Independence of the B-KK Isomorphism of Infinite Prime

Alexei Kanel-Belov and Andrey Elishev

Abstract

We investigate a certain class of ind-group homomorphisms between the automorphism group of the $n$-th complex Weyl algebra and the group of Poisson structure-preserving automorphisms of the commutative complex polynomial algebra in $2n$ variables. An open conjecture of Kanel-Belov and Kontsevich states that these automorphism groups are canonically isomorphic in characteristic zero, with the mapping discussed here possibly realizing the isomorphism. The goal of the present paper is to establish the independence of the said mapping of the choice of infinite prime - that is, the class $[p]$ of prime number sequences modulo some non-principal ultrafilter $U$ on the set of positive integers. This independence of non-constructible objects issue, which is quite intricate (often resolving negatively in similar settings), is closely related to the famous Dixmier conjecture on Weyl algebra endomorphisms as well as to a number of growth/dimension problems in universal algebraic geometry.

1 The problem

The Belov-Kontsevich conjecture $[\mathbb{I}]$, sometimes Kanel-Belov - Kontsevich conjecture, dubbed $B-KKC_n$ for positive integer $n$, seeks to establish a canonical isomorphism between algebra automorphism groups

$$\text{Aut}(A_{n,\mathbb{C}}) \simeq \text{Aut}(P_{n,\mathbb{C}}).$$

Here $A_{n,\mathbb{C}}$ is the $n$-th Weyl algebra over the field of complex numbers

$$A_{n,\mathbb{C}} = \mathbb{C}(x_1, \ldots, x_n, y_1, \ldots, y_n)/(x_ix_j - x_jx_i, y_iy_j - y_jy_i, y_ix_j - x_jy_i - \delta_{ij}),$$

and $P_{n,\mathbb{C}} \simeq \mathbb{C}[z_1, \ldots, z_{2n}]$ is the commutative polynomial ring viewed as a $\mathbb{C}$-algebra and equipped with the standard Poisson bracket:

$$\{z_i, z_j\} = \omega_{ij} \equiv \delta_{i,n+j} - \delta_{i+n,j}$$

The automorphisms from $\text{Aut}(P_{n,\mathbb{C}})$ preserve the Poisson bracket.

Let $\zeta_i$, $i = 1, \ldots, 2n$ denote the standard generators of the Weyl algebra (the images of $x_j$, $y_i$ under the canonical projection). The filtration by total degree on $A_{n,\mathbb{C}}$ induces a filtration on the automorphism group:

$$\text{Aut}^{\leq N} A_{n,\mathbb{C}} := \{f \in \text{Aut}(A_{n,\mathbb{C}}) \mid \deg f(\zeta_i), \deg f^{-1}(\zeta_i) \leq N, \forall i = 1, \ldots, 2n\}.$$

The obvious maps

$$\text{Aut}^{\leq N} A_{n,\mathbb{C}} \to \text{Aut}^{\leq N+1} A_{n,\mathbb{C}}$$

are Zariski-closed embeddings, the entire group $\text{Aut}(A_{n,\mathbb{C}})$ is a direct limit of the inductive system formed by $\text{Aut}^{\leq N}$ together with these maps. The same can be said for the symplectomorphism group $\text{Aut}(P_{n,\mathbb{C}})$.

The Belov-Kontsevich conjecture admits a stronger form, with $\mathbb{C}$ being replaced by the rationals. The latter conjecture will not be treated here in any way.

1
Since Makar-Limanov [6], [7], Jung [4] and van der Kulk [5], the B-KK conjecture is known to be true for \( n = 1 \). The proof is essentially a direct description of the automorphism groups. Such a direct approach however seems to be completely out of reach for all \( n > 1 \). Nevertheless, at least one known candidate for isomorphism may be constructed in a rather straightforward fashion. The idea is to start with an arbitrary Weyl algebra automorphism, lift it after a shift by a certain automorphism of \( \mathbb{C} \) to an automorphism of a larger algebra (of formal power series with powers taking values in the ring \( \ast \mathbb{Z} \) of hyperintegers) and then restrict to a subset of its center isomorphic to \( \mathbb{C}[z_1, \ldots, z_{2n}] \).

This construction goes back to Tsuchimoto [9], who devised a morphism \( \text{Aut}(A_{n,\mathbb{C}}) \to \text{Aut}(P_{n,\mathbb{C}}) \) in order to prove the stable equivalence between the Jacobian and the Dixmier conjectures. It was independently considered by Kontsevich and Kanel-Belov [2], who offered a shorter proof of the Poisson structure preservation which does not employ \( p \)-curvatures. It should be noted, however, that Tsuchimoto’s thorough inquiry into \( p \)-curvatures has exposed an array of problems of independent interest, in which certain statements from the present paper might well appear.

The construction that we describe in detail in the next section differs from that of Tsuchimoto in one aspect: a Weyl algebra automorphism \( f \) may in fact undergo a shift by an automorphism of the base field \( \gamma : \mathbb{C} \to \mathbb{C} \) prior to being lifted, and this extra procedure is homomorphic. Taking \( \gamma \) to be the inverse nonstandard Frobenius automorphism (see below), we manage to get rid of the coefficients of the form \( a^{[p]} \), with \( [p] \) an infinite prime, in the resulting symplectomorphism. The key result here is that for a large subgroup of automorphisms, the so-called tame automorphisms, one manages to completely eliminate the dependence of the whole construction on the choice of the infinite prime \( [p] \). Also, the resulting ind-group morphism \( \phi_{[p]} \) is an isomorphism of the tame subgroups. In particular, for \( n = 1 \) all automorphisms of \( A_{1,\mathbb{C}} \) are tame (Makar-Limanov’s theorem), and the map \( \phi_{[p]} \) is the conjectured canonical isomorphism.

These observations motivate the question whether for any \( n \) the group homomorphism \( \phi_{[p]} \) is independent of infinite prime.

2 The homomorphism

2.1 Ultrafilters and infinite primes

Let \( \mathcal{U} \subset 2^\mathbb{N} \) be an arbitrary non-principal ultrafilter on the set of all positive numbers (\( \mathbb{N} \) will almost always be regarded as the index set in this note). Let \( \mathbb{P} \) be the set of all prime numbers, and let \( \mathbb{P}^\mathbb{N} \) denote the set of all sequences \( p = (p_m)_{m \in \mathbb{N}} \) of prime numbers.

We refer to a generic set \( A \in \mathcal{U} \) as an index subset in situations involving the restriction \( p_A : A \to \mathbb{P} \). We will call a sequence \( p \) of prime numbers \( \mathcal{U} \)-stationary if there is an index subset \( A \in \mathcal{U} \) such that its image \( p(A) \) consists of one point.

A sequence \( p : \mathbb{N} \to \mathbb{P} \) is bounded if the image \( p(\mathbb{N}) \) is a finite set. Thanks to the ultrafilter finite intersection property, bounded sequences are necessarily \( \mathcal{U} \)-stationary.

Any non-principal ultrafilter \( \mathcal{U} \) generates a congruence

\[ \sim_{\mathcal{U}} \subseteq \mathbb{P}^\mathbb{N} \times \mathbb{P}^\mathbb{N} \]

in the following way. Two sequences \( p^1 \) and \( p^2 \) are \( \mathcal{U} \)-congruent iff there is an index subset \( A \in \mathcal{U} \) such that for all \( m \in A \) the following equality holds:

\[ p^1_m = p^2_m. \]

The corresponding quotient

\[ \ast \mathbb{P} \equiv \mathbb{P}^\mathbb{N} / \sim_{\mathcal{U}} \]
contains as a proper subset the set of all primes $\mathbb{P}$ (naturally identified with classes of $\mathcal{U}$-stationary sequences), as well as classes of unbounded sequences. The latter are referred to as nonstandard, or infinitely large, primes. We will use both names and normally denote such elements by $[p]$, mirroring the convention for equivalence classes. The terminology is justified, as the set of nonstandard primes is in one-to-one correspondence with the set of prime elements in the ring $^*\mathbb{Z}$ of nonstandard integers in the sense of Robinson [16].

Indeed, one may utilize the following construction, which was thoroughly studied in [17]. Consider the ring $\mathbb{Z}^\omega = \prod_{m \in \mathbb{N}} \mathbb{Z}$ - the product of countably many copies of $\mathbb{Z}$ indexed by $\mathbb{N}$. The minimal prime ideals of $\mathbb{Z}^\omega$ are in bijection with the set of all ultrafilters on $\mathbb{N}$ (perhaps it is opportune to remind that the latter is precisely the Stone-Cech compactification $\beta \mathbb{N}$ of $\mathbb{N}$ as a discrete space). Explicitly, if for every $a = (a_m) \in \mathbb{Z}^\omega$ one defines the support complement as

$$\theta(a) = \{ m \in \mathbb{N} | a_m = 0 \}$$

and for an arbitrary ultrafilter $\mathcal{U} \in 2^{\mathbb{N}}$ sets

$$(\mathcal{U}) = \{ a \in \mathbb{Z}^\omega | \theta(a) \in \mathcal{U} \},$$

then one obtains a minimal prime ideal of $\mathbb{Z}^\omega$. It is easily shown that every minimal prime ideal is of such a form. Of course, the index set $\mathbb{N}$ may be replaced by any set $I$, after which one easily gets the description of minimal primes of $\mathbb{Z}^I$ (since those correspond to ultrafilters, there are exactly $2^{2^{|I|}}$ of them if $I$ is infinite and $|I|$ when $I$ is a finite set). Note that in the case of finite index set all ultrafilters are principal, and the corresponding $(\mathcal{U})$ are of the form $\mathbb{Z} \times \cdots \times (0) \times \cdots \times \mathbb{Z}$ - a textbook example.

Similarly, one may replace each copy of $\mathbb{Z}$ by an arbitrary integral domain and repeat the construction above. If for instance all the rings in the product happen to be fields, then, since the product of any number of fields is von Neumann regular, the ideal $(\mathcal{U})$ will also be maximal.

The ring of nonstandard integers may be viewed as a quotient (an ultrapower)

$$\mathbb{Z}^\omega/(\mathcal{U}) = ^*\mathbb{Z}.$$

The class of $\mathcal{U}$-congruent sequences $[p]$ corresponds to an element (also an equivalence class) in $^*\mathbb{Z}$, which may as well as $[p]$ be represented by a prime number sequence $p = (p_m)$, only in the latter case some but not too many of the primes $p_m$ may be replaced by arbitrary integers. For all intents and purposes, this difference is insignificant. Also, observe that $[p]$ indeed generates a maximal prime ideal in $^*\mathbb{Z}$: if one for (any) $p \in [p]$ defines an ideal in $\mathbb{Z}^\omega$ as

$$(p, \mathcal{U}) = \{ a \in \mathbb{Z}^\omega | \{ m | a_m \in p_m \mathbb{Z} \} \in \mathcal{U} \},$$

then, taking the quotient $\mathbb{Z}^\omega/(p, \mathcal{U})$ in two different ways, one arrives at an isomorphism

$$^*\mathbb{Z}/([p]) \simeq \left( \prod_m \mathbb{Z}_{p_m} \right)/(\mathcal{U}),$$

and the right-hand side is a field by the preceding remark. For a fixed non-principal $\mathcal{U}$ and an infinite prime $[p]$, we will call the quotient

$$\mathbb{Z}_{[p]} \equiv ^*\mathbb{Z}/([p])$$

the nonstandard residue field of $[p]$. Under our assumptions this field has characteristic zero.

\footnote{also cf. [12]}
2.2 Algebraic closure of nonstandard residue field

We have seen that the objects $[p]$ - the infinite prime - behaves similarly to the usual prime number in the sense that a version of a residue field corresponding to this object may be constructed. Note that the standard residue fields are contained as a degenerate case in this construction, namely if we drop the condition of unboundedness and instead consider $U$-stationary sequences, we will arrive at a residue field isomorphic to $\mathbb{Z}_p$, with $p$ being the image of the stationary sequence in the chosen class. The fields of the form $\mathbb{Z}_p[[p]]$ are a realization of what is known as pseudofinite field, cf. [14].

The nonstandard case is surely more interesting. While the algebraic closure of a standard residue field is countable, the nonstandard one itself has the cardinality of the continuum. Its algebraic closure is also of that cardinality and has characteristic zero, which implies that it is isomorphic to the field of complex numbers. We proceed by demonstrating these facts.

Proposition 2.1. For any infinite prime $[p]$ the residue field $\mathbb{Z}_p[[p]]$ has the cardinality of the continuum.

Proof. It suffices to show there is a surjection

$$h^*: \mathbb{Z}_p[[p]] \rightarrow \mathbb{P},$$

where $\mathbb{P} = \{0,1\}^\omega$ is the Cantor set given as the set of all countable strings of bits with the 2-adic metric

$$d_2(x, y) = 1/k, \quad k = \min\{m \mid x_m \neq y_m\}.$$

The map $h^*$ is constructed as follows. If $\mathfrak{F} \subset \mathbb{P}$ is the subset of all strings with finite number of ones in them, and

$$e: \mathbb{Z}_+ \rightarrow \mathfrak{F}, \quad e \left( \sum_{k<m} f_k 2^k \right) = (f_1, \ldots, f_{m-1}, 0, \ldots)$$

is the bijection that sends a nonnegative integer to its binary decomposition, then for a class representative $a = (a_m) \in [a] \in \mathbb{Z}_p[[p]]$ set $h^*(a)$ to be the (unique) ultralimit of the sequence of points $\{x_m = e(a_m)\}$. The correctness of this map rests on the property of the Cantor set being Hausdorff quasi-compact. Surjectivity is then established directly: consider an arbitrary $x \in \mathbb{P}$. For each $m \in \mathbb{N}$ the set

$$\mathfrak{P}_m = \{e(0), e(1), \ldots, e(p_m - 1)\}$$

consists of $p_m$ distinct points. Let $x_m$ be the nearest to $x$ point from this set with respect to the 2-adic metric. The sequence $(x_m)$ is unbounded, so that for every $m \in \mathbb{N}$ the index subset

$$A_m = \{k \in \mathbb{N} \mid p_k > 2^m\}$$

belongs to the ultrafilter $\mathcal{U}$. It is easily seen that for every $k \in A_m$ one has:

$$d_2(x, x_k) < 1/m$$

But that effectively means that the sequence $(x_m)$ has the ultralimit $x$, after which $a_m = e^{-1}(x_m)$ yields the desired preimage.

As an immediate corollary of this proposition and the well-known Steinitz theorem, one has

Theorem 2.2. The algebraic closure $\mathbb{Z}_p[[p]]$ of $\mathbb{Z}_p[[p]]$ is isomorphic to the field of complex numbers. 

2There is a general statement on cardinality of ultraproduct due to Frayne, Morel, and Scott [14]. We believe the proof of this particular instance may serve as a neat example of what we are dealing with in the present paper.
We now fix the notation for the aforementioned isomorphisms in order to employ it in the next section.

For any nonstandard prime \([p] \in {}^*\mathbb{P}\) fix an isomorphism \(\alpha_{[p]} : \mathbb{C} \to \mathbb{Z}_{[p]}\) coming from the preceding theorem. Denote by \(\Theta_{[p]} : \mathbb{Z}_{[p]} \to \mathbb{Z}_{[p]}\) the nonstandard Frobenius automorphism - that is, a well-defined field automorphism that sends a sequence of elements to a sequence of their \(p_m\)-th powers:

\[
(x_m) \mapsto (x_{p_m}^m).
\]

The automorphism \(\Theta_{[p]}\) is identical on \(\mathbb{Z}_{[p]}\); conjugated by \(\alpha_{[p]}\), it yields a wild automorphism of complex numbers, as by assumption no finite power of it (as always, in the sense of index subsets \(A \in \mathcal{U}\)) is the identity homomorphism.

### 2.3 Extension of the Weyl algebra

The \(n\)-th Weyl algebra \(A_{n,\mathbb{C}} \simeq A_{n,\mathbb{F}_{[p]}}\) can be realized as a proper subalgebra of the following ultraproduct of algebras

\[
A_n(\mathcal{U}, [p]) = \left( \prod_{m \in \mathbb{N}} A_{n,\mathbb{F}_{p_m}} \right) / \mathcal{U}.
\]

Here for any \(m\) the field \(\mathbb{F}_{p_m} = \mathbb{Z}_{p_m}\) is the algebraic closure of the residue field \(\mathbb{Z}_{p_m}\). This larger algebra contains elements of the form \((\zeta^m_{[m]})_{m \in \mathbb{N}}\) with unbounded \(|I_m|\) - something which is not present in \(A_{n,\mathbb{F}_{[p]}}\), hence the proper embedding. Note that for the exact same reason (with degrees \(|I_m|\) of differential operators having been replaced by degrees of minimal polynomials of algebraic elements) the inclusion

\[
\mathbb{Z}_{[p]} \subseteq \left( \prod_{m \in \mathbb{N}} \mathbb{F}_{p_m} \right) / \mathcal{U}
\]

is also proper.

It turns out that, unlike its standard counterpart \(A_{n,\mathbb{C}}\), the algebra \(A_n(\mathcal{U}, [p])\) has a huge center described in this proposition:

**Proposition 2.3.** The center of the ultraproduct of Weyl algebras over the sequence of algebraically closed fields \(\{\mathbb{F}_{p_m}\}\) coincides with the ultraproduct of centers of \(A_{n,\mathbb{F}_{p_m}}\):

\[
C(A_n(\mathcal{U}, [p])) = \left( \prod_{m \in \mathbb{N}} C(A_{n,\mathbb{F}_{p_m}}) \right) / \mathcal{U}.
\]

The proof is elementary and is left to the reader. As in positive characteristic the center \(C(A_{n,\mathbb{F}_{p}})\) is given by the polynomial algebra

\[
\mathbb{F}_p[x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p] \simeq \mathbb{F}_p[\xi_1, \ldots, \xi_{2n}],
\]

there is an injective \(\mathbb{C}\)-algebra homomorphism

\[
\mathbb{C}[\xi_1, \ldots, \xi_{2n}] \to \left( \prod_{m \in \mathbb{N}} \mathbb{F}_{p_m}[\xi^{(m)}_1, \ldots, \xi^{(m)}_{2n}] \right) / \mathcal{U}
\]

from the algebra of regular functions on \(A_{2n,\mathbb{C}}^n\) to the center of \(A_n(\mathcal{U}, [p])\), evaluated on the generators in a straightforward way:

\[
\xi_i \mapsto [(\xi^{(m)}_i)_{m \in \mathbb{N}}].
\]
Just as before, this injection is proper.

Furthermore, the image of this monomorphism (the set which we will simply refer to as the polynomial algebra) may be endowed with the canonical Poisson bracket. Recall that in positive characteristic case for any \( a, b \in \mathbb{Z}_p[\xi_1, \ldots, \xi_{2n}] \) one can define

\[ \{ a, b \} = -\pi \left( \frac{[a_0, b_0]}{p} \right). \]

Here \( \pi : A_{n, Z} \to A_{n, Z_p} \) is the modulo \( p \) reduction of the Weyl algebra, and \( a_0, b_0 \) are arbitrary lifts of \( a, b \) with respect to \( \pi \). The operation is well defined, takes values in the center and satisfies the Leibnitz rule and the Jacobi identity. On the generators one has

\[ \{ \xi_i, \xi_j \} = \omega_{ij}. \]

The Poisson bracket is trivially extended to the entire center \( F_p[\xi_1, \ldots, \xi_{2n}] \) and then to the ultraproduct of centers. Observe that the Poisson bracket of two elements of bounded degree is again of bounded degree, hence one has the bracket on the polynomial algebra.

### 2.4 Endomorphisms and symplectomorphisms

The point of this construction lies in the fact that thus defined Poisson structure on the (injective image of) polynomial algebra is preserved under all endomorphisms of \( A_n(U, [p]) \) of bounded degree. Every endomorphism of the standard Weyl algebra is specified by an array of coefficients \((a_{i,j})\) (which form the images of the generators in the standard basis); these coefficients are algebraically dependent, but with only a finite number of bounded-order constraints. Hence the endomorphism of the standard Weyl algebra can be extended to the larger algebra \( A_p(U, [p]) \). The restriction of any such obtained endomorphism on the polynomial algebra \( \mathbb{C}[\xi_1, \ldots, \xi_{2n}] \) preserves the Poisson structure. In this setup the automorphisms of the Weyl algebra correspond to symplectomorphisms of \( A_p^{2n} \). Here we are only interested in this correspondence between automorphisms and symplectomorphisms.

Example. If \( x_i \) and \( y_i \) are standard generators, then one may perform a linear symplectic change of variables:

\[
\begin{align*}
  f(x_i) &= \sum_{j=1}^n a_{ij}x_j + \sum_{j=1}^n a_{i,n+j}y_j, \quad i = 1, \ldots, n, \\
  f(d_i) &= \sum_{j=1}^n a_{i+n,j}x_j + \sum_{j=1}^n a_{i+n,n+j}y_j, \quad a_{ij} \in \mathbb{C}.
\end{align*}
\]

In this case the corresponding polynomial automorphism \( f^c \) of

\[
\mathbb{C}[\xi_1, \ldots, \xi_{2n}] \cong \mathbb{C}[x_1^{[p]}, \ldots, x_n^{[p]}, y_1^{[p]}, \ldots, y_n^{[p]}]
\]

acts on the generators \( \xi \) as

\[ f^c(\xi_i) = \sum_{j=1}^{2n} (a_{ij})^{[p]} \xi_j, \]

where the notation \((a_{ij})^{[p]}\) means taking the base field automorphism that is conjugate to the nonstandard Frobenius via the Steinitz isomorphism.

Let \( \gamma : \mathbb{C} \to \mathbb{C} \) be an arbitrary automorphism of the field of complex numbers. Then, given an automorphism \( f \) of the Weyl algebra \( A_{n,C} \) with coordinates \((a_{i,j})\), one can build another algebra automorphism using the map \( \gamma \). Namely, the coefficients \( \gamma(a_{i,j}) \) define a new automorphism \( \gamma_*(f) \) of the Weyl algebra, which is of the same degree as the original
one. In other words, every automorphism of the base field induces a map $\gamma: A_n,\mathbb{C} \to A_n,\mathbb{C}$ which preserves the structure of the ind-object. It obviously is a group homomorphism.

Now, if $P_n,\mathbb{C}$ denotes the commutative polynomial algebra with Poisson bracket, we may define an ind-group homomorphism $\phi: \text{Aut}(A_n,\mathbb{C}) \to \text{Aut}(P_n,\mathbb{C})$ as follows. Previously we had a morphism $f \mapsto f^c$, however as the example has shown it explicitly depends on the choice of the infinite prime $[p]$. We may eliminate this dependence by pushing the whole domain $\text{Aut}(A_n,\mathbb{C})$ forward with a specific base field automorphism $\gamma$, namely $\gamma = \Theta^{-1}_{[p]}$ - the field automorphism which is Steinitz-conjugate with the inverse nonstandard Frobenius, and only then constructing the symplectomorphism $f^c$ as the restriction to the (nonstandard) center. For the subgroup of tame automorphisms such as linear changes of variables this procedure has a simple meaning: just take the $[p]$-th root of all coefficients $(a_{i,l})$ first. We thus obtain a group homomorphism which preserves the filtration by degree and is in fact well-behaved with respect to the Zariski topology on $\text{Aut}$ (indeed, the filtration $\text{Aut}^N \subset \text{Aut}^{N+1}$ is given by Zariski-closed embeddings). Formally, we have a proposition:

**Proposition 2.4.** There is a system of morphisms

$$\phi_{[p],N}: \text{Aut}^N(A_n,\mathbb{C}) \to \text{Aut}^N(P_n,\mathbb{C}).$$

such that the following diagram commutes for all $N \leq N'$:

$$\begin{array}{ccc}
\text{Aut}^N(A_n,\mathbb{C}) & \xrightarrow{\phi_{[p],N}} & \text{Aut}^N(P_n,\mathbb{C}) \\
\downarrow^{\nu_{NN'}} & & \downarrow^{\nu_{NN'}} \\
\text{Aut}^{N'}(A_n,\mathbb{C}) & \xrightarrow{\phi_{[p],N'}} & \text{Aut}^{N'}(P_n,\mathbb{C})
\end{array}$$

The corresponding direct limit of this system is given by $\phi_{[p]}$, which maps a Weyl algebra automorphism $f$ to a symplectomorphism $f^c_{\Theta}$.

The Kontsevich-Belov conjecture then states:

**Conjecture 2.5.** $\phi_{[p]}$ is a group isomorphism.

Injectivity may be established right away.

**Theorem 2.6.** $\phi_{[p]}$ is an injective homomorphism.

(See [1] for the fairly elementary proof). Surjectivity, however, if true at all (which we believe to be the case), has proven itself to be a much tougher problem.

### 3 Independence of infinite prime

Let us at first assume that the Belov-Kontsevich conjecture holds, with $\phi_{[p]}$ furnishing the isomorphism between the automorphism groups. This would be the case if all automorphisms in $\text{Aut}(A_n,\mathbb{C})$ were tame, which is unknown at the moment for $n > 1$.

The main result of the paper is as follows:

**Theorem 3.1.** If one assumes that $\phi_{[p],N}$ is surjective for any infinite prime $[p]$, then $\Phi_N$ is necessarily an identity morphism.
Let \([p]\) and \([p']\) be two distinct classes of \(U\)-congruent prime number sequences - that is, two distinct infinite primes. We then have the following diagram:

\[
\begin{array}{ccc}
\text{Aut}(A_{n,C}) & \overset{\phi_{[p]}}{\longrightarrow} & \text{Aut}(P_{n,C}) \\
\downarrow_{\text{isom}} & & \downarrow_{\text{isom}} \\
\text{Aut}(A_{n,C}) & \overset{\phi_{[p']}\cdot \text{Aut}(A_{n,C})} {\longrightarrow} & \text{Aut}(P_{n,C})
\end{array}
\]

with all arrows being isomorphisms. Vertical isomorphisms answer to different presentations of \(C\) as \(\mathbb{Z}/[p]\) and \(\mathbb{Z}/[p']\). The corresponding automorphism \(C \rightarrow \mathbb{Z}/[p]\) is denoted by \(\alpha_{[p]}\) for any \([p]\).

The fact that all the arrows in the diagram are isomorphisms allows one instead to consider a loop of the form

\[\Phi : \text{Aut}(A_{n,C}) \rightarrow \text{Aut}(A_{n,C}).\]

Furthermore, as it was noted in the previous section, the morphism \(\Phi\) belongs to \(\text{Aut}(\text{Aut}(A_{n,C})).\)

We need to prove that \(\Phi\) is a trivial automorphism. The first observation is as follows.

**Proposition 3.2.** The map \(\Phi\) is a morphism of algebraic varieties.

**Proof.** Basically, this is a property of \(\phi_{[p]}\) (or rather its unshifted version, \(f_p \mapsto f'_p\)). More precisely, it suffices to show that, given an automorphism \(f_p\) of the Weyl algebra in positive characteristic \(p\) with coordinates \((a_{i,I})\), its restriction to the center (a symplectomorphism) \(f'_p\) has coordinates which are polynomials in \((a_{i,I}^p)\).

The switch to positive characteristic and back is performed for a fixed \(f \in \text{Aut}(A_{n,C})\) on an index subset \(A_f \in U\).

Let \(f\) be an automorphism of \(A_{n,C}\) and let \(N = \deg f\) be its degree. The automorphism \(f\) is given by its coordinates \(a_{i,I} \in C, i = 1, \ldots, 2n, I = \{i_1, \ldots, i_{2n}\}\), obtained from the decomposition of algebra generators \(\zeta_i\) in the standard basis of the free module:

\[f(\zeta_i) = \sum_{i,I} a_{i,I} \zeta^I, \quad \zeta^I = \zeta_{i_1} \cdots \zeta_{i_{2n}}.\]

Let \((a_{i,I,p})\) denote the class \(\alpha_{[p]}(a_{i,I}), p = (p_m)\), and let \(\{R_k(a_{i,I} | i, I) = 0\}_{k=1, \ldots, M}\) be a finite set of algebraic constraints for coefficients \(a_{i,I}\). Let us denote by \(A_1, \ldots, A_M\) the index subsets from the ultrafilter \(U\), such that \(A_k\) is precisely the subset, on whose indices the constraint \(R_k\) is valid for \((a_{i,I,p})\). Take \(A_f = A_1 \cap \ldots \cap A_M \in U\) and for \(m \in A_f\), define an automorphism \(f_{p_m}\) of the Weyl algebra in positive characteristic \(A_{n,F_{p_m}}\) by setting

\[f_{p_m}(\zeta_i) = \sum_{i,I} a_{i,I,p_m} \zeta^I.\]

All of the constraints are valid on \(A_f\), so that \(f\) corresponds to a class \([f_p]\) modulo ultrafilter \(U\) of automorphisms in positive characteristic. The degree of every \(f_{p_m}\) \((m \in A_f)\) is obviously less than or equal to \(N = \deg f\).

Now consider \(f \in \text{Aut}^{\leq N}(A_{n,C})\) with the index subset \(A_f\) over which its defining constraints are valid. The automorphisms \(f_{p_m} = f : A_{n,F_p} \rightarrow A_{n,F_p}\) defined for \(m \in A_f \in U\) provide arrays of coordinates \(a_{i,I,p}\). Let us fix any valid \(p_m = p\) denote by \(F_{p^k}\) a finite subfield of \(F_p\) which contains the respective coordinates \(a_{i,I,p}\) (one may take \(k\) to be equal to the maximum degree of all minimal polynomials of elements \(a_{i,I,p}\) which are algebraic over \(\mathbb{Z}_p\)).
Let \( a_1, \ldots, a_s \) be the transcendence basis of the set of coordinates \( a_{i,I,p} \) and let \( t_1, \ldots, t_s \) denote \( s \) independent (commuting) variables. Consider the field of rational functions:

\[
F_{p^k}(t_1, \ldots, t_s).
\]

The vector space

\[
\text{Der}_{\mathbb{Z}_p}(F_{p^k}(t_1, \ldots, t_s), F_{p^k}(t_1, \ldots, t_s))
\]

of all \( \mathbb{Z}_p \)-linear derivations of the field \( F_{p^k}(t_1, \ldots, t_s) \) is finite dimensional with \( \mathbb{Z}_p \)-dimension equal \( ks \) and a particular basis given by elements \( \{ e_a D_{t_b} \mid a = 1, \ldots, k, \ b = 1, \ldots, s \} \) where \( e_a \) stand for the basis of the field \( F_{p^k} \) regarded as a vector space over \( \mathbb{Z}_p \), and \( D_{t_b} \) is the partial derivative with respect to the variable \( t_b \).

Set \( a_1, \ldots, a_s = t_1, \ldots, t_s \) (i.e. consider an \( s \)-parametric family of automorphisms), so that the rest of the coefficients \( a_{i,I,p} \) are algebraic functions of \( s \) variables \( t_1, \ldots, t_s \). We need to show that the coordinates of the corresponding symplectomorphism \( f_p^{c_0} \) are annihilated by all derivations \( e_a D_{t_b} \).

Let \( \delta \) denote a derivation of the Weyl algebra induced by an arbitrary basis derivation \( e_a D_{t_b} \) of the field. For a given \( i \), let us introduce the short-hand notation

\[
a = f_p(\zeta_i), \quad b = \delta(a).
\]

We need to prove that

\[
\delta(f_i(\zeta_i)) = \delta(f_p(\zeta^p_i)) = 0.
\]

In our notation \( \delta(f_p(\zeta^p_i)) = \delta(a^p) \), so by Leibnitz rule we have:

\[
\delta(f_p(\zeta^p_i)) = ba^{p-1} + aba^{p-2} + \cdots + a^{p-1}b.
\]

Let \( \text{ad}_x : A_{n,\bar{\mathbb{F}}_p} \to A_{n,\bar{\mathbb{F}}_p} \) denote a \( \mathbb{Z}_p \)-derivation of the Weyl algebra corresponding to the adjoint action (all Weyl algebra derivations are inner!):

\[
\text{ad}_x(y) = [x, y].
\]

We will call an element \( x \in A_{n,\bar{\mathbb{F}}_p} \) locally ad-nilpotent if for any \( y \in A_{n,\bar{\mathbb{F}}_p} \) there is an integer \( D = D(y) \) such that

\[
\text{ad}^D_x(y) = 0.
\]

All algebra generators \( \zeta_i \) are locally ad-nilpotent. Indeed, one could take \( D(y) = \deg y + 1 \) for every \( \zeta_i \).

If \( f \) is an automorphism of the Weyl algebra, then \( f(\zeta_i) \) is also a locally ad-nilpotent element for all \( i = 1, \ldots, 2n \). That means that for any \( i = 1, \ldots, 2n \) there is an integer \( D \geq N + 1 \) such that

\[
\text{ad}^D_{f_p(\zeta_i)}(\delta(f_p(\zeta_i))) = \text{ad}^D_{\zeta_i}(b) = 0.
\]

Now, for \( p \geq D + 1 \) the previous expression may be rewritten as

\[
0 = \text{ad}^{p-1}_{a^p}(b) = \sum_{l=0}^{p-1} (-1)^l \binom{p-1}{l} a^l ba^{p-1-l} \equiv \sum_{l=0}^{p-1} a^l ba^{p-1-l} \pmod{p},
\]

and this is exactly what we wanted.

We have thus demonstrated that for an arbitrary automorphism \( f_p \) of the Weyl algebra in positive characteristic, the coordinates of the corresponding symplectomorphism \( f_p^{c_0} \) are given by polynomials over \( p \)-th powers of the coordinates of \( f_p \), so long as the characteristic \( p \) is greater than \( \deg f_p + 1 \). As the sequence \( (\deg f_{p^n}) \) is bounded from above by \( N \) for
all $m \in A_f$, we see that there is an index subset $A_f^* \subseteq U$ such that the coordinates of the symplectomorphism $f^p_m$ for $m \in A_f^*$ are polynomials over $p_m$-th powers of $a_{i,I,p_m}$. This implies that $f^c$ in characteristic zero is given by coefficients polynomial in $a_{i}[a_{i,I}]^p$ as desired.

It follows after shifting by the inverse nonstandard Frobenius that $\Phi$ is an endomorphism of the algebraic variety $\text{Aut}(A_{n,C})$. \hfill \Box

The automorphism $\Phi$ acting on elements $f \in \text{Aut}(A_{n,C})$, takes the set of coordinates $(a_{i,I})$ and returns an array $(G_{i,I}(a_{k,K}))$ of the same length. All functions $G_{i,I}$ are algebraic by the above proposition. It is convenient to introduce a partial ordering on the set of coordinates. We say that $a_{i,I}$ is higher than $a_{i,I'}$ (for the same generator $i$) if $|I| < |I'|$ and we leave pairs with $i \neq j$ or with $|I| = |I'|$ unconnected. We define the dominant elements $a_{i,I}$ (or rather, dominant places $(i,I)$) to be the maximal elements with respect to this partial ordering, and subdominant elements to be the elements covered by maximal ones (in other words, for fixed $i$, subdominant places are the ones with $|I| = |I_{\text{max}}| - 1$).

The following observation follows straight from the fact that we are dealing with algebra automorphisms.

**Lemma 3.3.** Functions $G_{i,I}$ corresponding to dominant places $(i,I)$ are identities:

$$G_{i,I}(a_{k,K}) = a_{i,I}.$$  

**Proof.** Indeed, it follows from the commutation relations that for any $i = 1, \ldots, 2n$ and $f^p, p = p_m, m \in A_f \subseteq U$, the highest-order term in $f^p_i(\xi_i) = f^p_i(\zeta^p_i) = f^p_i(\zeta_i)^p$ has the coefficient $a_{i,I,p}$. The shift by the inverse Frobenius then kills the $p$ power, so that we are left with $a_{i,I}$. Evidently, it is independent of the choice of $[p]$. \hfill \Box

Let us now once and for all fix $N \geq 1$ and consider

$$\Phi_N : \text{Aut}^{\leq N} A_{n,C} \rightarrow \text{Aut}^{\leq N} A_{n,C}$$

the restriction of $\Phi$ to the subvariety $\text{Aut}^{\leq N} A_{n,C}$, which is well defined by the above lemma. The morphism corresponds to an endomorphism of the ring of functions

$$\Phi_N^* : \mathcal{O}(\text{Aut}^{\leq N} A_{n,C}) \rightarrow \mathcal{O}(\text{Aut}^{\leq N} A_{n,C})$$

Let us take a closer look at the behavior of $\Phi_N$ (and of $\Phi_N^*$, which is essentially the same up to an inversion), specifically at how $\Phi_N$ affects one-dimensional families of automorphisms. Let $X_N$ be the set of all algebraic curves of automorphisms from $\text{Aut}^{\leq N} A_{n,C}$; by virtue of Lemma 3.3 it makes sense to consider the subset of all curves with fixed dominant places - we denote this subset by $X_N^{(1)}$, and, for that same matter, the subsets $X_N^{(k)}$ of curves with fixed places of the form $(i,I')$, which are away from a dominant place by a path of length at most $(k - 1)$. Thus $X_N^{(k)} = X_N^{(1)}$.

The morphism $\Phi_N$ yields a map

$$\bar{\Phi}_N : X_N \rightarrow X_N$$

and its restrictions

$$\bar{\Phi}_{N}^{(k)} : X_N^{(k)} \rightarrow X_N.$$  

Our immediate goal is to prove that for all attainable $k$ we have

$$\bar{\Phi}_{N}^{(k)} : X_N^{(k)} \rightarrow X_N^{(k)},$$

i.e. the map $\Phi_N$ preserves the terms corresponding to non-trivial differential monomials.
Despite minor abuse of language, we will call the highest non-constant terms of a curve in $X_N^{(k)}$ dominant, although they cease to be so when that same curve is regarded as an element of $X_N$.

Let $A \in X_N$ be a generic algebraic curve. Coordinate-wise $A$ answers to an array $(a_i, I(\tau))$ of coefficients parameterized by some indeterminate. By Lemma 3.3, $\Phi_N$ leaves the (coefficients corresponding to) dominant places of this curve unchanged, so we may well set $A \in X_N^{(1)}$. In fact, it is easily seen that the subdominant terms are not affected by $\Phi_N$ either; this is the case thanks to the commutation relations that define the Weyl algebra: for every legitimate $p$ in the ultraproduct decomposition, after one raises to the $p$-th power one should perform a reordering within monomials, which degrades the cardinality $|I|$ by an even number. Therefore, nothing contributes to the image of any subdominant term other than that subdominant term itself, which thus stays the same under $\Phi_N$. We are then to consider the image $\tilde{\Phi}_N^{(2)}(A) \in X_N^{(2)}$.

Again, let us consider a positive characteristic $p$ within the ultraproduct decomposition, and suppose the curve $A$ (or rather its component answering to the chosen element $p$) has a number of poles attained on dominant terms. Let us pick among these poles the one of the highest order $k$, and let $(i_0, I_0)$ be its place. By definition of an automorphism of Weyl algebra as an array of coefficients, the number $i_0$ does not actually carry any meaningful data, so that we are left with a pair $(k, |I_0|)$. As we can see, this pair is a maximal element from two different viewpoints; in fact, this is nothing but a vertex of a Newton polygon taken over the appropriate field, with the discrete valuation given by $|I|$. The coordinate function $a_{i_0, I_0}$ corresponding to this pole admits a decomposition

$$a_{i_0, I_0} = \frac{a-k}{t^k} + \cdots,$$

with $t$ a local parameter. Acting upon this curve by the morphism $\Phi_N$ consists of two steps: first, we raise everything in the $p$-th power and then assemble the components within the ultraproduct decomposition, then we take the preimage, which is essentially the same as taking the $p'$-root, but with respect to a different ultraproduct decomposition. The order of the maximal pole gets multiplied by an integer after the first step and divided by the same integer after the second one. By maximality, there are no other terms that can contribute to the resulting place in $\tilde{\Phi}_N^{(2)}(A)$. It therefore does not change under $\Phi_N$.

We may process the rest of the dominant (with respect to $X_N^{(2)}$) terms similarly: indeed, it suffices to pick a different curve in general position. We then move down to $X_N^{(k)}$ with higher $k$ and argue similarly.

After we have exhausted the possibilities with non-constant terms, we arrive at the conclusion that all that $\Phi_N$ can do is permute the irreducible components of $\text{Aut}^N A_{n,\mathbb{C}}$. That in turn implies there exists a positive integer $l$ such that $\Phi_N^l = \text{Id}$.

In fact, the preceding argument gives us more than just the observation that $\Phi_N$ is unipotent. Let $\Phi_{N,M}$ denote the linear map of finite-dimensional vector spaces obtained by restricting $\Phi_N^*$ to regular functions of total degree less than or equal to $M$. Then the following proposition holds.

**Proposition 3.4.** If $\lambda$ is an eigenvalue of $\Phi_{N,M}^*$, then $\lambda = 1$.

**Proof.** Indeed, should there exist $\lambda_0 \neq 1$, we may find an exceptional curve whose singularity changes under $\Phi_N$. 

\footnote{With respect to $X_N^{(2)}$, i.e. the highest terms that actually change - see above where we specify this convention.}
We are almost done: we have shown that the finite-dimensional components that make up the pullback $\Phi_N^*$ are similar to the identity maps. There is still the possibility that some of these components have nontrivial Jordan blocks. However, the consideration of order and residue of the poles of algebraic curves allows to exclude that possibility as well.

**Proposition 3.5.** If $\Phi_{N,M}^*$ is as before, then its matrix in the standard basis is the identity matrix.

**Proof.** Assume the contrary. Then there exists a set of coordinates $\{a_1, \ldots, a_s\}$ such that

\[
\begin{align*}
\Phi_{N,M}^*(a_1) &= a_1 + a_2, \\
\Phi_{N,M}^*(a_2) &= a_2 + a_3, \\
&\vdots \\
\Phi_{N,M}^*(a_{s-1}) &= a_{s-1} + a_s, \\
\Phi_{N,M}^*(a_s) &= a_s.
\end{align*}
\]

We may choose a curve in such a way that it will have poles delivered by both $a_{s-1}$ and $a_s$. But then $\Phi_{N,M}^*$ changes the value of the residue at $a_{s-1}$, which stands in contradiction with the preceding argument. \(\blacksquare\)

Therefore, all finite-dimensional maps $\Phi_{N,M}^*$ are identities, which means that the limit $\Phi_N^*$ is an identity as well. This proves the main theorem.

**4 Discussion**

The investigation of decomposition of polynomial algebra-related objects into ultraproducts over the prime numbers $\mathbb{P}$ leads to a problem of independence of the choice of infinite prime. In the case of the Tsuchimoto-Belov-Kontsevich homomorphism the answer turns out to be affirmative, although there are other constructions (for instance, the $p$-determinant over the Weyl algebra, which was introduced in [8] and further studied in [15]), which are of algebraic or even polynomial nature but for which the independence fails. The reason for such arbitrary behavior may have something to do with certain growth functions (in which case the situation is akin to the one described in, where one has a non-injective endomorphism $f_p : A_n, \mathbb{F}_p \to A_n, \mathbb{F}_p$, whose degree however grows with $p$, which disallows for the construction of a naive counterexample to the Dixmier Conjecture given by the ultralimit). This problem needs to be further explored.

**Acknowledgments**

This work was supported by the RFBR grant 14-01-00548.

**References**


**Addresses:**
A.B.-K.: Mathematics Department, Bar-Ilan University, Ramat-Gan, 52900, Israel
kanel@mccme.ru

A.E.: Department of Discrete Mathematics, Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Moscow Region, 141700, Russian Federation
elishev@phystech.edu