ON PROPERTIES OF SOME
OF THE EIGENELEMENTS AND ASSOCIATED ELEMENTS
OF SELFADJOINT QUADRATIC OPERATOR PENCILS

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We consider quadratic operator pencils

\[ L(\lambda) = I - \lambda B - \lambda^2 C, \]

where \( B \) and \( C \) are selfadjoint bounded operators acting in a Hilbert space \( \mathcal{H} \), and \( I \) is the identity operator. Under the assumption that the discrete spectrum of the pencil is nonempty, we single out certain subsystems \( E^+ \) and \( E^- \) of the eigenelements and associated elements (EAE) of the pencil (1) and obtain assertions about minimality and completeness with respect to them.

In another part of the paper the pencil (1) is studied under the assumption

\[ (Bx, x)^2 > -4(Cx, x)(x, x) \quad \forall x \in \mathcal{H}, x \neq 0, \]

called the strong damping condition (see [1] and [2]). The systems \( E^+ \) and \( E^- \) for such pencils are minimal in \( \mathcal{H} \), and we also obtain conditions for them to be complete and to have the basis property. As a corollary we obtain assertions about the basis property for certain trigonometric systems of functions.

1. With the quadratic pencil (1) we associate the operators

\[ \mathcal{L} = \begin{pmatrix} B & C \\ I & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & I \\ I & -B \end{pmatrix} \]

acting in the Hilbert space \( \mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H} \).

The operator \( \mathcal{L} \) is known [2] to be \( G \)-selfadjoint, i.e., \( (G\mathcal{L})^* = G\mathcal{L} \). The discrete spectrum of the pencil \( L(\lambda) \) is denoted by \( \sigma_d(L) \) (\( \mu \in \sigma_d(L) \) if \( \mu^{-1} \) is a normal eigenvalue (EV) of \( \mathcal{L} \)). To a number \( \mu \in \sigma_d(L) \) there corresponds a canonical (in the Keldysh sense [3]) system of EAE of the pencil \( L(\lambda) \):

\[ y^0_j, \ldots, y^{p_j}_j, \quad j = 1, \ldots, N. \]

To the system (3) of EAE there corresponds the dual canonical system

\[ z^0_j, \ldots, z^{p_j}_j, \quad j = 1, \ldots, N, \]

which is uniquely determined from the form of the principal part of the expansion of the operator-valued function \( L^{-1}(\lambda) \) in a neighborhood of the pole \( \lambda = \mu \) (see [3] and [4]). If \( \mu \in \sigma_d(L) \cap \mathbb{R} \), then it is assumed that the canonical system (3) is chosen to be normal, i.e., satisfies the equalities

\[ y^\nu_j = \varepsilon_j z^\nu_j, \quad \varepsilon_j = \pm 1, \quad \nu = 0, 1, \ldots, p_j, \quad j = 1, \ldots, N. \]

The existence of a normal canonical system is ensured by Lemma 1.1 in [4].

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From the normal canonical system (3) corresponding to an EV $\mu \in \sigma_d(L) \cap R$ we extract subsystems

$$y_j^0, \ldots, y_j^N, \quad j = 1, \ldots, N,$$

$$y_j^0, \ldots, y_j^N, \quad j = 1, \ldots, N,$$

where the numbers $\kappa_j^{(+)}$ are defined as follows:

$$\kappa_j^{(+)} = \begin{cases} l_j & \text{if } p_j = 2l_j + 1 \text{ or } p_j = 2l_j \text{ and } \varepsilon_j > 0 \quad (\varepsilon_j < 0), \\ l_j - 1 & \text{if } p_j = 2l_j \text{ and } \varepsilon_j < 0 \quad (\varepsilon_j > 0). \end{cases}$$

We tacitly assume that for $\kappa_j^{(-)} = -1$ the elements of the chain (3) do not appear in the subsystem (4) (the subsystem (5)).

**DEFINITION.** The collection of all canonical systems (3) of the pencil $L(\lambda)$ corresponding to EV's $\mu \in \sigma_d(L)$, $\Im \mu > 0$, combined with all subsystems (4) (respectively, (5)) corresponding to EV's $\mu \in \sigma_d(L) \cap R$, is called the first (second) subsystem of EAE of the pencil $L(\lambda)$, and is denoted by $E^+ (E^-)$.

**REMARK 1.** In the case $\ker C = 0$ it is natural to call the systems $E^+$ and $E^ -$ the first and second halves of the EAE of $L(\lambda)$ (see [3] and [4]).

**THEOREM 1.** The systems $E^+$ and $E^-$ for the pencil (1) are minimal in the space $\mathcal{H}$.

**PROOF.** For definiteness we consider $E^-$. Denote by $\mathcal{L}^+$ the closure in $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ of all the elements of the form $\tilde{y}_k^\nu = \{y_k^\nu, \lambda_k y_k^\nu + y_k^\nu \}$, where $y_k^\nu \in E^-$ (here and below it is assumed that $y_k^\nu = 0$ if $\nu < 0$). The elements $\tilde{y}_k^\nu$ form a system of EAE of the operator $\mathcal{L}$; therefore, the subspace $\mathcal{L}^+$ is invariant with respect to $\mathcal{L}$. In $\mathcal{H}^2$ we consider the indefinite inner product $[\hat{x}, \hat{x}] = (G\hat{x}, \hat{x})$ and show that $\mathcal{L}^+$ is $G$-nonnegative, i.e., $[\hat{x}, \hat{x}] \geq 0$ for all $x \in \mathcal{H}$.

In order not to introduce additional multi-indices we assume that the EV's of the pencil $L(\lambda)$ are counted as many times as there are chains of EAE corresponding to them. Since $L(\lambda)$ is selfadjoint, it can be assumed that the nonreal EV's are indexed and the corresponding canonical systems are chosen in such a way that $\lambda_k = \overline{\lambda}_k$ and $y_k^\nu = z_k^\nu \overline{z_k^\nu}$. Depending on the situation, the elements $\hat{x} \in \mathcal{H}^2$ will be written in a row or in a column.

The biorthogonality relations of Keldysh for $L(\lambda)$ can be written in the form (see [4], Russian p. 42 (English p. 112))

$$\left(\begin{array}{c} y_k^h \\ \lambda_j y_k^h + y_k^{h-1} \end{array}\right) \left(\begin{array}{c} B + \overline{\lambda}_m C \\ Cz_m^s \end{array}\right) = -\delta_{k,m} \delta_{h,p_m-s}.$$

The definition of the EAE of $L(\lambda)$ gives us the equalities

$$(I - \overline{\lambda}_m B - \overline{\lambda}_m^2 C)z_m^s - (B + 2\overline{\lambda}_m C)z_m^{s-1} - Cz_m^{s-2} = 0, \quad s = 0, 1, \ldots, p_m,$$

from which it is not hard to get (see again [4], loc. cit.)

$$\left(\begin{array}{c} 0 & I \\ I & -B \end{array}\right) \left(\begin{array}{c} \tau_m^s \\ \rho_m^s \end{array}\right) = \overline{\lambda}_m \left(\begin{array}{c} (B + \overline{\lambda}_m C)z_m^s + Cz_m^{s-1} \\ Cz_m^s \end{array}\right),$$

where

$$\tau_m^s = \left\{\begin{array}{c} \overline{\lambda}_m z_m^s - 2\overline{\lambda}_m z_m^{s-1} + \ldots + (-1)^s(s + 1)\overline{\lambda}_m^{s-s}z_m^0 \\ z_m^s - \overline{\lambda}_m z_m^{s-1} + \ldots + (-1)^s\overline{\lambda}_m^{s-1}z_m^0 \end{array}\right\}$$

$$\rho_m^s = \left\{\begin{array}{c} \overline{\lambda}_m z_m^s - 2\overline{\lambda}_m z_m^{s-1} + \ldots + (-1)^s(s + 1)\overline{\lambda}_m^{s-s}z_m^0 \\ z_m^s - \overline{\lambda}_m z_m^{s-1} + \ldots + (-1)^s\overline{\lambda}_m^{s-1}z_m^0 \end{array}\right\}.$$

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Therefore the biorthogonality relations (6) can be transformed to the form
\[
[y^h_k, \bar{z}^s_m - 2 \bar{\lambda}^{-1} \bar{z}^s_m - s - 1 + \cdots + (-1)^s(s + 1)\bar{\lambda}^{-1} \bar{z}^0_m] = -|\lambda_m|^2 \delta_{k,m} \delta_{h,p_m - s}.
\]
The last relations and the definition of the system \(E^\perp\) give us that for all \(y^h_k, y^s_m \in E^\perp\)
\[
[y^h_k, y^s_m] = \begin{cases} 0 & \text{if } k \neq m \text{ or } k = m \text{ but } h + s < p_k, \\ |\lambda_k|^2 & \text{if } k = m \text{ and } h = s = p_k/2. \end{cases}
\]
It follows from these equalities that the subspace \(\mathcal{L}^+\) is G-nonnegative.

Let
\[
x_0 = \sum c_{k,h} y^h_k, \quad x_1 = \sum c_{k,h} (\lambda_k y^h_k + y^{h-1}_k),
\]
where \(y^h_k \in E^\perp\), and the sum is assumed to be finite and is over the indices \(k\) and \(h\). A device from [7] will be used to prove the estimate \(\|x_0\| \leq M\|x_1\|\), where the constant \(M\) does not depend on the numbers \(c_{k,h}\). We have that \(\bar{x} = \{x_0, x_1\} \in \mathcal{L}^+\); therefore,
\[
(7) \quad \|[\mathcal{L} \bar{x}, \bar{x}]\| \leq [\bar{x}, \bar{x}]^{1/2} [\mathcal{L} \bar{x}, \bar{x}]^{1/2},
\]
since the quadratic form \([\bar{x}, \bar{x}]\) is nonnegative on \(\mathcal{L}^+\), while the subspace \(\mathcal{L}^+\) is invariant with respect to \(\mathcal{L}\). By the equalities
\[
[\mathcal{L} \bar{x}, \bar{x}] = (x_0, x_0) - (C x_1, x_1),
\]
\[
[\bar{x}, \bar{x}] = 2 \text{Re}(x_0, x_1) - (B x_1, x_1),
\]
\[
[\mathcal{L} \bar{x}, \bar{x}] = 2 \text{Re}(x_0, C x_1) + (B x_0, x_0),
\]
we get the estimate \(\|x_0\| \leq M\|x_1\|\) from (7). From this estimate it follows at once (see [8]) that the system \(\{\lambda_k y^k_k + y^{k-1}_k\}, y^k_k \in E^\perp\), is minimal in \(\mathcal{F}\). But is is then easy to get (see [9]) that the system \(E^\perp\) is also minimal in \(\mathcal{F}\). Theorem 1 is proved.

Remark 2. The question of minimality of the systems \(E^+\) and \(E^-\) was considered in [4]–[6] under the assumption that the number of real EV's is finite along with certain other additional assumptions.

2. Below we present some theorems whose detailed proofs will be given elsewhere. Where possible (except for Theorem 6), we give the ideas of the proofs.

Let \(C = C_+ - C_-\), where \(C_+ = P^+ C = C P^+ \geq 0\), and \(P^\pm\) are orthogonal projections such that \(P^+ + P^- = I\). Denote by \(\mathcal{F}_{\theta} (\mathcal{F}_{0} = \mathcal{F})\) the scale of Hilbert spaces generated by the operator \(|C|^{-1} > 0\), where \(|C| = C_+ + C_-\) (in order not to make additional reservations we assume that \(\text{Ker} C = 0\)).

Theorem 2. Suppose that for some \(q > 0\) the resolvent \(L^{-1}(\lambda)\) of the pencil (1) is a meromorphic operator-valued function of order less than \(q\) and that there exist rays \(\gamma_j (j = 1, \ldots, n)\) symmetric relative to the real axis which separate the complex plane into sectors of angle at most \(\pi/q\) on which the estimate
\[
(8) \quad \| |C|^{1/2} L^{-1}(\lambda) \| \leq M |\lambda|^{-1}
\]
holds as \(|\lambda| \to \infty\). Assume that \(\text{Ker} C = 0\).

Then the systems \(E^+\) and \(E^-\) are complete in the space \(\mathcal{F}_{-1/4}\).

The proof is obtained by studying the function
\[
F(\lambda) = (L^{-1}(\lambda)|C|^{1/4} f, |C|^{1/4} f),
\]
where the element \(f \in \mathcal{F}\) is such that \((f, |C|^{1/4} y^h_k) = 0 \forall y^h_k \in E^+(E^-)\). Using an interpolation theorem and the equality
\[
|C| L^{-1}(\lambda)|C| \varphi = -\lambda^{-2} C \varphi - \lambda^{-1} |C| L^{-1}(\lambda) (BJ \varphi - \lambda^{-1} J \varphi), \quad J = P^+ - P^-,
\]
we can get the estimate \(F(\lambda) = o(|\lambda|^{-1})\) as \(\lambda \to \infty\) on the rays \(\{\gamma_j\}\) and then use the method in [4].
THEOREM 3. Suppose that the conditions of Theorem 2 hold with the difference that instead of (8) the estimate \( \| L^{-1}(\lambda) \| \leq M \) holds on the rays \( \{ \gamma_j \} \). Assume that the operator \( C_\lambda \) is finite-dimensional and that the estimates

\[
\| L^{-1}(\lambda)B \| \leq M|\lambda|^{-1}, \quad P^+L(\lambda)P^+ \geq \delta P^+, \quad \delta > 0,
\]

hold for \( \lambda \in \mathbb{R} \) and \( |\lambda| \) sufficiently large.

Then the systems \( E^+ \) and \( E^- \) are complete in \( \mathfrak{S} \).

The proof is according to the following scheme. Let \( f \perp y_k^h \in E^+(E^-) \). Then the function \( F(\lambda) = (L^{-1}(\lambda)f, f) \) has the following properties:

1. The function \( F(\lambda) \) is meromorphic and uniformly bounded by \( f \) in \( E' \) and \( E'' \) of \( \mathfrak{S} \).

2. The decomposition \( F(\lambda) = a\lambda^{-1} + O(\lambda^{-2}) \) is valid in a neighborhood of \( \infty \).

3. \( F(\lambda) = (P^+L^{-1}(\lambda)P^+f, f) + F_1(\lambda) \), where \( F_1(\lambda) = O(\lambda^{-2}) \) for \( \lambda \in \mathbb{R} \), \( |\lambda| \to \infty \), and \( (P^+L^{-1}(\lambda)P^+f, f) \geq 0 \) for \( \lambda \in \mathbb{R}, \ |\lambda| > r_0 \).

It follows from properties 1–3) that \( F(\lambda) \equiv 0 \) and \( f = 0 \).

DEFINITION. A subsystem \( E \) of the system of \( \mathfrak{S} \) of \( \mathfrak{S} \) is said to be admissible if \( E^+ \cap E^- \subseteq E \subseteq E^+ \cup E^- \). A system \( E' \) is said to be complementary to \( E \) if \( E' \cap E = E^+ \cap E^- \), while \( E' \cup E = E^+ \cup E^- \).

THEOREM 4. Suppose that an admissible system \( E \) is minimal in the space \( \mathfrak{S} \) (\( \mathfrak{S}_{1/2} \)) and that the conditions of Theorem 2 hold.

Then a complementary system \( E' \) is complete in the space \( \mathfrak{S}_{-1/2}(\mathfrak{S}) \).

To prove completeness in \( \mathfrak{S}_{-1/2} \) we consider the functions

\[
F_{m,s}(\lambda) = (L^{-1}(\lambda)|C|^{1/2}f, g_m),
\]

where \( f \perp y_k^h \in E' \), and \( \{g_m\} \) is the system biorthogonal to \( E \), i.e., \( (y_k^h, g_m^s) = \delta_{km}\delta_{sh} \).

The functions \( F_{m,s}(\lambda) \) have only one pole \( \lambda = \lambda_m \), and the functions \( F_{m,s}(\lambda) \) of the conditions of Theorem 2 we can deduce that \( F_{m,s}(\lambda) \equiv 0 \). Then the vector-valued function \( L^{-1}(\lambda)|C|^{1/2}f \) is entire; hence \( f = 0 \).

THEOREM 5. Suppose that the pencil \( L(\lambda) \) has finitely many real \( EV \)'s, and the admissible system \( E \) is complete in \( \mathfrak{S} \), while the complementary \( E' \) is complete in \( \mathfrak{S}_{-1/2} \).

Then \( E' \) is minimal in \( \mathfrak{S}_{-1/2} \), and \( E \) is minimal in \( \mathfrak{S} \). In particular, under the conditions of Theorem 3 the systems \( E^+ \) and \( E^- \) are minimal in \( \mathfrak{S}_{-1/2} \).

The proof is carried out in the following way. Under the conditions of Theorem 5 it is possible to use the biorthogonality relations (6) to obtain the factorization

\[
L(\lambda) = [I - \lambda(B - Z)][I - \lambda Z],
\]

where \( Z = K|C|^{1/2} \), \( K \) is bounded on \( \mathfrak{S} \), and the system of EAE of \( E \) coincides with some admissible system \( E_1 \subseteq E' \). Next we use the

LEMMA. If the factorization (10) is valid and \( E_1 \) is a system of EAE of \( Z \), then the complementary system \( E'_1 \) is a system of EAE of the operator \( B - Z^* \).

If the inclusion \( E_1 \subseteq E' \) is strict, then so is the inclusion \( E'_1 \subseteq E \). But \( E \) is complete in \( \mathfrak{S} \), while \( E'_1 \) is minimal in \( \mathfrak{S} \); hence \( E'_1 = E \).

REMARK 3. Under the conditions of Theorem 3, the factor \( Z \) responsible for the system \( E^+ (E^-) \) has a representation \( Z = K|C|^{1/2} \), where \( K \) is bounded and boundedly invertible in \( \mathfrak{S} \) (see [9]). The last circumstance enables us to obtain analogues of Theorem 1 in [9] and Theorem 5.1 in [4].

REMARK 4. The Theorems 1–5 remain true if instead of the pencil (1) we consider a pencil \( L(\lambda) = A - \lambda B - \lambda^2 C \) with the operator \( A = A^* \) bounded in \( \mathfrak{S} \), and require that
there exist a $\lambda_0 \in \mathbb{R}$ such that $L(\lambda_0)$ is invertible and $L(\lambda_0) > 0$. A change arises in the proof of the factorization in Theorem 5. Instead of (6) we use the biorthogonality relations

\[(y_k^h, |C| y_m^a) = 2(y_k^h, C_+ y_m^n) + (\mu_k^{-1} \eta_k^h, \mu_k^{-1} a_0 \eta_m^a),\]

where

\[\eta_k^h = y_k^h - \mu_k^{-1} y_k^h - \cdots - (-1)^{h} \mu_k^{-h} y_k^0, \quad \mu_k = \lambda_k + \lambda_0, \quad y_k^h, y_m^n \in E^+ \cap E^- .\]

Under the conditions of Theorem 5 it is possible to prove boundedness in $\mathcal{H}$ of the operator $K$ determined by $K(|C|^{1/2} y_k^h) = \mu_k^{-1} a_0^{1/2} \eta_k^h$, where $y_k^h \in E_1$ and $E^+ \cap E^- \subseteq E_1 \subseteq E'$ (for $A \neq I$ the same biorthogonality relations must also be used in Theorems 2.1 and 2.2 of [4]).

We mention also that Theorems 3 and 5 contain Theorem 4 of [6], whose proof in [6] contains an error.

3. It is known that a pencil (1) satisfying condition (2) has only simple real EV's (under the assumption that the spectrum of the pencil is discrete, i.e., all the nonzero points of the spectrum of $\mathcal{L}$ are normal EV's). A simple check shows that the system $E^+ (E^-)$ coincides for such a pencil with a system of eigenelements (EE's) corresponding to the negative (positive) EV's.

**THEOREM 6.** Suppose that the pencil (1) satisfies condition (2) and has a discrete spectrum.

If $\text{Ker } C = 0$, then both systems $E^+$ and $E^-$ are complete in $\mathcal{H}$. But if $\text{Ker } C \neq 0$, then for $x \in \text{Ker } C$ the form $(Bx, x)$ preserves its sign. In the case where $(Bx, x) < 0$ for $x \in \text{Ker } C$ the system $E^+$ is complete in $\mathcal{H}$, but the system $E^-$ is not complete; however, it becomes complete after the addition of elements forming a basis in the subspace $\text{Ker } C$. If $(Bx, x) > 0$ for $x \in \text{Ker } C$, then the systems $E^+$ and $E^-$ must be interchanged.

**THEOREM 7.** Suppose that the spectrum of the pencil (1) is discrete, condition (2) holds, and $C \geq 0$, $B \leq 0$ ($B \geq 0$).

Then the system $E^+ (E^-)$ forms a Riesz basis in $\mathcal{H}$, and the system $\{C^{1/2} y_k\}, y_k \in E^- (E^+)$, forms a Riesz basis in the space $\mathcal{H}^0$, where $\mathcal{H}^0 = \text{Im } C$.

The proof is carried out with the help of Theorem 6, an a priori estimate in [8] obtained by integration by parts, and a device in [9].

It is useful to mention a variant of Theorem 7 for pencils in the unbounded case.

**THEOREM 8.** Let

\[(11) \quad T(\lambda) = H - \lambda F - \lambda^2 Q, \]

where $H$ is a positive selfadjoint operator acting in $\mathcal{H}$ with discrete spectrum, $Q$ is an orthogonal projection, and $F$ is a symmetric operator whose domain satisfies the inclusion $\mathcal{D}(F) \supseteq \mathcal{D}(H^{1/2})$. Suppose that $F \leq 0$ ($\geq 0$), and $(Fx, x) < 0$ ($> 0$) for all $x \in \mathcal{D}(H^{1/2}) \cap \text{Ker } Q$.

Then the system $E^+ (E^-)$ consisting of EE's of the pencil $T(\lambda)$ corresponding to the positive (negative) EV's forms a Riesz basis in the space $\mathcal{H}^{1/2}$, where $\mathcal{H}^{1/2}$ is the scale of Hilbert spaces generated by the operator $H > 0$. For this system $Q(E^-) (Q(E^+))$ forms a Riesz basis in the space $\mathcal{H}^0$, where $\mathcal{H}^0 = \text{Im } Q$.

4. The spectral problem

\[(12) \quad -y'' - \lambda^2 y = 0, \quad y(0) = 0, \quad y'(\pi) - \lambda y(\pi) = 0\]

is considered in [10].
In the Hilbert space \( \mathcal{H} = L_2[0, \pi] \oplus \mathbb{C} \) we consider operators \( H, F, \) and \( Q \) defined on elements \( \tilde{y} \in \mathcal{D}(H) \)

\[
\mathcal{D}(H) = \{ \tilde{y} : \tilde{y} = \{ y, y(\pi) \}, y \in W^2_2[0, \pi], y(0) = 0 \},
\]

by the equalities

\[
H\tilde{y} = (-y''(\pi)), \quad F\tilde{y} = \{0, y(\pi)\}, \quad Q\tilde{y} = \{y, 0\}.
\]

Then the problem (12) will take the form (11), with

\[
E^+ = \left\{ \sin \left( n + \frac{1}{4} \right) x \right\}_{n=0}^{\infty}, \quad E^- = \left\{ \sin \left( n + \frac{1}{4} \right) x \right\}_{n=1}^{\infty}.
\]

Let \( W^1_{2,U} \) be the subspace of functions \( y(x) \in W^1_2[0, \pi] \) such that \( y(0) = 0 \). It can be shown (see [6], Proposition 1) that in the example considered \( W^1_{2,U} \) is isomorphic to \( \mathcal{H}_{1/2} \). As a corollary to Theorem 5 we get

**Proposition 1.** The system \( \{ \sin(n - \frac{1}{4})x \}_{n=1}^{\infty} \) forms a Riesz basis in \( L_2[0, \pi] \), and the system \( \{ n^{-1} \sin(n + \frac{1}{4})x \}_{n=0}^{\infty} \) forms a Riesz basis in \( W^1_{2,U} \).

We can also consider the spectral problem of the form (12) in which the condition \( y(0) = 0 \) is replaced by \( y'(0) = 0 \). This problem is representable in the form (11), but now with an operator \( H \geq 0 \). Despite this circumstance, Theorem 5 also remains in force for this concrete problem. Namely,

**Proposition 2.** The systems

\[
E^+ = \left\{ \cos \left( n + \frac{1}{4} \right) x \right\}_{n=0}^{\infty}, \quad E^- = 1 \cup \left\{ n^{-1} \cos \left( n - \frac{1}{4} \right) x \right\}_{n=1}^{\infty}
\]

form Riesz bases in the respective spaces \( L_2[0, \pi] \) and \( W^1_2[0, \pi] \).

**Remark 5.** The systems

\[
\left\{ \cos \left( n - \frac{1}{4} \right) x \right\}_{n=1}^{\infty}, \quad \left\{ \sin \left( n - \frac{1}{4} \right) x \right\}_{n=1}^{\infty}
\]

are complete and minimal in \( L_2 \), but are not Riesz bases in \( L_2 \). This is easy to get from results in [12]. What is more, these systems are not ordinary bases in \( L_2 \) (see [11]).

The system \( \{ \sin(n - \frac{1}{4})x \}_{n=1}^{\infty} \) was considered in [10], where the fact that it is a basis in \( L_p \) for \( p > 2 \) was mentioned. It follows from results in [11] that this system is a basis in \( L_p \) for \( p > 1 \). The fact that it is an unconditional basis in \( L_2 \) was apparently not noticed. Here the proof of this fact was obtained as a corollary to a general result, but we should mention that in view of the concreteness of the system it can also be obtained more simply without the use of operator methods. After the author communicated the results obtained here, B. S. Kashin suggested a simpler way of proving the estimates

\[
M_1 \sum |c_n|^2 \leq \left\| \sum c_n \sin \left( n - \frac{1}{4} \right) x \right\|_{L_2}^2 \leq M_2 \sum |c_n|^2.
\]

He observed that they can be obtained from a result of Ingham (see [12], or [14], p. 397). The completeness of this system in \( L_2 \) follows from a theorem of Levinson [13] (p. 6).

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BIBLIOGRAPHY


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