Quasi-capture in Hill problem

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Introduction

The origin of irregular satellites of the giant planets has been discussed for long period. And now there is no doubt that these satellites were capture. We have no time and possibilities to give the review of all the models of capturing process, but it is possible to point out the following main approaches:

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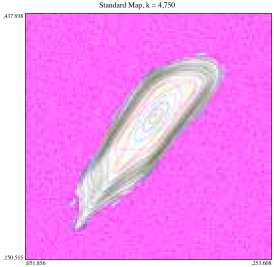
- Collisional capture scenario developed by F. Marzari and collegues
- The model of quasi-satellite capture investigated by L. Zelenyi, A. Neishtadt, V. Sidorenko
- The model of temporary capture based on stickiness phenomenon opened by G. Contopoulos

The aim of the talk

The aim of the talk is to demonstrate the presence of stickiness phenomenon in one of the most simple celestial mechanical model — Hill problem, and to discuss the application of the problem to outer satellites of the giant planets.

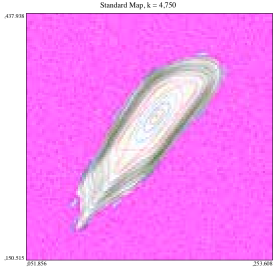
Stickiness phenomenon



Standart Map example

The picture demonstrate many iterations of points with different behavior: regular and chaotic.

Stickiness phenomenon



Definition

The stickiness is a temporary concentration of chaotic orbits in particular region of the phase space.

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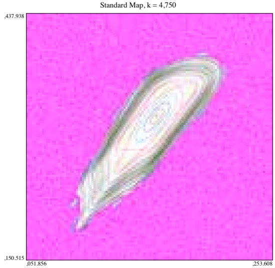

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Definition

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The first example of stickiness phenomenon was proposed by G. Contopoulos in 1971. It was called by Karney in 1983 and was investigated in many works till last 30-ty years.

Types of stickiness

Two types of stickiness are distinguished now:

- Stickiness around an island of stability or near the last invariant torus
- Stickiness near the asymptotic curve of unstable periodic solution which can extend into the region of chaotic motion

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- Stickiness near the asymptotic curve of unstable periodic solution which can extend into the region of chaotic motion

The most interesting stickiness effects around an island of stability appear when the last invariant curve surrounding the island is destroyed, as the perturbation increases. Then a new last invariant curve is formed closer to the center of the island and the former last invariant curve becomes a cantorus with infinite holes. Orbits between this cantorus and the new last invariant curve stay for some time in this region before they escape into the chaotic sea.

Cantori

It is a well known fact (C. Efthymiopoulos et al. *CM&DA*, **73: 221-230**, 1999) that stickiness phenomenon occures due to the presence of so called cantori which are the remain parts of the invariant tori.
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Invariant torus of the map is destroyed by the perturbation and form the Cantor sets of points on the section surface.

Definition

Cantorus is an invariant set consisting of infinite points but these points don't form continuos line and leave small gaps.
The rotation number of cantorus is equal to the rotation number of the corresponding invariant torus and thus it is possible to approximate this cantorus with resonances of suitable multiplicity.

Cantori and temporary capture

Partial barrier

A cantorus forms only a partial barrier to chaotic orbits which can penetrate into and abandon the vicinity of the last invariant torus passing through the gaps of the cantorus. Such a behavior of sticky orbits may be used for the explanation of temporary capture phenomenon. G. Zaslavsky, who worked in IKI, investigated such cases.

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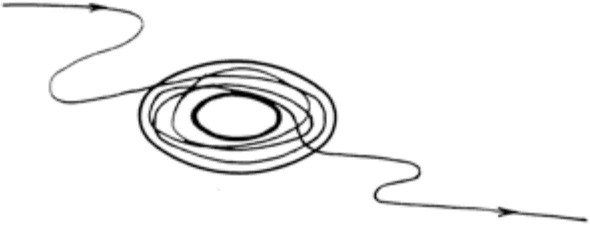

Chaotic flux minimization

Noble numbers

The rotation number ω of an invariant torus is defined by the continuous fraction:

$$\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} = [a_0, a_1, a_2, \ldots], \text{ where } a_i \in \mathbb{N}.$$

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The most robust invariant tori

Invariant tori with the noble rotation number are locally the most robust tori which create rather stiff obstacles to chaotic orbits and by that minimize chaotic flux (see R.S. MacKay et al. *Nonlinearity*, 5:867-888,1992)

Cantorus approximation

Each invariant torus may be approximated with the resonances (periodical orbits) with suitable multiplicity. Cantorus with noble rotation number $\omega = [a_0, a_1, \ldots, 1, 1, 1, \ldots]$ is approximated with resonances which multiplicity form the sequence of so called Farey fractions.

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E.g. for rotation number $\omega = (\sqrt{5} - 1)/2 = [0, 1, 1, \ldots]$ one can obtain the sequence:

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots$$

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Boundary islands surround the stability area of periodic orbit. Parameter change leads to appearance of resonant islands chain out of stability area.

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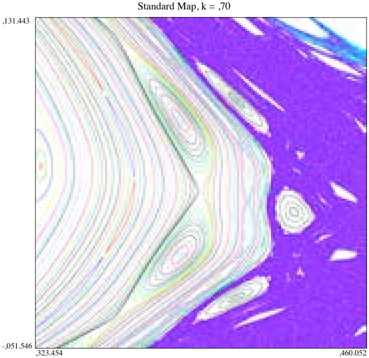
The figure demonstrate the appearance of boundary islands for the standard map with control parameter $K = 0.70$.



From quasi-capture to capture

We considered the basic mechanisms of stickiness phenomenon in Hamiltonian systems with two degrees of freedom. The presence of integral of energy makes possible only temporary capture for nondissipative systems. Due to the time-reversal symmetry of equations of motion one can state that sticky trajectories have penetrated earlier into the stickiness region and stay there for a long time.
In real planet systems one has to take into account many other factors that may transform temporary capture to total capture (e. g. tidal forces, presence of protoatmosphere and so on).

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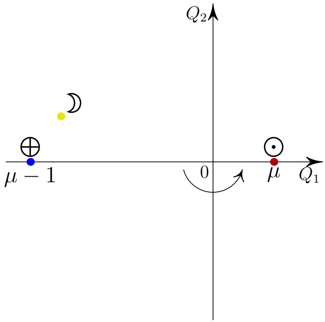

Restricted three-body problem

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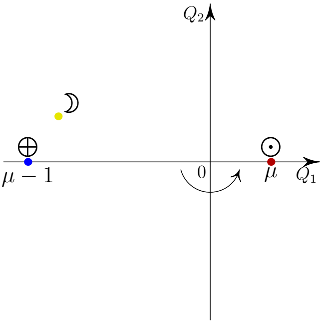

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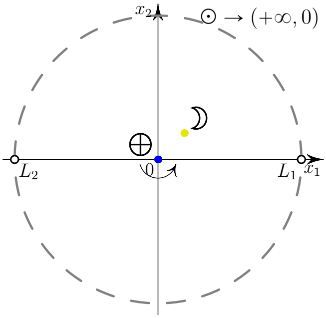
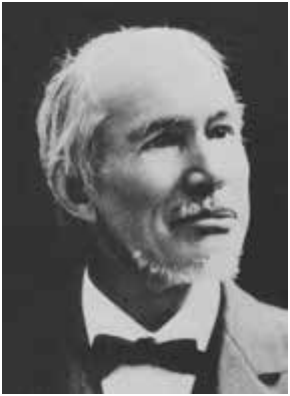


Hill problem

Hill problem contibutor

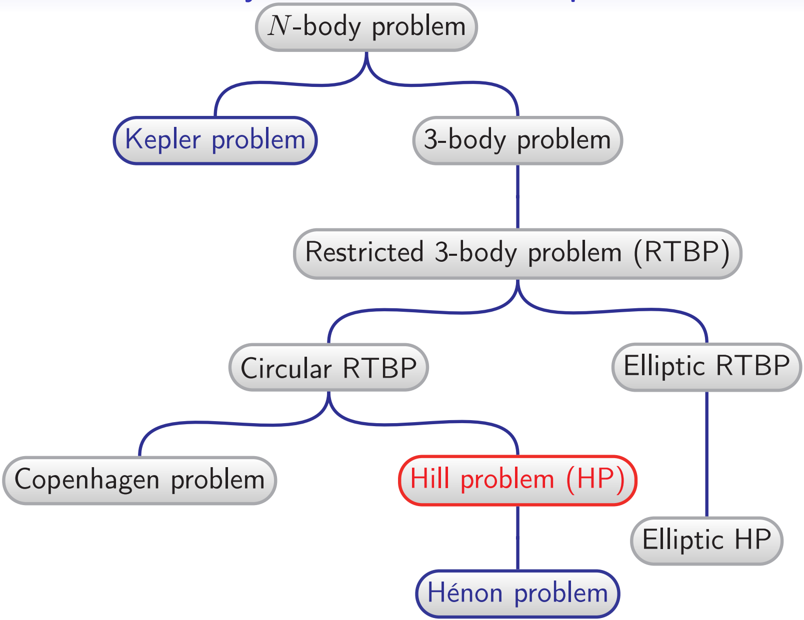
It has a lot of applications and originally was proposed by George Hill for the Moon motion theory.The significant contribution in studying periodic solutions of the Hill problem was provided by Prof. Michel Hénon.



George Hill

Michel Hénon

Hierarchy of celestial mechanics problems



Hill problem Hamiltonian

$$H = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 + \frac{1}{2}\left(x_1^2 + x_2^2\right) - \frac{3}{2}x_1^2 + \frac{1}{r}$$

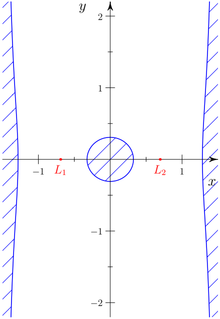
- Kinetic energy
- Potential of Coriolis forces
- Potential of centrifugal force
- Gravitational potential of the Sun
- Potential of the central body

Regions of possible motion

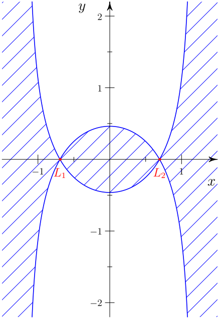
The canonical equations of Hill problem Hamiltonian has first integral called Jacobi integral

$$J = 3x_1^2 - 2/|\mathbf{x}| - \dot{x}_1^2 - \dot{x}_2^2 = C,\ C = -2H.$$

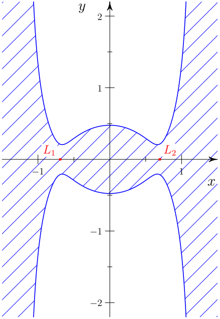
Regions of possible motion (Hill regions)



1) $C = 6.5$



2) $C = 3^{4/3}$



3) $C = 4.2$

Hill problem properties

Hill problem has two symmetries of extended phase space given by linear transformations

$$
\begin{aligned}
\Sigma_1 &: (t, x_1, x_2, y_1, y_2) \rightarrow (-t, x_1, -x_2, -y_1, y_2) \\
\Sigma_2 &: (t, x_1, x_2, y_1, y_2) \rightarrow (-t, -x_1, x_2, y_1, -y_2)\,, \\
\Sigma_{12} \equiv \Sigma_1 \circ \Sigma_2 &: (t, x_1, x_2, x_1, x_2) \rightarrow (t, -x_1, -x_2, -y_1, -y_2)
\end{aligned}
$$

- Symmetric periodic orbits of Hill problem can be extended into the periodic orbits of the restricted three-body problem (L. Perko)
- Periodic orbits with period $T \neq 2\pi$ can be extended into orbits of general three-body problem (K. Meyer)

Sticky region investigation

The region around the family of direct satellite orbits g' and the family of retrograde orbits f. The structure of the phase space in the vicinity of this family is shown on the figure with Poincare section of phase flow with plane $y = 0$.

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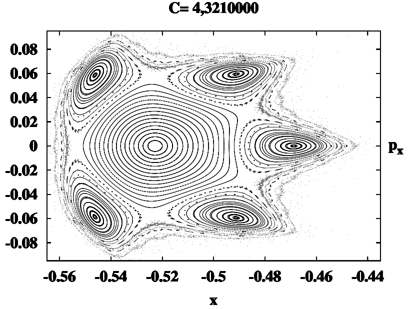

Cantorus with $\omega = [0,4,1,1,\ldots]$

The robust cantorus with rotation number $\omega = [0,4,1,1,\ldots] \approx 0.2165423$ can be approximated with resonances with such multiplicities:

$$\frac{1}{4}, \frac{1}{5}, \frac{2}{9}, \frac{3}{14}, \frac{5}{23}, \ldots$$

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Each of resonances mentioned above appears in the vicinity of the periodic solutions family $\boldsymbol{g'}$ and its islands of stability move away from $\boldsymbol{g'}$ and disappear in chaotic sea.

For the value $C = 4.321$ the sticky region was found. It is placed in the outer space of the last invariant curve and consists of chaotic trajectories with different time of escape. Many of these orbits make more then 10^4 revolutions before escaping into the chaotic sea.

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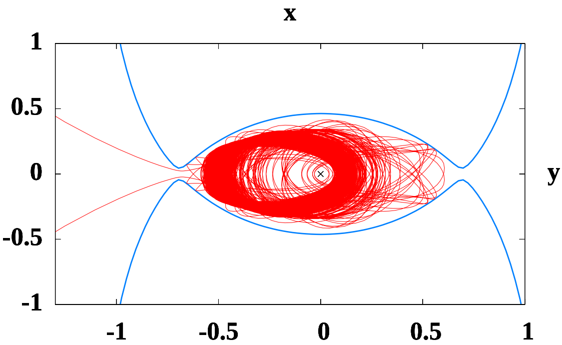
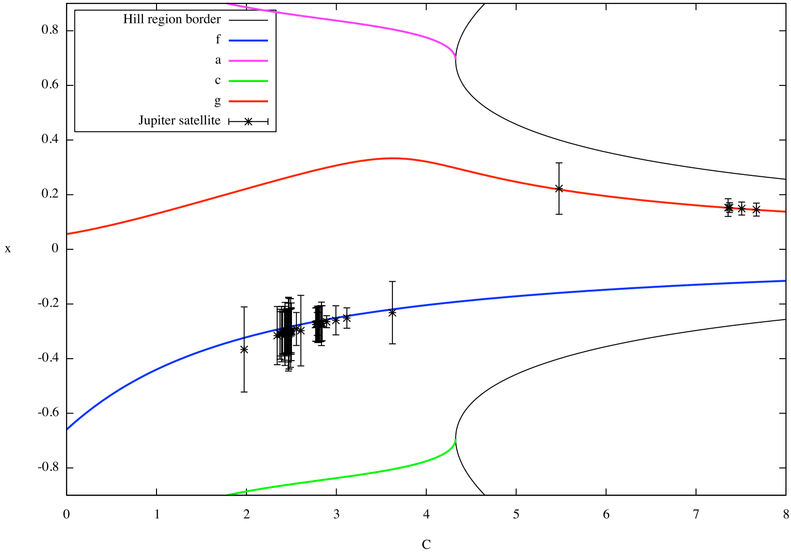


Natural satellites of the giant planets

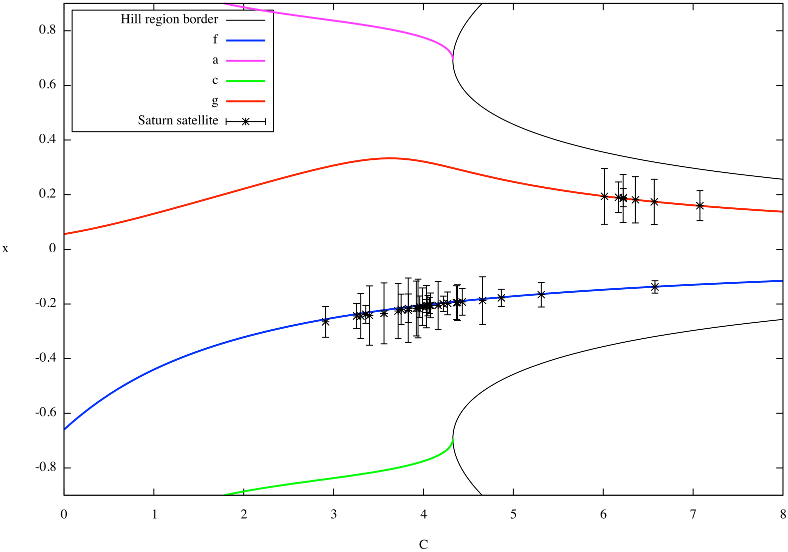
The contemporary database of natural satellites of Solar system's giant planets were used from the Natural Satellite Data Center of IMCCE, Paris and SAI, Moscow and from database of NASA JPL. The main parameters of the satellite orbits such as semi-major axis, eccentricity and orbit inclination were computed into the corresponding Hill unit of length equals to $\mu^{1/3}a'$, where μ is the ratio of the mass of the planet to the total mass of Sun and the planet, a' is the semi-major axis of the planet's orbit.

We have checked parameters of 59 Jupiter's outer satellites, 38 outer Saturn's outer satellites, 9 Uranus' outer satellites and 5 Neptune's outer satellites.

Jupiter satellites distribution



Saturn satellites distribution



Results

There was found only one Jupiter's satellite S/2003 J2, which orbital motion is near the inner boundary of stability island around the family f of retrograde satellite orbits.

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