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COMPOUND POISSON LAW GENERALIZED BY NEGATIVE BINOMIAL DISTRIBUTION

A. G. Belov, V. Ya. Galkin, and M. V. Ufimtsev

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For the compound Poisson law generalized by the negative binomial distribution we derive the explicit representation of the probability function, the finite-difference recurrence, and expressions for the derivatives with respect to all parameters. Explicit representations of ordinary and factorial cumulants to sixth order are given and asymptotic normality of the distribution is proved. The distribution functions are constructed and analyzed for the most typical parameter values. Used to solve direct and inverse problems by moment methods.

Introduction

Generalized Poisson distributions provide models of multiple processes in many areas of science [2–5]. Let us consider one such generalization.

Given is a discrete random variable (r.v.)

$$\zeta = \zeta_1 + 2\zeta_2 + \ldots + k\zeta_k$$

which is the weighted sum of k independent Poisson r.v.s $\zeta_j \sim \mathrm{Po}(\lambda)$. The characteristic function (c.f.) for ζ is

$$\varphi_\zeta(t) = Ee^{it\zeta} = \prod_{j=1}^{k} \exp\left\{\lambda(e^{ijt} - 1)\right\} = \exp\left\{\lambda \sum_{j=1}^{k} (e^{ijt} - 1)\right\}$$

and its distribution function (d.f.) $q_n = P\{\zeta = n\}$ is written as a $(k-1)$-dimensional sum with relationships [2, 6]

$$q_n = e^{-k\lambda} \sum_{I_1^{(k)}=n} \lambda^{J_1^{(k)}} \prod_{j=1}^{k} \left(i_j!\right)^{-1} = e^{-k\lambda} \sum_{i_j=0,\, j=k,\ldots,2}^{\left[\left(n-I_{j+1}^{(k)}\right)/j\right]} \frac{\lambda^{n-I_2^{(k)}+J_2^{(k)}}}{\left(n-I_2^{(k)}\right)!} \prod_{j=2}^{k} \left(i_j!\right)^{-1}, \quad n = 0, 1, \ldots .$$

Here $I_l^{(j)} = \sum_{\alpha=l}^{j} \alpha i_\alpha$, $J_l^{(j)} = \sum_{\alpha=l}^{j} i_\alpha$ for all integer $l \le j$, $i_l, \ldots, i_j \ge 0$, $[z]$ is the whole part of z.

Now consider the conditional r.v. ξ/ζ that follows the negative binomial distribution, $\xi/\zeta \sim NB(r\zeta, p)$, with the c.f.

$$\varphi_{\xi/\zeta}(t) = \frac{p^{r\zeta}}{(1-qe^{it})^{r\zeta}},$$

where $r\zeta$ is the number of "successes", p is the probability of "success" in a single trial, $q = 1 - p$ is the probability of "no success", $0 < r$ is a real number, $0 < p < 1$, $0 < q < 1$. Using a different system of param-

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eters $Q = 1/p,\ P = q/p,\ P > 0,\ Q - P = 1$ [7, 8], we write the c.f. as

$$\varphi_{\xi/\zeta}(t) = \left(Q - Pe^{it}\right)^{-r\zeta}.$$

For the r.v. ξ following a compound Poisson law of order k generalized by a negative binomial distribution $\xi \sim P_k NB(\lambda, r, P)$ the c.f. is

$$\varphi_\xi(t) = \varphi(t) = \exp\left\{\lambda \sum_{j=1}^{k} \left\{\left(Q - Pe^{it}\right)^{-jr} - 1\right\}\right\} \tag{1}$$

and the corresponding probability generating function (p.g.f.) is

$$\psi_\xi(z) = \psi(z) = \exp\left\{\lambda \sum_{j=1}^{k} \left\{(Q - Pz)^{-jr} - 1\right\}\right\} = \exp\left\{\lambda \sum_{j=1}^{k} \left(h_j(z) - 1\right)\right\}, \tag{2}$$

where $h_j(z) = (Q - Pz)^{-jr}$ $(|z| \le 1)$ is the p.g.f. of some negative binomial r.v. $X_j \sim NB(rj, P)$ with the d.f. [7]

$$\pi_m^{(j)} = P\left\{X_j = m\right\} = \frac{\Gamma(rj + m)}{\Gamma(rj)m!} P^m Q^{-jr-m}.$$

This d.f. has the obvious recurrences

$$\pi_{m+1}^{(j)} = \frac{rj + m}{m+1} \frac{P}{Q} \pi_m^{(j)} = \frac{rj}{m+1} \sum_{l=0}^{m} \left(\frac{P}{Q}\right)^{l+1} \pi_{m-l}^{(j)}, \tag{3}$$

where $\pi_0^{(j)} = Q^{-jr}$, $m = 0, 1, \dots$, and $\Gamma(x)$ is the gamma function,

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(1) = 1.$$

To prove the second equality in (3), we have

$$\pi_{m+1}^{(j)} = \frac{1}{(m+1)!} \frac{d^{m+1}}{dz^{m+1}} h_j(z)\bigg|_{z=0} = \frac{j\,\mathrm{Pr}}{(m+1)!} \frac{d^m}{dz^m} \frac{h_j(z)}{Q - Pz}\bigg|_{z=0}$$

$$= \frac{j\,\mathrm{Pr}}{(m+1)!} \sum_{l=0}^{m} C_m^l \frac{d^l}{dz^l} \frac{1}{Q - Pz} \frac{d^{m-l} h_j(z)}{dz^{m-l}}\bigg|_{z=0} = \frac{jr}{m+1} \sum_{l=0}^{m} \frac{P^{l+1}}{Q^{l+1}} \pi_{m-l}^{(j)}.$$

1. Explicit Form

Let $A_l = \sum_{j=1}^{l} j a_j$, $B_l = \sum_{j=1}^{l} a_j$, where $a_j \ge 0$ are integers.

Assertion 1. *The d.f.* $p_n = P\{\xi = n\}$ *has the representation*

$$p_n = p_0\left(\frac{P}{Q}\right)^n \sum_{m=0}^{n} \lambda^m \sum_{l=0}^{m} \frac{\left(k - \sum_{j=1}^{k} Q^{-jr}\right)^{m-l}}{(m-l)!} \sum_{B_{k+1}=l} \frac{(-k)^{a_{k+1}}}{a_{k+1}! Q^{rA_k}} \sum_{J_1^{(k)}=n} \prod_{j=1}^{k} \frac{\Gamma\left(jra_j + i_j\right)}{a_j! i_j! \Gamma\left(jra_j\right)}, \tag{4}$$

where $p_0 = \exp\left\{\lambda \sum_{j=1}^{k} \left(Q^{-jr} - 1\right)\right\}$, $n = 0, 1, \ldots$.

Proof. Inverting the p.g.f.

$$p_n = \frac{1}{n!} \frac{d^n}{dz^n} \psi(z)\bigg|_{z=0}, \quad n = 0, 1, \ldots, \tag{5}$$

and using (2), we obtain p_0 by setting $z = 0$. Now represent the p.g.f. (2) in the form $\psi(z) = \psi(y(z)) = \exp\{\lambda y(z)\}$, where $y(z) = \sum_{j=1}^{k} (Q - Pz)^{-jr} - k$, and apply the formula of [9, p. 78] for the n th derivative of a compound function:

$$\frac{d^n}{dz^n} \psi(y(z)) = \sum_{m=0}^{n} \frac{1}{m!} \frac{\partial^m \psi(y)}{\partial y^m} \sum_{l=0}^{m} C_m^l (-y)^{m-l} \frac{d^n}{dz^n} y^l(z). \tag{6}$$

In our case, we have

$$\frac{\partial^m}{\partial y^m} (\psi(y))\bigg|_{z=0} = \lambda^m \psi(y)\big|_{z=0} = \lambda^m p_0, \quad y(0) = \sum_{j=1}^{k} Q^{-jr} - k,$$

and using the multinomial expansion formula [10, p. 152]

$$\left(\sum_{j=1}^{k+1} u_j\right)^n = n! \sum_{B_{k+1}=n} \prod_{j=1}^{k+1} \frac{u_j^{a_j}}{a_j!}$$

followed by the generalized Leibniz formula for the n th derivative of the product [2, p. 33]

$$\frac{d^n}{dx^n} \prod_{j=1}^{k} u_j(x) = n! \sum_{J_1^{(k)}=n} \prod_{j=1}^{k} \frac{1}{i_j!} \frac{d^{i_j}}{dx^{i_j}} u_j(x)$$

for $z = 0$, we obtain

$$\frac{d^n}{dz^n} y^l(z)\bigg|_{z=0} = l! \sum_{B_{k+1}=l} \frac{(-k)^{a_{k+1}}}{a_{k+1}!} \frac{d^n}{dz^n} \prod_{j=1}^{k} \frac{(Q - Pz)^{-jra_j}}{a_j!}\Bigg|_{z=0}$$

$$= l! \sum_{B_{k+1}=l} \frac{(-k)^{a_{k+1}}}{a_{k+1}!} n! \sum_{J_1^{(k)}=n} \prod_{j=1}^{k} \frac{1}{i_j!} \frac{1}{a_j!} P^{i_j} Q^{-ra_j - i_j} \left(jra_j\right)\left(jra_j + 1\right) \times \ldots \times \left(jra_j + i_j - 1\right)$$

$$= \; l!n!\left(\frac{P}{Q}\right)^n \sum_{B_{k+1}=l} \frac{(-k)^{a_{k+1}}}{a_{k+1}!Q^{rA_k}} \sum_{J_1^{(k)}=n} \prod_{j=1}^{k} \frac{\Gamma\left(jra_j + i_j\right)}{a_j!i_j!\Gamma\left(jra_j\right)}.$$

Substituting these expressions in (6) for $z = 0$, we obtain (4).

The representation (4) is much too complex for regular use. We accordingly derive a recurrence form of this representation.

2. Recurrence Form

Assertion 2. *For all* $n = 0, 1, \dots,$ *we have*

$$(n+1)p_{n+1} \; = \; \lambda \sum_{i=0}^{n} p_i (n+1-i) \sum_{j=1}^{k} \pi_{n+1-i}^{(j)}. \tag{7}$$

Proof. From (5), differentiating (2) once and applying the Leibniz formula we obtain

$$p_{n+1} \; = \; \frac{1}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \exp\left\{\lambda \sum_{j=1}^{k} \left(h_j(z) - 1\right)\right\}\Bigg|_{z=0}$$

$$= \; \frac{\lambda r P}{(n+1)!} \frac{d^n}{dz^n} \frac{\psi(z)}{Q - Pz} \sum_{j=1}^{k} j h_j(z)\Bigg|_{z=0}$$

$$= \; \frac{\lambda r P}{(n+1)!} \sum_{i=0}^{n} C_n^i \frac{d^i \psi(z)}{dz^i} \sum_{l=0}^{n-i} C_{n-i}^l \frac{d^l}{dz^l} \frac{1}{Q - Pz} \frac{d^{n-i-l}}{dz^{n-i-l}} \sum_{j=1}^{k} j h_j(z)\Bigg|_{z=0}$$

$$= \; \frac{\lambda r}{n+1} \sum_{i=0}^{n} p_i \sum_{l=0}^{n-i} \frac{P^{l+1}}{Q^{l+1}} \sum_{j=1}^{k} j \pi_{n-i-l}^{(j)}.$$

Using (3) and setting $m = n - i$, we obtain (7).

Relationship (7) and (3) substantially simplify the evaluation of the d.f. p_n.

3. Derivatives with Respect to Parameters

Assertion 3. *For* $n = 0, 1, \dots,$ *we have the following three equalities:*

$$\frac{\partial p_n}{\partial \lambda} \; = \; \sum_{i=0}^{n} p_i \sum_{j=1}^{k} \pi_{n-i}^{(j)} - k p_n, \qquad \frac{\partial p_n}{\partial P} \; = \; \frac{1}{P}\{ n p_n - (n+1) p_{n+1} \},$$

$$\frac{\partial p_n}{\partial r} \; = \; \lambda \sum_{i=0}^{n} p_i \left\{ \sum_{l=1}^{n-i} \frac{1}{l} \left(\frac{P}{Q}\right)^l \sum_{j=1}^{k} j \pi_{n-i-l}^{(j)} - \ln Q \sum_{j=1}^{k} j \pi_{n-i}^{(j)} \right\}.$$

Proof. From representation (5) and the Leibniz formula we have

$$\frac{\partial p_n}{\partial \lambda} = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\frac{\partial \psi}{\partial \lambda}\bigg|_{z=0} = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left\{ \psi(z)\sum_{j=1}^{k} h_j(z) - k\psi(z) \right\}\bigg|_{z=0}$$

$$= \frac{1}{n!}\frac{\partial^n}{\partial z^n}\psi(z)\sum_{j=1}^{k} h_j(z)\bigg|_{z=0} - kp_n$$

$$= \sum_{i=0}^{n}\frac{d^i\psi(z)}{i!dz^i}\sum_{j=1}^{k}\frac{d^{n-i}h_j(z)}{(n-i)!dz^{n-i}}\bigg|_{z=0} - kp_n$$

$$= \sum_{i=0}^{n} p_i \sum_{j=1}^{k}\pi_{n-i}^{(j)} - kp_n .$$

Similarly,

$$\frac{\partial p_n}{\partial P} = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\frac{\partial \psi}{\partial P}\bigg|_{z=0} = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left\{\frac{(z-1)}{P}\frac{\partial\psi(z)}{\partial z}\right\}\bigg|_{z=0} = \frac{1}{P}\{np_n - (n+1)p_{n+1}\},$$

$$\frac{\partial p_n}{\partial r} = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left\{\lambda\psi(z)\sum_{j=1}^{k}\frac{\partial}{\partial r}h_j(z)\right\}\bigg|_{z=0}$$

$$= \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left\{-\lambda\psi(z)\ln(Q-Pz)\sum_{j=1}^{k} jh_j(z)\right\}\bigg|_{z=0}$$

$$= -\lambda\left\{\sum_{i=0}^{n}\frac{\partial^i\psi(z)}{i!\partial z^i}\sum_{l=0}^{n-i}\frac{\partial^l \ln(Q-Pz)}{l!\partial z^l}\frac{\partial^{n-i-l}}{(n-i-l)!\partial z^{n-i-l}}\sum_{j=1}^{k} jh_j(z)\right\}\bigg|_{z=0}$$

$$= -\lambda\sum_{i=0}^{n} p_i\left\{\ln Q\sum_{j=1}^{k} j\pi_{n-i}^{(j)} - \sum_{l=1}^{n-i}\frac{1}{l}\left(\frac{P}{Q}\right)^l\sum_{j=1}^{k} j\pi_{n-i-l}^{(j)}\right\}.$$

4. Cumulants

Only the first initial moment and the second central moment are considered in [1]. Let us now write out the other moments, e.g., the ordinary and the factorial cumulants κ_m and $(\kappa)_m$. The factorial cumulant generating function (f.c.g.f.) $w(t) = \ln\psi(1+t)$ from (1) and the relationship $Q - P = 1$ is

$$w(t) = \lambda\sum_{j=1}^{k}\left\{(1-Pt)^{-jr} - 1\right\}. \tag{8}$$

Inverting (8), we obtain for $t = 0$ $\left((\kappa)_0 = 0\right)$

$$(\kappa)_m = \frac{d^m}{dt^m} w(t)\Big|_{t=0} = \frac{d^m}{dt^m} \ln \psi(1+t)\Big|_{t=0} = \lambda \sum_{j=1}^{k} \frac{\Gamma(jr+m)}{\Gamma(jr)} P^m, \quad m = 1, 2, \dots .$$

In particular, using the standard equalities [11, p. 36]

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}, \quad \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}, \quad \sum_{j=1}^{k} j^3 = \frac{k^2(k+1)^2}{4},$$

$$\sum_{j=1}^{k} j^4 = \frac{k(k+1)}{2}\frac{(2k+1)}{15}\left(3k^2+3k-1\right), \quad \sum_{j=1}^{k} j^5 = \frac{k(k+1)}{2}\frac{k(k+1)}{6}\left(2k^2+2k-1\right),$$

$$\sum_{j=1}^{k} j^6 = \frac{k(k+1)}{2}\frac{(2k+1)}{21}\left(3k^4+6k^3-3k+1\right),$$

we successively obtain

$$(\kappa)_1 = \lambda \mathrm{Pr} \sum_{j=1}^{k} j = \lambda \mathrm{Pr} \frac{k(k+1)}{2},$$

$$(\kappa)_2 = \lambda P^2 r \sum_{j=1}^{k} j(jr+1) = \lambda P^2 r \frac{k(k+1)}{6} K_1(k,r),$$

$$(\kappa)_3 = \lambda P^3 r \sum_{j=1}^{k} j(jr+1)(jr+2) = \lambda P^3 r \frac{k(k+1)}{4} K_2(k,r),$$

$$(\kappa)_4 = \lambda P^4 r \sum_{j=1}^{k} j(jr+1)(jr+2)(jr+3) = \lambda P^4 r \frac{k(k+1)}{30} K_3(k,r),$$

$$(\kappa)_5 = \lambda P^5 r \sum_{j=1}^{k} j(jr+1)(jr+2)(jr+3)(jr+4) = \lambda P^5 r \frac{k(k+1)}{12} K_4(k,r),$$

$$(\kappa)_6 = \lambda P^6 r \sum_{j=1}^{k} j(jr+1)(jr+2)(jr+3)(jr+4)(jr+5) = \lambda P^6 r \frac{k(k+1)}{84} K_5(k,r),$$

where we use the following notation for the polynomials:

$$K_1(k,r) = r(2k+1) + 3,$$

$$K_2(k, r) = r^2k(k+1) + 2r(2k+1) + 4,$$

$$K_3(k, r) = r^3(2k+1)(3k^2+3k-1) + 45r^2k(k+1) + 55r(2k+1) + 90,$$

$$K_4(k, r) = r^4k(k+1)(2k^2+2k-1) + 4r^3(2k+1)(3k^2+3k-1) + 105r^2k(k+1) + 100r(2k+1) + 144,$$

$$K_5(k, r) = 2r^5(2k+1)(3k^4+6k^3-3k+1) + 105r^4k(k+1)(2k^2+2k-1)$$
$$+ 238r^3(2k+1)(3k^2+3k-1) + 4725r^2k(k+1) + 3836r(2k+1) + 5040.$$

The expressions for κ_m follow from the inversion [12]

$$\kappa_m = \frac{d^m}{ds^m}\theta(s)\bigg|_{s=0}, \quad m = 1, 2, \dots,$$

where the cumulant generating function (c.g.f.) is

$$\theta(s) = \ln\psi(e^s) = \lambda\sum_{j=1}^{k}\left\{\left(Q - Pe^s\right)^{-jr} - 1\right\} = \lambda\sum_{j=1}^{k}\left\{y^{-jr} - 1\right\} = F(y(s)),$$
$$y(s) = y = Q - Pe^s. \tag{9}$$

To invert (9) for $s = 0$ we apply the Bruno formula [13, p. 48] for the nth derivative of the compound function $\theta(s) = F(y(s))$:

$$\frac{d^m}{ds^m}\theta(s)\bigg|_{s=0} = m!\sum_{n=0}^{m}\frac{d^n}{dy^n}F(y)\sum_{\substack{I_1^{(m)}=m,\\ J_1^{(m)}=n}}\prod_{l=1}^{m}\frac{\left(\frac{d^l}{ds^l}y(s)\right)^{i_l}}{(l!)^{i_l}\, i_l!}\Bigg|_{s=0}$$

$$= \lambda m!\sum_{n=1}^{m}(-1)^n\sum_{j=1}^{k}\frac{\Gamma(jr+n)}{\Gamma(jr)}\frac{1}{y^{jr+n}}\sum_{\substack{I_1^{(m)}=m,\\ J_1^{(m)}=n}}\prod_{l=1}^{m}\frac{\left(-Pe^s\right)^{i_l}}{(l!)^{i_l}\, i_l!}\Bigg|_{s=0}$$

$$= \lambda m!\sum_{n=1}^{m}(-1)^n\sum_{j=1}^{k}\frac{\Gamma(jr+n)}{\Gamma(jr)}\sum_{\substack{I_1^{(m)}=m,\\ J_1^{(m)}=n}}\prod_{l=1}^{m}\frac{(-P)^{i_l}}{(l!)^{i_l}\, i_l!}.$$

Successively substituting $m = 1, 2, \dots$ we obtain the cumulants κ_m.

Let us apply the relationship of κ_m with factorial cumulants $(\kappa)_m$ in terms of Stirling numbers of first and second kind $s_m^{(j)}$ and $\sigma_m^{(j)}$ respectively [12, 14]:

$$(\kappa)_m = \sum_{j=0}^{m} s_m^{(j)} \kappa_j, \qquad \kappa_m = \sum_{j=0}^{m} \sigma_m^{(j)} (\kappa)_j \tag{10}$$

and also Stirling numbers of second kind [6, p. 49] $\sigma_m^{(j)}$, $m = 1, 2, \ldots, 6$, $\sigma_0^{(0)} = 1$, $j = 1, 2, \ldots, 6$, collected in Table 1.

Table 1

$\sigma_1^1 = 1$					
$\sigma_2^1 = 1$	$\sigma_2^2 = 1$				
$\sigma_3^1 = 1$	$\sigma_3^2 = 3$	$\sigma_3^3 = 1$			
$\sigma_4^1 = 1$	$\sigma_4^2 = 7$	$\sigma_4^3 = 6$	$\sigma_4^4 = 1$		
$\sigma_5^1 = 1$	$\sigma_5^2 = 15$	$\sigma_5^3 = 25$	$\sigma_5^4 = 10$	$\sigma_5^5 = 1$	
$\sigma_6^1 = 1$	$\sigma_6^2 = 31$	$\sigma_6^3 = 90$	$\sigma_6^4 = 65$	$\sigma_6^5 = 15$	$\sigma_6^6 = 1$

Substituting $(\kappa)_m$ and $\sigma_m^{(j)}$ in the second equality in (10), we successively obtain after simple manipulations

$$\kappa_1 = \alpha_1 = \lambda \Pr \frac{k(k+1)}{2},$$

$$\kappa_2 = \mu_2 = \lambda \Pr \frac{k(k+1)}{6}(3 + PK_1(k, r)),$$

$$\kappa_3 = \lambda \Pr \frac{k(k+1)}{4}\left(2 + 2PK_1(k, r) + P^2 K_2(k, r)\right),$$

$$\kappa_4 = \lambda \Pr \frac{k(k+1)}{30}\left(15 + 35PK_1(k, r) + 45P^2 K_2(k, r) + P^3 K_3(k, r)\right),$$

$$\kappa_5 = \lambda \Pr \frac{k(k+1)}{12}\left(6 + 30PK_1(k, r) + 75P^2 K_2(k, r) + 4P^3 K_3(k, r) + P^4 K_4(k, r)\right),$$

$$\kappa_6 = \lambda \Pr \frac{k(k+1)}{84}\left(42 + 434PK_1(k, r) + 1890P^2 K_2(k, r) + 182P^3 K_3(k, r) + 105P^4 K_4(k, r) + P^5 K_5(k, r)\right),$$

where $\alpha_1 = E\xi$ and $\mu_2 = E(\xi - \alpha_1)^2$ are the first initial moment and the second central moment of the r.v. ξ.

5. Asymptotic Behavior

Assertion 4. *For constant* r, P *and* $\lambda \to \infty$, *the r.v.* ξ *is asymptotically normal with the parameters* α_1, μ_2.

Proof. Consider the c.f. for the standardized r.v. $\tau = (\xi - \alpha_1)/\sqrt{\mu_2}$. Applying the c.f. $\varphi_\xi(t)$ (1), we obtain [10, p. 133]

$$\varphi_\tau(t) = \exp\left\{-\frac{it\alpha_1}{\sqrt{\mu_2}}\right\}\varphi_\xi\left(\frac{t}{\sqrt{\mu_2}}\right).$$

Applying the Taylor formula to $\varphi_\xi(t)$ and using (1), we obtain

$$\varphi_\tau(t) = \exp\left\{-\frac{it\alpha_1}{\sqrt{\mu_2}} + \lambda\sum_{j=1}^{k}\left(\left(1 - P\left(\frac{it}{\sqrt{\mu_2}} - \frac{t^2}{2!\mu_2} + O\left(\lambda^{-3/2}\right)\right)\right)^{-jr} - 1\right)\right\}$$

$$= \exp\left\{-\frac{it\alpha_1}{\sqrt{\mu_2}} + \lambda\frac{itP}{\sqrt{\mu_2}}\sum_{j=1}^{k} jr - \lambda\frac{t^2}{2!\mu_2}\left(P\sum_{j=1}^{k} jr + P^2\sum_{j=1}^{k} jr(jr+1)\right) + O\left(\lambda^{-1/2}\right)\right\}$$

$$= \exp\left\{-\frac{it\alpha_1}{\sqrt{\mu_2}} + \frac{it\alpha_1}{\sqrt{\mu_2}} - \frac{t^2\mu_2}{2!\mu_2} + O\left(\lambda^{-1/2}\right)\right\}$$

$$= \exp\left\{-\frac{t^2}{2} + O\left(\lambda^{-1/2}\right)\right\}.$$

By the Levy theorem [10, p. 134] the last equality implies convergence in distribution of the r.v. τ to the standard normal distribution $\mathrm{N}(0,1)$. Returning to the original r.v. ξ, we apply the strong reproducibility of the normal distribution [10, p. 172] to prove the assertion.

6. Evaluation of the Distribution Function

The recurrences (3), (7) have been used to evaluate the d.f. p_n (4) for typical values of the parameters λ, r, P, k. The probability values are plotted along the vertical axis and joined by a smooth curve (instead of a histogram) for argument values $n = 0, 1, \ldots$ marked along the horizontal axis.

Figures 1a–1c for $k = 1$ illustrate the behavior of the Poisson – Pascal distribution function [7]. Figures 1, 2a–2c demonstrate the asymptotic normality of the d.f. p_n as $\lambda \to \infty$. We see that for small λ (λ = 1, 2) the distribution function "sags" at $n = 1$ (see Figs. 1, 2a–2c). As p decreases the "sagging" becomes noticeable also for large λ (λ = 4, 6), and the d.f. itself is "smeared" in n. For $k = 2$ and small λ (λ = 1, 2) the "sagging" is more pronounced (compare Fig. 1b with Fig. 2b, Fig. 1c with Fig. 2c).

As k increases (Figs. 3a, 3b), or as λ increases for small p (Figs. 1b, 1c, 2b, 2c), or as r increases (Figs. 4a, 4b), the "sagging" of the broken d.f. p_n at the point $n = 1$ becomes less pronounced.

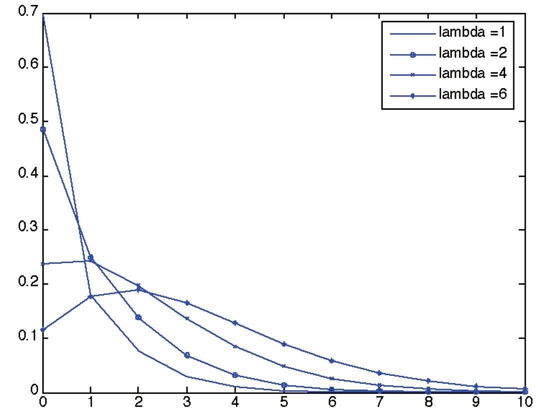

Fig. 1a. $k = 1,\ p = 0.8,\ r = 2.$

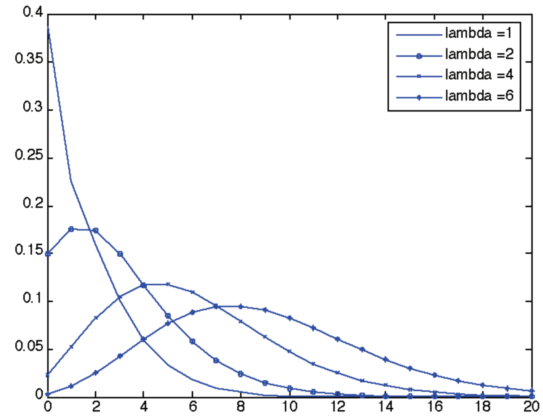

Fig. 2a. $k = 2,\ p = 0.8,\ r = 2.$

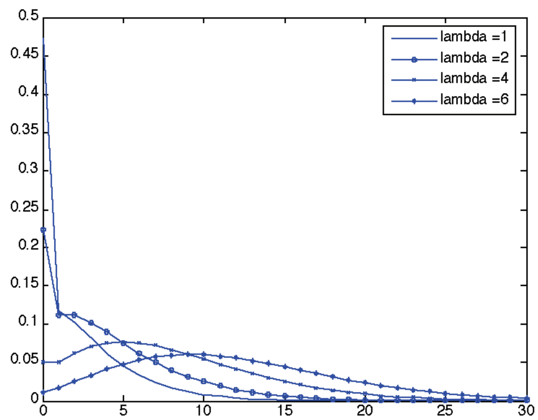

Fig. 1b. $k = 1,\ p = 0.5,\ r = 2.$

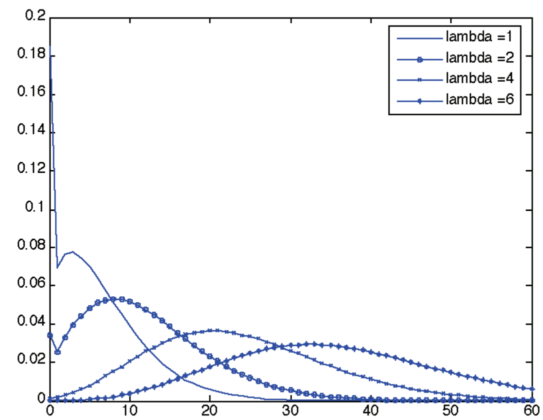

Fig. 2b. $k = 2,\ p = 0.5,\ r = 2.$

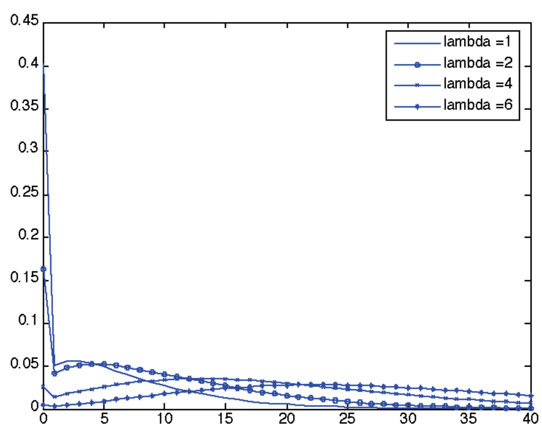

Fig. 1c. $k = 1,\ p = 0.3,\ r = 2.$

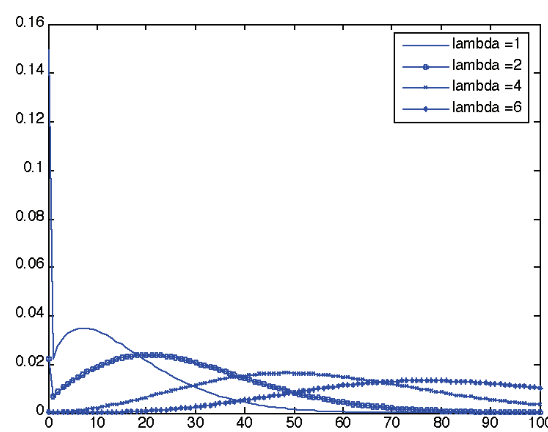

Fig. 2c. $k = 2,\ p = 0.3,\ r = 2.$

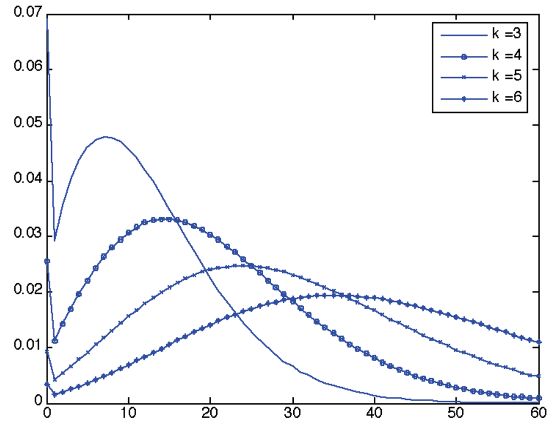

Fig. 3a. λ = 1, $p = 0.5$, $r = 2$.

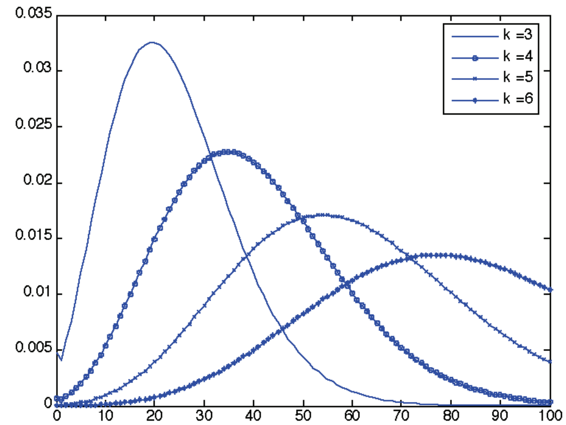

Fig. 3b. λ = 2, $p = 0.5$, $r = 2$.

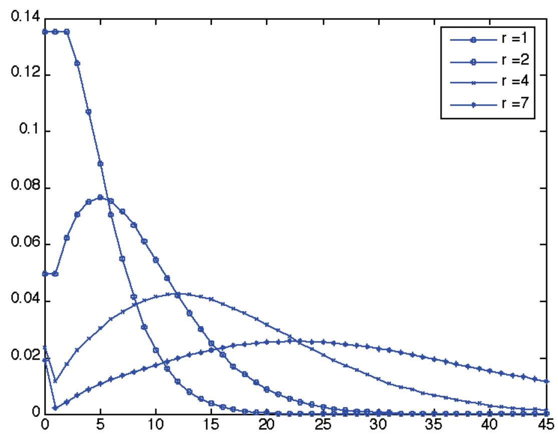

Fig. 4a. $k = 1$, λ = 4, $p = 0.5$.

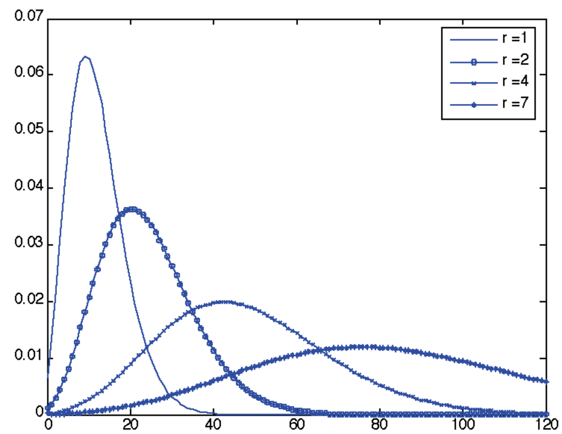

Fig. 4b. $k = 2$, λ = 4, $p = 0.5$.

The calculations and the graphs did not reveal any special structure of the d.f. at other points. There is no justification to attribute the break to the "multiplicity" effect characteristic of a number of other compound Poisson distributions [6, 15].

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