series $\sum_{r=1}^{\infty} a_r$. If the CPD is defined by the p.g.f. (2), then this sum is $\lambda(1 - \Psi(0)) = \lambda(1 - f_0)$. The algorithm (16) evaluates p_n when $\exp\{-\max_{1 \leq r \leq k} a_r\}$ is not the

TABLE 2

n	p_n	(4)	(12)	(15)	(16)	(17)	(18)	Double precision
0	0,006737	8	9	9	9	9	9	9469990855
1	0,030320	5	8	8	8	8	8	7614958846
2	0,071590	2	7	7	7	7	7	6868652831
3	0,117492	3	9	9	9	9	9	9507965528
4	0,15007	65	72	72	71	72	72	72413624427
5	0,15856	75	80	80	79	80	80	81073855090
6	0,143938	7	9	9	7	9	8	9540995388
7	0,115184	8	7	7	5	7	7	7715476334
8	0,08278	41	37	38	37	38	37	38032579862
9	0,054190	6	2	2	1	2	1	2095787301
10	0,03266	43	39	40	39	40	39	39746362272
11	0,01828	92	89	89	89	89	89	89177674320
12	0,009580	6	3	3	3	3	3	3420491393
13	0,004723	3	1	1	1	1	1	1120760430
14	0,002202	6	4	5	4	5	4	4533136666
15	0,000975	7	6	6	6	6	6	6101325029
Total	0,99931	87	92	95	85	95	90	98433636572

machine zero (p_0 may be the machine zero in this case). The dependence of the evaluated p_n on smallness of p_0 is weakened in algorithms (12), (15), (17) if we change to $\ln p_n$.

Let us demonstrate the evaluation of the compound Poisson distribution by each of the algorithms described above. For $k = 1$ for the compound Poisson distribution $p_n = e^{-a_1} a_1^n/n!$ the algorithms (12), (15), and (17) reduce to the procedure

$$p_0 = \exp\{-a_1\},\ b_0 = 1,\ b_n = a_1 b_{n-1}/n,\ p_n = p_0 b_n,\ n = 1, 2, \ldots, \tag{19}$$

and the algorithms (16) and (18) reduce to the recursion

$$p_0 = \exp\{-a_1\},\quad p_n = a_1 p_{n-1}/n,\quad n = 1, 2, \ldots . \tag{20}$$

The results obtained for p_n for $a_1 = 5$ on the ES-1045 computer in single and double precision are presented in Table 1.

Double-precision computation by the procedures (19) and (20) produces identical results up to the 16th digit after the decimal point, inclusive. Accepting these values of the distribution function as true, we can estimate the error of the algorithms in single-precision computation. We see from Table 1 that (19) and (20) produce values of the Poisson distribution function that coincide (up to rounding) with the tabulated values (see [9]), with the exception of p_{10} (due to a misprint in [9]). The results differ starting with the 7th digit due to rounding errors. The accuracy of the inversion algorithm (4) obviously depends on the number N of points where the distribution function is evaluated. Thus, it is only for $N \geq 64$ that we obtain six correct decimal places for the first 16 points in single-precision computation.

The specific features of algorithms (12), (15)-(18) obviously have an essential effect only for $k \geq 2$. For $k = 2$, for the Hermite distribution function [10, 11]

$$p_n = e^{-a_1 - a_2} \sum_{i=0}^{[n/2]} a_1^{n-2i}/(n-2i)!\ a_2^i/i!,\ n = 0, 1, \ldots . \tag{21}$$

The computation results with $a_1 = 4.5$, $a_2 = 0.5$ obtained on the ES-1045 computer are given in Table 2.

As in the previous case, when the input values a_1 and a_2 are exactly given, all the algorithms produce values of the distribution function (21) that coincide up to the sixth decimal digit (up to rounding). The differences in higher digits are attributable to rounding errors.

The same picture is observed for the evaluation of the Neyman type A distribution with two parameters λ_1, λ_2 [11, 12] with the p.g.f.

$$\Phi(z) = \exp\{\lambda_1 (e^{\lambda_2(z-1)} - 1)\} = \exp\Big\{\sum_{r=1}^{\infty} a_r (z^r - 1)\Big\},$$

where $a_r = \lambda_1 e^{-\lambda_2} \lambda_2^r/r!$. The results obtained for the corresponding distribution function

$$p_n = e^{-\lambda_1} \frac{\lambda_2^n}{n!} \sum_{r=0}^{\infty} \frac{(\lambda_1 e^{-\lambda_2})^r}{r!} r^n,\ n = 0, 1, \ldots$$

TABLE 3

n	p_n	(4)	(12)	(17)	(18)	Double precision
0	0,042400	0	2	2	2	1747986612
1	0,077990	4	7	7	7	7630525148
2	0,11072	28	33	33	33	33773469912
3	0,12870	46	50	50	50	52146942847
4	0,131282	2	4	4	4	6848831338
5	0,121442	0	1	1	1	4197128772
6	0,10396	70	68	69	69	71930918091
7	0,083514	9	6	6	7	9687065808
8	0,06357	10	06	06	07	09388114603
9	0,04619	41	37	37	38	39819498639
10	0,032227	7	4	4	4	5581023135
11	0,02168	52	49	49	48	49887620505
12	0,01412	51	48	48	48	49021920053
13	0,008933	9	7	7	7	7473532265
14	0,005500	8	7	7	7	7100207643
15	0,003304	5	3	3	3	3733745820
Total	0,99556	62	52	53	55	79968531191

by algorithms (4), (12), (17), (18) and the values obtained in double precision for $\lambda_1 = 5, \lambda_2 = 1$ are presented in Table 3. Note that here, contrary to the previous examples, the coefficients a_r are computed.

Our examples of the evaluation of the distribution function p_n for three typical representatives of the CPD class show that, with a_r, $r = 1, 2, \ldots$, known exactly or with machine accuracy, the values of p_n are computed correctly to six decimal places by all algorithms on ES-1045.

LITERATURE CITED

1. G. P. Patil and S. W. Joshi, A Dictionary and Bibliography of distributions, Oliver and Boyd, Edinburgh (1968).
2. Yu. V. Prokhorov and Yu. A. Rozanov, Probability Theory [in Russian], Nauka, Moscow (1973).
3. V. S. Korolyuk, N. I. Portenko, A. V. Skorokhod, and A. F. Turbin, Handbook of Probability Theory and Mathematical Statistics [in Russian], Moscow (1985).
4. C. S. Beightler, L. G. Mitten, and G. L. Nemhauser, "A short table of z-transforms and generating functions," Oper. Res., **9**, No. 4, 574-578 (1961).
5. T. Koneko and B. Liu, "Accumulation of round-off error in fast Fourier transforms," Commun. ACM, **17**, No. 4, 637-654 (1970).
6. H. Buhlmann, "Numerical evaluation of the compound Poisson distribution: recursion or fast Fourier transform?" Scand. Actuarial J., No. 2, 116-126 (1984).
7. W. Feller, An Introduction to Probability Theory and Its Application [Russian translation], Vol. 1, Mir, Moscow (1964).
8. H. Nussbaumer, Fast Fourier Transform and Convolution Algorithms, Springer, New York (1982).
9. L. N. Bol'shev and N. V. Smirnov, Tables of Mathematical Statistics [in Russian], Nauka, Moscow (1983).
10. A. W. Kemp and C. D. Kemp, "Some properties of Hermite distribution," Biometrika, **52**, 381-394 (1965).
11. A. G. Belov, V. Ya. Galkin, and M. V. Ufimtsev, Probabilistic-Statistical Problems in Experimental Separation of Multiple Processes [in Russian], Moscow State Univ. (1985).
12. J. Neyman, "On a new class of 'contagious' distributions, applicable in entomology and bacteriology," Ann. Math. Stat., **10**, 35-57 (1939).