Phil Mag Letters 2003 Vol. 83, No. 3, 197-203

Topological Interlocking of Platonic Solids: A Way to New Materials and Structures

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ABSTRACT

The structural integrity of natural and engineered materials relies on chemical or mechanical bonding between the building blocks of which they consist. Materials whose building blocks are not joined, but rather interlocked topologically, possess remarkable mechanical and functional properties. We show that identical elements, in the shape of the five platonic solids, can be arranged into layer-like structures in which they are interlocked topologically. It is shown that truncated icosahedra (buckyballs) can also be arranged in a layer with topological interlocking. The geometrical possibility of such assemblies opens up interesting avenues in the design of structures and materials.

§1. INTRODUCTION

In natural or man-made materials, the 'building blocks' are held together by some sort of bonding, be it at molecular, crystallite or larger scale. It has been a common belief that, in the absence of bonding or joining, any structure made up of convex blocks can be disassembled by removing blocks one by one. While true in two dimensions, this does not necessarily hold in three dimensions. A very special type of structural organization is *topological interlocking* for which the building blocks are locked in their positions by purely geometrical constraints. It was found (Glickman 1984) - and recently rediscovered (Dyskin et al. 2001a) - that this is possible with identical regular tetrahedra. Interlocking is achieved by arranging them in a way shown in Figure 1a: in this assembly none of the blocks can be removed from the layer they form, apart from those at the periphery. By constraining them externally the entire structure is kept in one piece. (No external constraint is required if the tetrahedra are appropriately deformed so that the edges of the layer are linked to form a closed shape, eg a torus or a sphere. Of course, such an assembly cannot sustain isotropic dilation.) The structures under consideration are to be distinguished from the historically first known interlocking structure - the arch - which provides interlocking in only one direction, since its building blocks can be removed by upward displacement.

The beauty of topological interlocking is that the blocks are held together without being physically joined. This means that cracks cannot propagate from one block to another: they will be arrested at the boundaries between the blocks. Furthermore, the structure keeps its integrity when an individual element or a large percentage of elements are missing (Dyskin *et al*. 2001b). A material engineered in this way will possess an enhanced fracture resistance and high damage tolerance. It is remarkable that interlocking is achieved with the simplest platonic shape, *viz*. the tetrahedron. One can ask whether other simple polyhedra permit interlocking as well. Below we demonstrate that all five platonic shapes (tetrahedron, cube, octahedron, dodecahedron and icosahedron) can be assembled in interlocking structures.

§2. PRINCIPLE OF TOPOLOGICAL INTERLOCKING

To understand the architecture of the interlocking assemblies, we first look at a layer-like arrangement of tetrahedra and consider the evolution of a planar section parallel to the layer as it is moved in the normal direction. Squares forming a regular pattern in the middle section (magenta lines on figure 1a, b) transform to rectangles (blue) and finally degenerate into lines (edges of the tetrahedra) as the section plane moves upwards. If the section plane moves downwards, a similar evolution of shape will occur, with the rectangles rotated by 90°.

Interlocking in this assembly can be rationalized by considering a reference block whose sections are highlighted in figure 1b. It cannot be removed from the assembly by upward displacement, since its middle section is obstructed by sections 1 and 2 of the adjacent blocks. It is the inclined faces of these adjacent blocks that prevent the upward movement of the reference block. A similar argument holds for downward displacement which is blocked by the other two neighbours (represented by sections 3 and 4). Obviously, sidewise movement of the reference block is also prevented by the adjacent ones. It can be proved that the reference block cannot be removed by any combination of displacements and rotations either. Figure 1c shows a physical model of the assembly.

Using the above principle of cross-section evolution, one can construct interlocking arrangements of the other platonic solids. In our example with tetrahedra, the middle section plane was fully tiled with squares. Furthermore, we note that at least two pairs of faces of neighbouring blocks are needed to prevent the reference block from moving upwards and downwards. This means that no interlocking arrangements based on plane tiling by regular triangles exist. The only other regular tiling is provided by regular hexagons (honeycomb pattern). Three platonic solids have hexagonal sections permitting stacking in a honeycomb pattern: the cube, the octahedron and the dodecahedron; see figure 2. The corresponding assemblies of interlocked cubes and octahedra are sketched in figure 3.

As evident from figure 2, the middle section of a reference element cannot pass through the ‘channel’ formed by successive sections of its neighbours in either direction. Thus, the reference element is locked topologically by virtue of its shape and the mutual arrangement of elements. (The criterion of translational interlocking in a layer can be rigorously formulated by considering polygons formed by the intersections of the extensions of the element faces constrained by the element's neighbours with a plane parallel to the layer. An element is locked if, and only if, by continuously shifting the section plane in either direction the corresponding polygon eventually degenerates to a segment or a point.)

We now turn to the last remaining platonic solid, *viz*. the icosahedron. This shape does not have regular hexagonal cross-sections that would permit interlocking. Instead, interlocking can be achieved by packing of icosahedra with their regular decagonal central sections coplanar, figure 4. It is remarkable that the same type of arrangement can be generated by dodecahedra since they also possess regular decagonal central sections. Although the arrangement of the middle sections in this case differs from the honeycomb structures shown in figure 2 – it does not even provide a complete covering of the plane – the same interlocking principle holds.

Mechanical properties of these assemblies are dictated by their symmetry. As a honeycomb-based assembly (figure 2) possesses the sixth order symmetry group in the middle plane, it is planar isotropic in that plane (see, for example, Landau and Lifshitz 1959). Therefore, the whole assembly can be considered as nearly planar isotropic. The other structures (figures 1, 4) possess two orthogonal symmetry planes (normal to the assembly plane) and are therefore planar orthotropic, the assembly of tetrahedra having equal compliances in two mutually perpendicular directions (see, for example, Landau and Lifshitz 1959).

This completes the catalogue of the interlocking arrangements for the platonic solids. Obviously, certain truncations of these solids will leave the interlocking property unaffected (cf. examples of such truncations for the case of the tetrahedra in Glickman (1984) and Dyskin *et al*. (2001)). A further example is the truncated icosahedron (buckyball). Since buckyballs have the same decagonal middle sections as icosahedra, they can be arranged in an interlocking structure, figures 4 and 5 – as long as they can be considered as solid bodies. This arrangement is akin to a {111} monolayer of the face-centred cubic structure observed experimentally (Onoe *et al*. 2002). Of course, this is no more than a resemblance – in the experimentally observed monolayers there is a separation between C_{60} molecules. Moreover, the latter possess rotational degrees of freedom. What figures 4 and 5 suggest is a *geometric* possibility of structures in which buckyballs are interlocked. If monolayers of interlocked buckyballs existed in nature or could be synthesised, some unusual properties, in particular enhanced mechanical stiffness, could be expected.

Note that the buckyball has yet another middle section - a regular dodecagon – which gives rise to a further type of interlocking layer. In this case, two opposite hexagonal faces of buckyballs are oriented parallel to the layer they form. (This property is ‘inherited’ from interlocking of dodecahedra shown in figure 2, since the dodecahedron can be reconstructed from the buckyball by extending the pentagonal faces of the latter.) It can be expected that the existence of two types of positions in which buckyballs interlock can facilitate formation of out-of-plane interlocking.

§3. DISCUSSION

We have demonstrated that there exists a variety of shapes permitting topological interlocking. It is interesting to note that it can be realised with all platonic solids. We believe that topological interlocking has a great practical significance, as it opens new ways in design of structures and materials. Efficient technologies for their manufacturing are yet to be developed. Techniques based on the layer-by-layer deposition of material (for example, using selective laser sintering, Estrin et al. (2002a)) to produce the *entire structure* appear to be most promising. To avoid bonding between the elements, an ‘inflated’ pre-assembly with gaps between the elements is produced first, followed by 'deflation' that brings the elements into contact.

Being a purely geometric property, topological interlocking is scale independent. Multifarious applications ranging from architectural design to novel materials can thus be envisaged. For example, the periodicity of the interlocking structures, coupled with their tolerance to 'vacancies' (Estrin et al. (2002b), Khor et al. (2002)), may prove useful in developing materials with special acoustic or optical properties (such as, for example, photonic crystals, Yablonovitch (1994)). Another prospective application is seen in assembling tetrahedron- or cube-shaped microelements for electronic packaging (Gracias *et al*. 2000) by interlocking them in a structure in the way proposed. Finally, partitioning into unconnected, yet interlocked, elements offers the possibility of developing multifunctional materials by integrating geometrically identical, but functionally different, 'building blocks' in one assembly.

ACKNOWLEDGEMENTS

Support from the Australian Research Council through a Discovery Grant is acknowledged. Y. E and A. K-B acknowledge financial support from the University of Western Australia through the Department of Civil and Resource Engineering (Y. E) and a Gledden Fellowship (A. K-B). The authors are grateful to B. H. G. Brady, J. Dell, M. Fahey, L. Faraone, E. A. Katz and M. F. Randolph for stimulating discussions.

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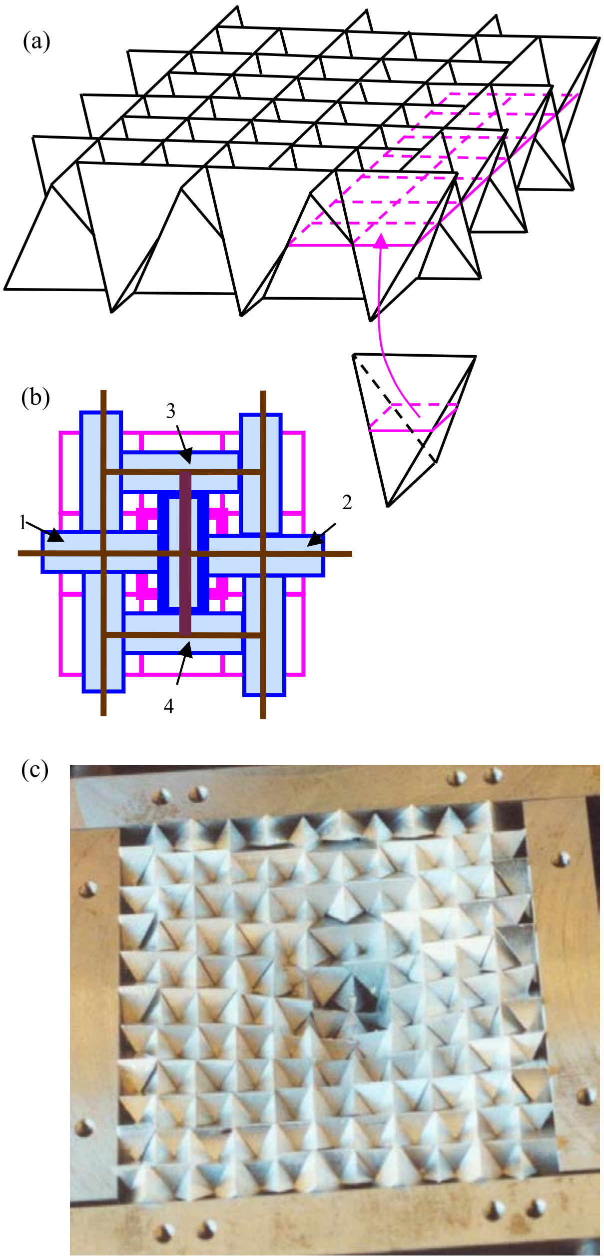

Figure 1. Layer-like assembly of interlocked tetrahedra. (a) Overview of assembly. (b) Evolution of the planar section through the assembly. Magenta lines indicate a fragment of the middle section, shaded blue rectangles correspond to the section moved from the middle section half way up, black and brown lines indicate the sections that have degenerated into the upper edges (seen when one looks on the assembly from the top). Bold lines delineate sections of a reference block. (c) Photograph of a model consisting of 100 aluminium tetrahedra 1 cm in size with a constraining frame after an indentation test. It is seen that a large residual deformation of assembly is possible without plastic deformation of individual tetrahedra.

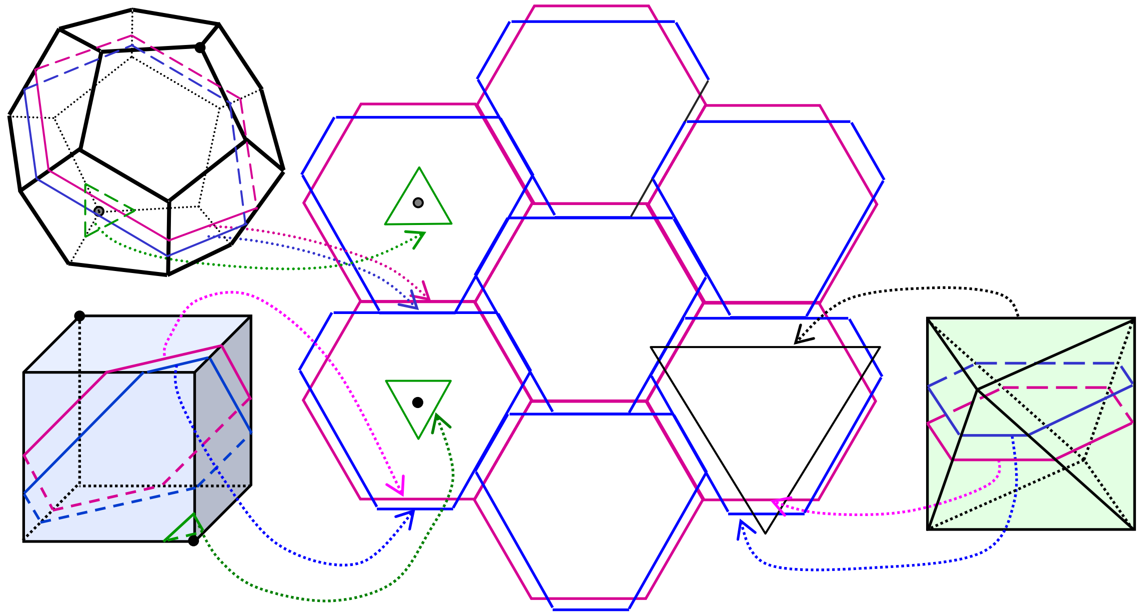

Figure 2. Hexagonal tiling at middle sections of assemblies of cubes, octahedra and dodecahedra. The regular hexagons in the middle section (shown in magenta) transform to distorted hexagons (blue) as the section moves away from the middle plane. For an octahedron the hexagonal section eventually degenerates to a regular triangle (black) coinciding with its top face. For a cube, the triangle (green) then reduces in size eventually degenerating to a point (vertex of the cube) shown in black. For a dodecahedron the hexagonal section degenerates to a triangle (green) and finally to a point (dodecahedron vertex) shown as a grey dot.

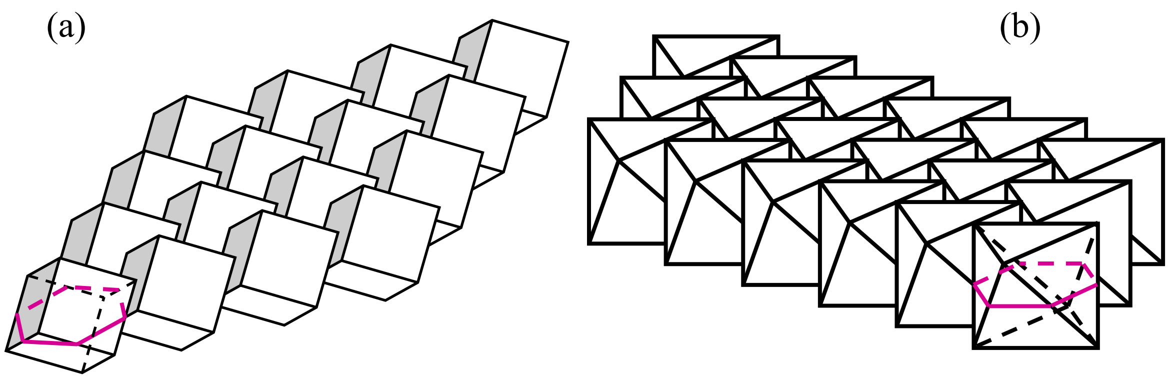

Figure 3. Fragments of assemblies of interlocked cubes and octahedra. (a) Cubes with their space diagonals aligned perpendicularly to the layer. (b) Octahedra resting on their faces. A regular hexagonal middle section is shown for a corner element.

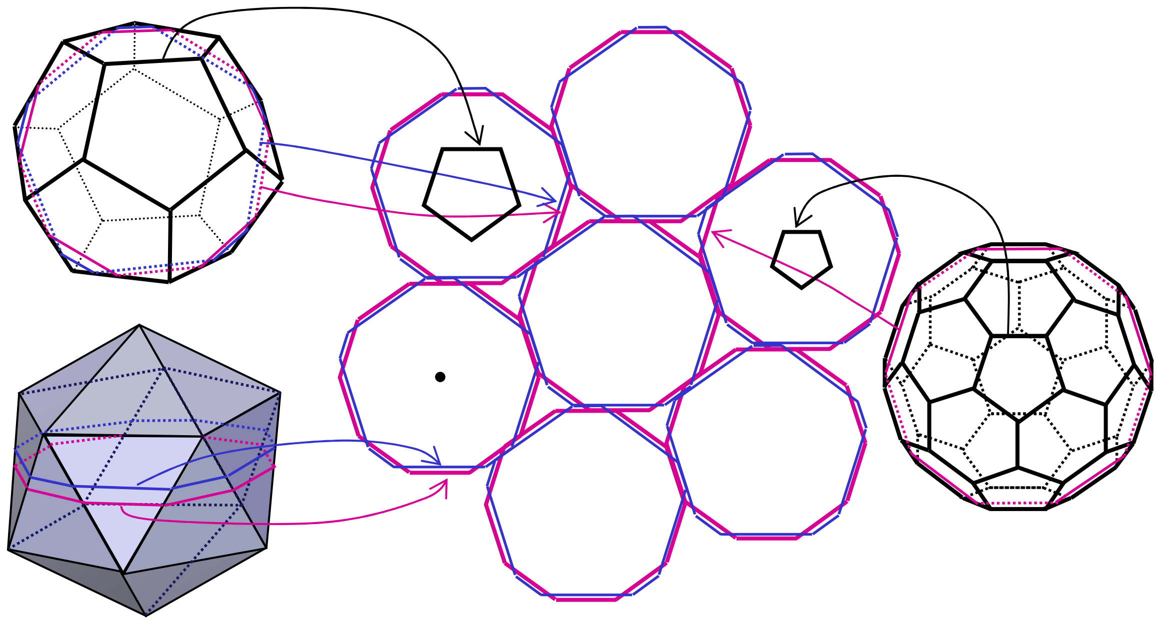

Figure 4. Decagon-based assembly of dodecahedra, icosahedra and truncated icosahedra (buckyballs). Tiling of the middle plane with regular decagons is shown in magenta, a parallel section (upper for the icosahedron and lower for the dodecahedron) is shown in blue. The upper, pentagonal faces of the dodecahedron and the buckyball are shown in black. A black dot in the left part of the decagonal section corresponds to the apex of the icosahedron.

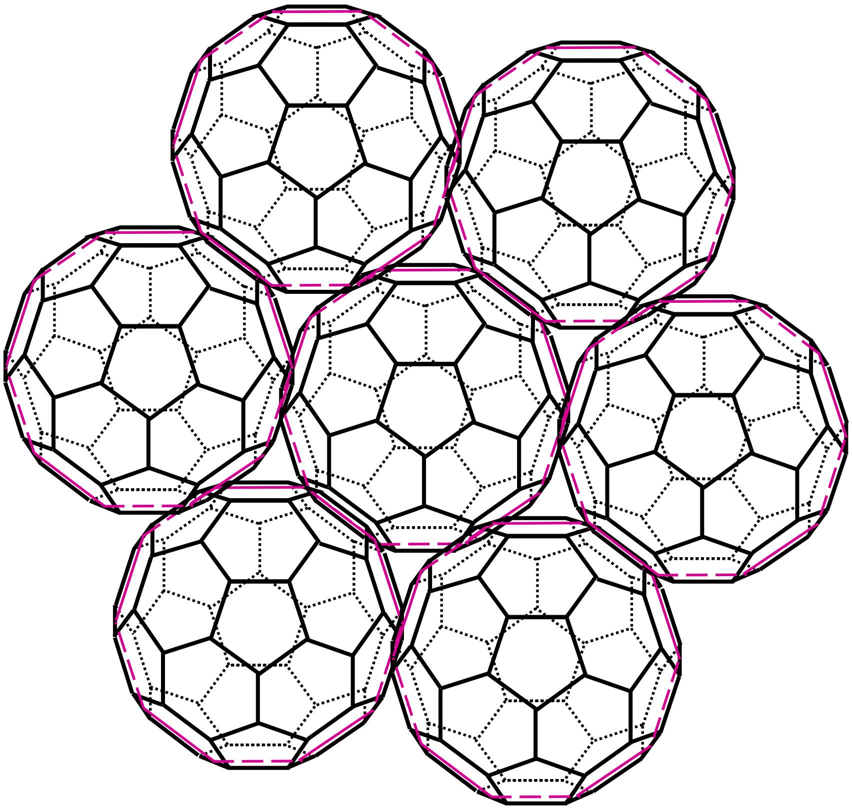

Figure 5. Fragment of an assembly of interlocked buckyballs. The buckyballs are resting on their pentagonal faces. The regular decagons in the middle section are shown in magenta.