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A rigorous mathematical formalism for calculating the propagation of light rays in the stationary post-Newtonian field of an isolated celestial body (or system of bodies) considered as a gravitational lens having a complex multipole structure is developed. Symmetric trace-free tensors are used in the definition of gravitational multipoles instead of the less convenient (in general situations) scalar and vector spherical harmonics. Two types of perturbations of light rays, caused correspondingly by the mass and spin multipoles, are analyzed in full detail. A new simple method of integration for the equations of light propagation is proposed. This method enables us for the first time to obtain complete expressions both for the relativistic time delay and for the angle of the total deflection of light in any order of multipole perturbations without restriction. The results thus obtained can be applied to the interpretation of the secondary weak gravitational lens effects produced by the solar system bodies, stars, binary pulsars, and galaxies where the influence of higher-order multipoles on the propagation of null rays may be important and measurable. The methods developed in the paper can be also applied to physical optics of multipole electromagnetic lenses and for calculation of propagation of gravitational waves through the curved space–time. As a particular application of the method the generalized equation for a multipole gravitational lens is derived using Cartesian coordinates and symmetric transverse-traceless tensors.

I. INTRODUCTION

The theoretical prediction and the observational appearance of relativistic effects associated with the physical process of light propagation in the gravitational fields of various celestial bodies have been a matter of considerable interest among physicists and astronomers for quite a while. The gravitational microlensing effects exhibited by point mass deflectors may be considered as the simplest particular example. Such effects are well understood and have been extensively exploited by the OGLE\(^1\) and MACHO\(^2\) collaborations in their search for dark matter in our galaxy.\(^3\) Gravitational macrolensing effects exhibited by nonspherically symmetric deflectors producing extended arclike images of numerous background galaxies\(^4\) are very useful for probing the mass distribution of galaxy clusters.\(^5\) The relativistic time delay between the brightness variations of two images in cosmological gravitational lenses is proportional to the Hubble time\(^6,7\) and, in combination with lens statistics, can be used to determine fundamental cosmological parameters.\(^8,9\)

Unconventional theories of gravity which describe gravity in terms of a metric and one or more additional (scalar, vector, etc.) fields, predict photon deflections and relativistic time delays which are different from the predictions of standard General Relativity Theory.\(^10,11\) Thus, careful measurements of relativistic effects involved in light propagation can serve as a powerful tool for discrimination between various theories of gravity. Indeed, a rather remarkable fact is that the
propagation of light is subject to the influence of both the time–time as well as the space–space components of the metric tensor in the first order of relativistic small parameter $\eta = GM/c^4 d$ (where $G$ is the universal gravitational constant, $c$ is the speed of light, $M$ is the characteristic mass of the body deflecting light rays, and $d$ is the impact parameter of the photon’s trajectory). This fact allows for a test of General Relativity Theory which is completely independent of the red-shift test and tests involving the relativistic advance of the perihelion of Mercury. Moreover, the detailed study of relativistic effects inherent in the propagation of radio signals in the gravitational field of binary pulsars has shown that the approach can be used to probe General Relativity Theory in the strong-field gravity regime with an unparalleled degree of accuracy. At present, the main relativistic effects in the light propagation are confirmed at the level 0.5% in solar system astrometric observations and at about the 0.8% level in observations of the strong gravitational field effects of (a neutron star) companion of pulsar PSR 1534+12.

New astrometric and astrophysical projects such as ground-based large baseline optical interferometers CHARA, VLTI, etc., the optical interferometer in space, POINTS, space VLBI observatories RADIOASTRON, and VSOP, will be realized in the foreseeable future. All of these instruments will have very precise observational accuracy attaining 1–10 microarcseconds and will yield substantially improved measurements of positions, motions, diameters, and images of stars, quasars, galaxies, and their clusters as well as new observational tests of general relativity in the solar system. In addition, pulsar timing techniques are developing quite rapidly and have now entered a phase of maximum impact on astronomical observations made with radio telescopes. It is expected that the invention of a nontraditional pulsar timing software-based dedispersion system developed by Joe Taylor and collaborators will eliminate the largest sources of systematic errors now present in the highest-precision timing experiments, greatly improving the quality of measurements obtained in the areas of relativistic gravity, fundamental astrometry, cosmology, and time-keeping metrology. Reliable processing all above mentioned highly precise observations will require an accounting of the more sophisticated relativistic effects in the propagation of light in the gravitational fields of the bodies.

The post-post-Newtonian perturbations from the spherically symmetrical part of a gravitational field, as well as the post-Newtonian perturbations caused by high-order gravitational multipoles of a system of bodies, deserve a special consideration. Testing post-post-Newtonian effects will give a new experimental access to the nonlinear structure of General Relativity Theory while the investigation of light propagation in a field of high-order multipoles will allow a “fine-structure” study of the matter distribution and rotational parameters of astronomical objects deflecting light rays. The latter effects may play an especially important role in the modelling of cosmological gravitational lenses since essential deviations from spherical symmetry are found in the mass distribution of clusters of galaxies. Quite possibly, measurement of higher-order multipole relativistic effects in the propagation of pulsar radio signals in nearly edge-on binary systems containing a black hole as the pulsar’s companion might give us an unbiased quantitative confirmation of the black hole’s existence. In addition, examination of relativistic effects in propagation of electromagnetic waves in the gravitational field of the solar system bodies has shown that the effects caused by rotation and oblateness of the bodies have a magnitude already accessible for measuring by the present day VLBI technique.

The post-post-Newtonian relativistic effects have been studied by many authors (see, for instance, Ref. 23 and references therein). As for the relativistic effects in a field of high-order multipoles, only the perturbations from the spin dipole and the mass quadrupole moments have been thoroughly investigated so far (see Refs. 24–26 and references therein). The main goal of the present paper is to generalize the results of previous authors and to obtain an exhaustive description of relativistic effects inherent in the propagation of light in the field of any mass and spin higher-order multipole. We propose a new method of integration for the equations of the propagation of light, essentially modernizing the approach used in Refs. 23–25. The foundation of the method is an extensive use of symmetric trace-free (STF) and symmetric transverse-traceless...
(STT) tensors. These tensors are very convenient for describing the multipole structure of a gravitational field and the projective geometry of a bundle of null rays created by the propagating electromagnetic (or gravitational) waves. Our results refine the treatment of the problem of propagation of null rays in a field of high-order gravitational multipoles. It is worth emphasizing that the method of integration of equations of propagation of light being developed in the present paper does not presuppose the smallness of the impact parameter of the light ray—the condition used to be imposed in the analysis of cosmological gravitational lenses. Therefore, our results are applicable to the wide range of possible relative configurations of lensing mass and observer and can be used both for the processing of highly precise astrometric observations and in the treatment cosmological gravitational lenses without any restrictions. We note also that the methods being considered here have analogs in the field of physical optics of multipole electromagnetic lenses, and could be applied there as well after making an appropriate replacement of scalar and vector gravitational potentials by their electromagnetic counterparts.

In Sec. II we give definitions of the mass and spin multipole moments of gravitational field using symmetric trace-free tensors. The equations of motion of the massless test particle (photon, graviton) are given in Sec. III. The very method of integration of the equations of motion is discussed in full detail in Sec. IV. Perturbations of trajectory of the particle are discussed in Sec. V. Finally, in Sec. VI, we give exact expressions for two basic relativistic effects, namely the gravitational time delay (Shapiro effect) and the deflection angle. The multipole as well as transparent gravitational lens equations are also derived as a methodological application. A definition and a short description of the properties of symmetric trace-free as well as symmetric transverse-traceless tensors are outlined in the Appendices.

II. MULTIPOLe MOMENTS OF GRAVITATIONAL FIELD

Let us consider the case of an isolated astronomical system composed of a gravitating body or a system of bodies. Far away from the system, the space–time is asymptotically flat. We suppose that space–time is covered by four-dimensional coordinates with the time coordinate \( t \), and three spatial coordinates \( x^i = x \) (all small Latin indices run 1,2,3). The origin of the spatial coordinates is chosen to be the center of mass of the system. The gravitational field of the system can be described in the first post-Newtonian approximation by ‘‘scalar’’ and ‘‘vector’’ gravitational potentials. We restrict ourselves to the consideration of a stationary situation when the gravitational potentials depend only on the spatial coordinates. It simplifies calculations when one can neglect all the time derivatives of the potentials. The nonstationary case makes calculations much more lengthy and cumbersome (see, for instance, Ref. 25). We used to work in geometrized units where \( G = c = 1 \).

The ‘‘scalar’’ potential \( U \) has the usual form

\[
U(x) = \int_V d^3x \sigma(x') |x-x'|^{-1},
\]

where \( \sigma(x) \) is the mass density distribution expressed through the energy-momentum tensor \( T^{\alpha\beta} (\alpha, \beta = 0,1,2,3) \):

\[
\sigma = c^{-2} (T^{00} + T^{aa}),
\]

and we denote the spatial components of vectors by bold letters.

The general multipole expansion of the ‘‘scalar’’ potential outside the gravitating system is given by

\[
U(x) = \frac{M}{r} + \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \left( \frac{D}{dx} \right)^{a_1 \ldots a_l} \frac{1}{r} \left( \frac{1}{r} \right),
\]

where \( r = |\mathbf{x}| \), \( M \) is the post-Newtonian mass of the system, and \( \mathcal{M}^{a_1 \cdots a_l} \) are the symmetric trace-free (STF) mass multipole moments of the system. They are defined as integrals: \( 30,31 \)

\[
M = \int_V \mathbf{d}^3 \mathbf{x} \, \sigma(\mathbf{x}),
\]

\[
\mathcal{M}^{a_1 \cdots a_l} = \sum_l \int_V \mathbf{d}^3 \mathbf{x} \, \sigma(\mathbf{x}) \mathcal{M}^{a_1 \cdots a_l},
\]

where \( \mathcal{M}^{a_1 \cdots a_l} = \mathcal{M}^{a_1 a_2 \cdots a_l} \), the acute angular brackets denote STF operator (see Appendix A), and \( V \) means the total volume of the isolated astronomical system under consideration. The dipole moment, \( \mathcal{M}^{a} \) is absent in the expansion \( 3 \) since we took the origin of the coordinates at the center of mass of the system.

The “vector” potential \( U^a \) has the form

\[
U^a(x) = \int_V \mathbf{d}^3 \mathbf{x} \, \sigma^a(\mathbf{x}) |\mathbf{x} - \mathbf{x}'|^{-1},
\]

where \( \sigma^a(x) \) is the current density distribution expressed through the energy-momentum tensor \( T^{\alpha \beta} \) \( (\alpha, \beta = 0,1,2,3) \) as follows:

\[
\sigma^a = c^{-1} T^{0a}.
\]

The multipole expansion of the “vector” potential of stationary gravitating systems is given by \( 27,30 \)

\[
U^a(x) = - \sum_{l=1}^{\infty} \frac{l(-1)^l}{(l+1)!} \varepsilon^{ijb} S^{b a_1 \cdots a_{l-1}} \frac{\partial^l}{\partial x^{a_1} \partial x^{a_2} \cdots \partial x^{a_{l-1}}} \left( \frac{1}{r} \right),
\]

where \( S^{a_1 \cdots a_l} \) are the STF spin multipole moments:

\[
S^{a_1 \cdots a_l} = \int_V \mathbf{d}^3 \mathbf{x} \mathcal{M}^{a_1 \cdots a_l} \varepsilon^{ijb} x^a a^b \sigma(\mathbf{x})
\]

and the acute angle brackets around spatial indices again indicate that they are STF spatial tensors.

### III. EQUATIONS OF MOTION OF PHOTON’S PROPAGATION

Let the motion of a photon be given by the mixed initial-boundary conditions \( 23 \)

\[
x(t_0) = x_0, \quad \frac{dx(-\infty)}{dt} = k,
\]

where \( k^2 = 1 \) and again one denotes the spatial components of vectors by bold letters. These conditions define the coordinates \( x_0 \) of the photon at the moment of emission \( t_0 \) and its velocity at an infinite distance from the origin of the spatial coordinates, which is equal to the physically measurable laboratory value of speed of light \( c \).

The equations of propagation for photons in a stationary gravitational field are thus merely the equations of null geodesics in vacuum, and are given in the first post-Newtonian approximation by the formulas \( 25 \)

\[
x(t) = x_0 + \int_0^t \frac{dx}{dt} \, dt,
\]

\[
\frac{dx}{dt} = \frac{d}{dt}(\mathbf{x}(\mathbf{r})),
\]

\[
\mathbf{x}(\mathbf{r}) = \mathbf{x}_0 + \int_0^r \frac{d\mathbf{x}}{d\tau} \, d\tau = \mathbf{r},
\]

\[
\frac{d\mathbf{x}}{d\tau} = \frac{\mathbf{k}}{c},
\]

where \( k^2 = 1 \) and \( \mathbf{r} \) is the position vector of the point at which the photon is emitted.

The equations of motion of a photon in a stationary gravitational field are given by the formulas \( 25 \)

\[
\frac{d\mathbf{x}}{d\tau} = \frac{\mathbf{k}}{c},
\]

\[
\mathbf{x}(\mathbf{r}) = \mathbf{x}_0 + \int_0^r \frac{d\mathbf{x}}{d\tau} \, d\tau = \mathbf{r},
\]

\[
\frac{d\mathbf{x}}{d\tau} = \frac{\mathbf{k}}{c},
\]

where \( k^2 = 1 \) and \( \mathbf{r} \) is the position vector of the point at which the photon is emitted.
\[
\ddot{x}^i = U_j + U\dot{r}^2 - 4U_j \ddot{x} \dot{x}^i - 4(U^i_j - U^j_i) \ddot{x}^i - 4U^i_j \dot{x}^i \dot{x}^j, \tag{11}
\]

where the dots over the vectors indicate differentiation with respect to time, and the comma indicates partial derivatives with respect to the spatial coordinates. Everywhere the repeated indices mean summation from 1 to 3. The given equations are the ordinary second-order differential equations. It depends not only on the "scalar" but the "vector" potential also, and, in that sense, is quite similar to the equation of propagation of electrons in electron optics when magnetic fields are involved and electromagnetic vector potential must be considered.\(^{32}\) In such situations (both in gravity and electromagnetism) the rays of photons or electrons are no longer orthogonal trajectories of the equiphase surfaces. So, the simple "method of triangles"\(^ {33}\) cannot be applied any more for the derivation of the equation of gravitational lens and one has to resort to rigorous mathematical technique.

The right-hand side of Eq. (11) includes also terms which depend on the coordinate velocity \(\dot{x}^i\) of the photon being close to \(c\). Thus, it is admissible to make the replacement \(\dot{x}^i = k^i\) on the right-hand side of the equation. The resulting equation is

\[
\ddot{x} = 2U_i - 4U_j \dot{k}^i k^j - 4(U^i_j - U^j_i) k^j - 4U^i_j \dot{k}^i k^j. \tag{12}
\]

This equation is to be solved by iterations to get a perturbed trajectory of the photon propagating through the gravitational field of the astronomical system under consideration.

To accomplish this, one substitutes the expressions (3) and (8) for the gravitational potentials into the right-hand side of Eq. (12), yielding

\[
\ddot{x} = 2(k, j - P_{ij}) \left[ M + \sum_{\ell=2}^{\infty} \frac{(-1)^{\ell}}{\ell!} \left( j^{a_1 \cdots a_{\ell-1}} \frac{\partial}{\partial x^{a_1 \cdots a_{\ell-1}}} \right) \left( \frac{x^j}{r^2} \right) - 4 \sum_{l=1}^{\infty} \frac{(-1)^l}{(l+1)!} \right]
\times (\epsilon_{a_1 b_1} k_j - \epsilon_{a_1 b_1} k_c) P_{ij} \left( j^{b_1 \cdots a_{\ell-1}} \frac{\partial}{\partial x^{b_1 \cdots a_{\ell-1}}} \right) \left( \frac{x^j}{r^2} \right), \tag{13}
\]

where

\[
P_{ij} = \delta_{ij} - k_i k_j
\]

is the projection operator onto the plane being orthogonal to the vector \(k^i\), \(\epsilon_{ijk}\) is the fully antisymmetric Levi–Cita symbol, and all the quantities on the right-hand side are considered in the Newtonian approximation only. Let us note that the projection operator \(P_{ij}\) has only two algebraically independent components. One should integrate the obtained equation two times with respect to time to get a perturbed photon’s trajectory.

**IV. METHOD OF INTEGRATION OF THE EQUATIONS OF MOTION**

To integrate Eq. (13) we resort to an approximation method. In the Newtonian approximation, the unperturbed trajectory of the light ray is a straight line:

\[
\chi^i(t) = x^i_0(t) = x^i_0 + k^i(t - t_0), \tag{15}
\]

where \(t_0\) is a moment of the photon emission from the point \(x_0^i\), and \(k^i = k\) is a constant unit vector being tangent to the nonperturbed trajectory. In the Newtonian approximation, the coordinate speed of the photon \(\dot{x}^i = k^i\).

It is convenient to introduce a new independent parameter \(\tau\) along the photon’s trajectory according to the rule

\[
\tau = k \cdot x = t - t_0 + k \cdot x_0, \tag{16}
\]

where the dot symbol ‘‘‘’ between two vectors denotes the Euclidean dot product of two vectors. The moment $t_0$ of the signal’s emission corresponds to the numerical value of the parameter $\tau_0 = k \cdot x_0$, and the moment $\tau=0$ corresponds to the moment of the closest approach of the unperturbed trajectory of the photon to the origin of the coordinate system. Using parameter $\tau$, the equation of unperturbed trajectory can be represented as

$$x^i(\tau) = x^i_0(\tau) = k^i \tau + \xi^i,$$

and the distance $r$ of the photon from the origin of coordinate system is given by

$$r = r_A(\tau) = (\tau^2 + d^2)^{1/2},$$

where the constant transverse vector $\xi^i = \xi = k \times (x_0 \times k)$, $d = |\xi|$, and the symbol ‘‘×’’ between two vectors denotes the Euclidean cross product. The vector $\xi^i$ is the impact parameter vector being orthogonal to the unperturbed trajectory. The relations

$$r + \tau = d^2/(r - \tau), \quad r_0 + \tau_0 = d^2/(r_0 - \tau_0),$$

are also useful for comparing our results with those obtained elsewhere (for example, see Refs. 23 and 25), and for transforming the resulting expressions.

To complete the calculations, it is convenient to use the next decomposition of three-dimensional spatial partial derivatives:

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial \xi^i} + k^i \frac{\partial}{\partial \tau}. \tag{20}$$

The formula (20) is a covariant vector representation of the contravariant vector

$$dx^i = d\xi^i + k^i\,d\tau, \tag{21}$$

and can be easily confirmed using the rules of contraction of vectors and covectors:23

$$\left\langle \frac{\partial}{\partial x^i}, dx^j \right\rangle = \delta^i_j. \tag{22}$$

As a consequence of the formulas (20)–(22), we obtain the useful result

$$\frac{\partial \xi^i}{\partial \xi^j} = P^i_j = P_{ij} = P_{ij}. \tag{23}$$

It is worthy to note again that coordinates $\xi^i$ lie at the plane being orthogonal to the vector $k^i$. Therefore, only two of three $\xi^1, \xi^2, \xi^3$ are, in fact, independent. It may help to explain the formula (23) from the geometrical point of view.

Let us stress also that the new variables $\xi^i$ and $\tau$ can be considered as being completely independent. Thus, with the decomposition (20) at our disposal, the expression for the $l$th STF partial derivative is

$$\frac{\partial^l}{\partial x^{a_1} \cdots \partial x^{a_l}} = \sum_{p=0}^l \frac{l!}{p!(l-p)!} k_{a_1} \cdots k_{a_p} \hat{a}_{a_{p+1}} \cdots \hat{a}_{a_l} \frac{\partial^p}{\partial \xi^{a_p}}, \tag{24}$$

where a new shorthand notation $\hat{a}_a = \partial / \partial \xi^a$ has been used.

Integration with respect to time $t$ along the photon’s trajectory is now completely equivalent.
to integration with respect to the parameter \( \tau \). It may be simply confirmed that for any arbitrary function \( F(\tau, \xi^i) \), the following result is true:

\[
\int \frac{\partial}{\partial \tau} F(\tau, \xi^i) d\tau = F(\tau, \xi^i) + \frac{d}{dt} F(\tau, \xi^i),
\]

(25)

where \( C(\xi^i) \) is an arbitrary function of the constant impact parameter. Moreover, all partial derivatives with respect to \( \xi^i \) can be taken off the sign of time integrals when calculating them along the photon’s trajectory.

All of these remarks simplify the next calculations so drastically that the only integrals that need to be taken are

\[
\int_{\tau_0}^{\tau} \frac{d}{dt} \left( \frac{x^i}{r} \right) dt = A^i_{<a_1\ldots a_l>}(\tau)
\]

(26)

and

\[
\int_{\tau_0}^{\tau} \frac{d}{dt} \left( \frac{x^i}{r^j} \right) dt = B^i_{<a_1\ldots a_l>}(\tau) - B^i_{<a_1\ldots a_l>}(\tau_0).
\]

(27)

The tensor functions in the formulas under consideration are given by

\[
A^i_{<a_1\ldots a_l>}(\tau) = \hat{\partial}_{<a_1\ldots a_l>} \left[ \frac{\xi^i}{r} \left( \frac{\tau}{r} + 1 \right) - \frac{k^i}{r} \right] + \sum_{p=1}^{l} \frac{l^1}{p!(l-p)!} k^i_{<a_p}\hat{a}_{p+1}\ldots\hat{a}_{a_l}\frac{\partial^{l-p-1}}{\partial \tau^{l-p-1}} \left( \frac{x^i}{r^j} \right)
\]

(28)

and

\[
B^i_{<a_1\ldots a_l>}(\tau) = \hat{\partial}_{<a_1\ldots a_l>} \left[ \frac{\xi^i}{r} \frac{\tau}{r} - \frac{k^i}{r} \right] + \sum_{p=2}^{l} \frac{l^1}{p!(l-p)!} k^i_{<a_p}\hat{a}_{p+1}\ldots\hat{a}_{a_l}\frac{\partial^{l-p-2}}{\partial \tau^{l-p-2}} \left( \frac{x^i}{r^j} \right).
\]

(29)

To complete differentiation in formulas (28) and (29) the function \( r \) is to be substituted for its value given by the expression (18). If necessary, after taking all corresponding derivatives, the variable \( \tau \) may be replaced by its original value \( \tau = k \cdot x \), and \( \tau_0 \) may be replaced by \( k \cdot x_0 \).

Now we are ready to discuss the relativistic perturbations of the photon’s trajectory in the field of higher-order gravitational multipoles of an isolated astronomical system.

V. PERTURBATIONS OF THE PHOTON’S Trajectory

It follows from Eq. (13) that a perturbed photon’s trajectory contains two types of perturbations arising from the existence of mass and spin multipole moments, respectively. They are added linearly on the right-hand side of this equation. Thus, the coordinate speed of the photon and its trajectory can be represented in the form

\[
\dot{x}^i(\tau) = k^i + \dot{x}^i_1(\tau) + \dot{x}^i_Q(\tau) + \dot{x}^i_K(\tau)
\]

(30)

and

\[
x^i(\tau) = k^i \tau + \xi^i_1(\tau) + x^i_Q(\tau) + x^i_K(\tau),
\]

(31)

where the function \( x^i_Q(\tau) \) relates to the Schwarzschild part of the gravitational field, the function
\( x'_M(\tau) \) describes the mass (quadrupole and higher) multipole perturbations, and \( x'_S(\tau) \) describes the spin induced perturbations. All of them are obtained in a straightforward manner after twofold integration of Eq. (13) using formulas (28) and (29).

The function \( x'_S(\tau) \) yields well-known perturbations of the photon’s trajectory which arise from the spherically symmetric distribution of matter are small enough.

\[
\dot{x}'_S(\tau) = -2M \left[ \frac{\xi'}{r} \left( \frac{r}{\tau} + 1 \right) + \frac{k'}{r} \right], \quad (32)
\]

\[
x'_S(\tau) = 2M \left[ k' \ln \frac{r_0 + \tau_0}{r + \tau} - \frac{\xi'}{(r + \tau)} (r - r_0 + \tau - \tau_0) \right]. \quad (33)
\]

The monopole perturbations have the largest magnitude among other multipole perturbations if deviations from the spherically symmetric distribution of matter are small enough.

**A. Mass multipole perturbations**

For higher-order mass multipole perturbations one has

\[
\dot{x}'_Q(\tau) = -2 \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \mathcal{F}_{a_1 \ldots a_l} \dot{\hat{a}}_{a_1} \ldots \hat{\partial}_{a_l} \left[ \frac{\xi'}{r^2} \left( \frac{r}{\tau} + 1 \right) - \frac{k'}{r^2} \right] + 2 \sum_{l=2}^{\infty} \sum_{p=1}^{l} \frac{(-1)^l}{p!(l-p)!} \mathcal{F}_{a_1 \ldots a_l} \hat{\partial}_{a_1} \ldots \hat{\partial}_{a_l} \left[ \frac{\xi'}{r^2} \left( \frac{r}{\tau} + 1 \right) - \frac{k'}{r^2} \right] \quad \text{and} \quad (34)
\]

\[
x'_Q(\tau) = Q'(\tau) - Q'(\tau_0), \quad (35)
\]

where

\[
Q'(\tau) = -2 \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \mathcal{F}_{a_1 \ldots a_l} \hat{\partial}_{a_1} \ldots \hat{\partial}_{a_l} \left[ \frac{\xi'}{r^2} \left( \frac{r}{\tau} + 1 \right) + k' \ln (r + \tau) \right] - 2 \sum_{l=2}^{\infty} \frac{(-1)^l}{(l-1)!} \mathcal{F}_{a_1 \ldots a_l} \hat{a}_{a_1} \ldots \hat{\partial}_{a_l} \left[ \frac{\xi'}{r^2} \left( \frac{r}{\tau} + 1 \right) + k' \ln (r + \tau) \right] + 2 \sum_{l=2}^{\infty} \sum_{p=1}^{l} \frac{(-1)^l}{p!(l-p)!} \mathcal{F}_{a_1 \ldots a_l} \hat{a}_{a_1} \ldots \hat{\partial}_{a_l} \left[ \frac{\xi'}{r^2} \left( \frac{r}{\tau} + 1 \right) + k' \ln (r + \tau) \right] \quad \text{and} \quad (36)
\]

For the pure quadrupole case, when \( l = 2 \), one obtains the formulas derived previously by Klioner and Kopeikin.

**B. Spin multipole perturbations**

Spin multipole perturbations can be expressed most appropriately in the form

\[
\dot{x}'_K(\tau) = -4 \sum_{l=1}^{\infty} \frac{(-1)^l}{l+1} \left( \epsilon_{ia} \epsilon_{bc} k_c P_{ij} \right) \mathcal{S}^{ba_1 \ldots a_{l-1} \hat{a}_l} A_{a_1 \ldots a_{l-1} \hat{a}_l}^{ij} (\tau) \quad (37)
\]

and

\[
x'_K(\tau) = -4 \sum_{l=1}^{\infty} \frac{(-1)^l}{l+1} \left( \epsilon_{ia} \epsilon_{bc} k_c P_{ij} \right) \mathcal{S}^{ba_1 \ldots a_{l-1} \hat{a}_l} A_{a_1 \ldots a_{l-1} \hat{a}_l}^{ij} (\tau) \times \left[ B_{a_1 \ldots a_{l-1} \hat{a}_l}^{ij} (\tau) - B_{a_1 \ldots a_{l-1} \hat{a}_l}^{ij} (\tau_0) \right] \quad (38)
\]

For a pure spin dipole (Kerr or Lense–Thirring) perturbation, when \( l = 1 \), the expressions (37) and (38) coincide precisely with the formulas given, for example, in Ref. 24.

VI. BASIC RELATIVISTIC EFFECTS

There are two main relativistic effects inherent in the propagation of photons through a field of high-order multipoles, namely, the gravitational (Shapiro) time delay and gravitational deflection of light. Using the technique developed above, we can treat them in a relatively simple way.

A. Time delay

The gravitational time delay is derived from Eq. (31), which can be more clearly expressed as

\[
x^i - x'^i = k^i (t - t_0) + x^i_S (\tau) + x^i_Q (\tau) + x^i_K (\tau).
\]

To obtain the expression for the time delay one multiplies this equation by itself and then finds a difference \( t - t_0 \). The result is

\[
t - t_0 = |x - x_0| + \Delta_S + \Delta_Q + \Delta_K,
\]

where \(|x - x_0|\) is the usual Euclidean distance between the points of emission, \( x_0 \), and reception, \( x \), \( \Delta_S \) is the classical Shapiro time delay produced by the spherically symmetric part of the gravitational field, \( \Delta_Q \) describes an additional time delay caused by the mass quadrupole, octupole, and high-order moments, and \( \Delta_K \) is the Kerr delay due to the stationary rotation of the gravitating system. Specifically, one has

\[
\Delta_S = 2M \ln \frac{r + \tau}{r_0 + \tau_0},
\]

\[
\Delta_Q = \sum_{l=2}^{\infty} \left( \frac{(-1)^l}{l!} \right) \mathcal{F}^{a_1 \cdots a_l} \partial_{a_1} \cdots \partial_{a_l} \ln \frac{r + \tau}{r_0 + \tau_0} + \sum_{l=2}^{\infty} \left( \frac{(-1)^l}{(l-1)!} \right) \mathcal{F}^{a_1 \cdots a_l} k_{a_1} \partial_{a_2} \cdots \partial_{a_l} \left[ \frac{1}{r} - \frac{1}{r_0} \right] + \sum_{l=2}^{\infty} \left( \frac{(-1)^l}{l!(l-p)!} \right) \mathcal{F}^{a_1 \cdots a_l} k_{a_1} \partial_{a_{p+1}} \cdots \partial_{a_l} \left[ \frac{\partial^{p-2}}{\partial \tau_0^{p-2}} \left( \frac{r_0}{r} \right) - \frac{\partial^{p-2}}{\partial \tau^{p-2}} \left( \frac{\tau}{r} \right) \right] \cdot
\]

\[
\Delta_K = -4 \epsilon_{iabk} \mathcal{S}^{<b_{a_1 \cdots a_l-1}>} \partial_{a_1} \cdots \partial_{a_l-1} \ln \frac{r + \tau}{r_0 + \tau_0} - 4 \epsilon_{iabk} \mathcal{S}^{<b_{a_1 \cdots a_l-1}>} k_{a_1} \partial_{a_2} \cdots \partial_{a_l} \frac{1}{r} \left[ \frac{1}{r_0} - \frac{1}{r} \right] + 4 \epsilon_{iabk} \mathcal{S}^{<b_{a_1 \cdots a_l-1}>} k_{a_1} \partial_{a_{p+1}} \cdots \partial_{a_l} \left[ \frac{\partial^{p-2}}{\partial \tau_0^{p-2}} \left( \frac{r_0}{r} \right) - \frac{\partial^{p-2}}{\partial \tau^{p-2}} \left( \frac{\tau}{r} \right) \right].
\]

The magnitudes of different components of the time delay can be estimated under conditions that

min(\frac{dr}{d\tau}, \frac{dr_0}{\tau}) \ll 1 as follows: \[ \Delta_5 \sim (GM/c^3)\ln(r_0/\ell^2), \quad \Delta_Q \sim (GM/c^3)(R/d)^\frac{1}{2} \varepsilon_1, \quad \Delta_K \sim (GM/c^3) \times (R/d)^\frac{1}{2} (v/c), \]
where \( R \) is the characteristic size of the lensing mass, \( v \) is the characteristic speed of its internal motions, and \( \varepsilon_1 \) is dimensionless value of the \( l \)th multipoles moment (\( \varepsilon_1 \ll 1 \)).

**B. Deflection of light**

1. **Angle of the total deflection**

Let us introduce the notation \( v^i = \hat{\chi}^i(\infty) \) and consider Eq. (30) in the limiting case when \( \tau = \infty \). In this limit, one has \( \tau = r \) and \( \tau \tau + 1 = 2 \). The numerical value of scalar product \( (v \cdot \n) \), where the unit vector \( n^i = \xi^i/d \), determines the cosine of the angle \( \pi/2 + \alpha \), where \( \alpha \) is the total deflection of light passing through the field of the gravitating system in question. We consider only the paraxial (i.e., small-angle) approximation where \( \sin \alpha = \alpha + O(\alpha^3) \). Thus, taking the limit \( \tau \to \infty \) in Eq. (30) and calculating the scalar product \( (v \cdot \n) \), one gets

\[ \alpha = \alpha_5 + \alpha_Q + \alpha_K, \]

where \( \alpha_5, \alpha_Q, \) and \( \alpha_K \) are the angles of the gravitational deflection of light caused by the Schwarzschild, mass, and spin multipoles, respectively. They are calculated using the equalities

\[ \hat{\phi}_{(a_1 \cdots a_i)} \frac{\xi^i}{d} = \hat{\phi}_{(a_1 \cdots a_i)} \frac{\xi^i}{d} = \hat{\phi}_{(a_1 \cdots a_i)} \ln d, \]

as well as formulas given in Appendix. In addition, one has introduced abbreviated notation \( \hat{\phi}_{(a_1 \cdots a_i)} = \hat{\phi}_{(a_1 \cdots a_i)} \), where henceforth the obtuse angular brackets around indices designate symmetric transverse-traceless (STT) tensors which are explained in more detail in Appendix B.

This yields

\[ \alpha_5 = \frac{4M}{d}, \]

\[ \alpha_Q = -\frac{4}{d} \sum_{l=2}^{\infty} \frac{(-1)^l}{(l-1)!} J^{(a_1 \cdots a_l)} \hat{\phi}_{(a_1 \cdots a_l)} \ln d, \]

\[ \alpha_K = \frac{8}{d} \varepsilon_{abc, k} \sum_{l=1}^{\infty} \frac{(-1)^l l^2}{(l+1)!} J^{(b a_1 \cdots a_{l-1})} \hat{\phi}_{(a a_1 \cdots a_{l-1})} \ln d. \]

The magnitudes of different components of the deflection angle are estimated as follows: \( \alpha_5 \sim GM/c^2 d, \alpha_Q \sim \alpha_5 (R/d) \varepsilon_1, \alpha_K \sim \alpha_5 (R/d) \varepsilon_1 (v/c) \). We are now able to obtain the equation for gravitational lensing in the field of higher-order multipoles.

2. **The multipole gravitational lens equation**

As a particular consequence of application of our method, the gravitational lens equation can be obtained directly from the formula (39), where when deriving the equation, we should neglect all quantities having magnitudes \( dr/d\tau \) and \( dr_0/\tau \) in comparison to the main terms. Taking into account the relationship (40) and designating \( K^i = (x^i - x_0^i)/|x - x_0| \), one obtains

\[ K^i = k^i - \frac{r}{R} \alpha^i + O\left(\frac{d}{r} \right) + O\left(\frac{d}{r_0} \right), \]
where \( r \) is the distance from the center-of-mass of the gravitating body being considered as a lens, \( R = |x - x_0| \) is the distance between the point of emission of photon, \( x_0 \), and observer being at the point \( x \), and \( \alpha' \) is the dimensionless vector relating to the angle \( \alpha \) of the total deflection of light by means of the simple formula \( \alpha'n_j = \alpha \). The exact expression for the vector \( \alpha' \) is

\[
\alpha' = \alpha'_S + \alpha'_Q + \alpha'_K,
\]

where

\[
\alpha'_S = \frac{4M}{d^2} \xi',
\]

\[
\alpha'_Q = 4 \sum_{l=2}^{\infty} \frac{(-1)^l}{l!} \mathcal{F}^{(a_1 \cdots a_l)} \hat{\partial}_{(a_1 \cdots a_l)} \ln d,
\]

\[
\alpha'_K = -8 e_{abc} \kappa \sum_{l=1}^{\infty} \frac{(-1)^l}{(l+1)!} \mathcal{F}^{(ba_1 \cdots a_{l-1})} \hat{\partial}_{(ba_1 \cdots a_{l-1})} \ln d.
\]

It is worthy to note that the trajectory of a light ray in the field of the higher-order multipoles can no longer be considered flat. There is not only a bending of the trajectory but also a torsion as a consequence of expression (53).

It is convenient to rewrite Eq. (49) using astronomical coordinates corresponding to the plane of the sky. For this purpose let us introduce a triad of the unit vectors \((I_0, J_0, K_0)\). The vector \( K_0 \) points from the observer toward the gravitational lens, and the vectors \( I_0 \) and \( J_0 \) lie in the plane of the sky orthogonal to the vector \( K_0 \). Vector \( I_0 \) is directed to the east, and \( J_0 \) points toward the north celestial pole. In this system, the vectors \( k \) and \( K \) can be represented as

\[
k = \xi' r_0 + N,
\]

\[
N = \eta' r_0 + K_0,
\]

\[
K = \eta' r_0 + K_0,
\]

where \( \xi' \) and \( \eta' \) are vectors lying in the plane of the sky, and the vector \( N' = -x_0/r_0 \). Finally, the lens equation results in

\[
\eta' = \xi' + \xi' - \frac{rr_0}{R} \alpha'(\xi).
\]

This equation describes the mapping from the object plane onto the image plane with the origin of the coordinate system being chosen to arbitrarily. It generalizes the well-known pointlike deflector equation for the event of higher-order gravitational multipoles.

3. The transparent gravitational lens equation

In cosmological gravitational lenses, light rays can propagate not only outside but also inside the gravitational lens. It is not difficult to generalize Eq. (57) to include the consideration of transparent gravitational lens. We suppose that the surface of the lens is smooth, and is described by the equation \( F(\xi, z) = 0 \), where the coordinate \( z \) is directed along the unperturbed trajectory of the photon, and \( \xi = (\xi' \xi^2) \) are the transverse coordinates. This equation may be inverted to obtain an explicit dependence \( z = f(\xi', \xi^2) \). After integrating Eq. (12), expanding it into series over transverse coordinates \((\xi', \xi^2)\), and omitting terms of order \( dl/r \) and \( dl/r_0 \), as previously, we obtain
\[ \alpha'(\xi) = 4 \hat{\chi}_i \psi(\xi), \] (58)

where the gravitational lens potential

\[ \psi(\xi) = \int_D d^2\xi' \Sigma(\xi') \ln |\xi - \xi'|, \] (59)

and \( D \) indicates a two-dimensional domain occupied by the surface mass density \( \Sigma(\xi) \):

\[ \Sigma(\xi) = \int_{f_+}(\xi) dz[\sigma(\xi, z) - 2k^i \sigma^i(\xi, z)]. \] (60)

Note that the gravitational lens equations (58) and (59) are the same as in the “classic” theory but the surface mass density, \( \Sigma(\xi) \), includes now both the mass and the current densities of matter. This is may be important for gravitational lenses in which the matter has extraordinary radial velocities like in active galactic nuclei (AGN) or relativistic stellar clusters. Because the surface of the lens has been assumed to be simple enough, the line of sight intersects it at only two points, \( f_+(\xi) \), and, \( f_-(\xi) \), for \( z > 0 \) and \( z < 0 \), respectively. In principle, more complicated configurations of the matter distribution inside the lens can be expected. In such an event, the integral representations of the surface densities are to be more complicated.

Multipole expansion of the potential \( \psi(\xi) \) in terms of the STT tensors (see Appendix B) is

\[ \psi(\xi) = m \ln d + I_+ - \sum_{l=1}^{\infty} \frac{2^{l-1}}{l} \int_{D^-(\xi)} d^2\xi' \frac{\hat{\xi}'(a_1 \cdots a_l)}{\hat{\xi}'}, \] (61)

where \( m \) is the mass enclosed by a tube whose cross section is a circle of radius \( d \):

\[ m = \int_{D^-} d^2\xi \Sigma(\xi). \] (62)

Meanwhile \( \hat{\xi}'(a_1 \cdots a_l) \) and \( \hat{\xi}'(a_1 \cdots a_l) \) are the internal and external multipole moments of the deflector. Both are STT tensors, and are given by the formulas

\[ I_+ = \int_{D^+} d^2\xi \Sigma(\xi) \ln d, \] (63)

\[ I_+^{(a_1 \cdots a_l)} = \int_{D^+} d^2\xi \Sigma(\xi) \frac{\hat{\xi}'(a_1 \cdots a_l)}{\hat{\xi}'}, \] (64)

\[ I_-^{(a_1 \cdots a_l)} = \int_{D^-} d^2\xi \Sigma(\xi) \frac{\hat{\xi}'(a_1 \cdots a_l)}{\hat{\xi}'}. \] (65)

Here \( D^- \) and \( D^+ \) designate the spatial cylindrical domains where \( |\xi'| < d \), and \( |\xi'| > d \). It is noteworthy to point out that, when \( D^- \) exceeds the cross section of the lens, the external multipole moments \( I_+^{(a_1 \cdots a_l)} \) exceed the internal multipole moments \( I_-^{(a_1 \cdots a_l)} \) coincide exactly with the multipole moments, \( I_-^{(a_1 \cdots a_l)} \) of the external gravitational field of the lens. Under these circumstances, the expression (58) can be converted exactly to (50).

We expect that STT tensors will find considerable utility when applied to the problem of inverting images in cosmological gravitational lenses. The multipole STT expansions are represented in Cartesian orthogonal coordinates of the observer plane and should be more appropriate.
for the analysis of charge-coupled-device (CCD) camera images of galaxies than the polar coor-

dinates and trigonometric functions\(^2\) no matter whether the deflector has a center of symmetry or not. This is explained by the fact that the technological structure of the CCD plate already forms an orthogonal coordinate system. So, the application of STT tensors excludes redundant intermediate trigonometrical transformations in the procedure of inverting the gravitational lens equation.

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APPENDIX A: SYMMETRIC TRACE-FREE TENSORS

The symmetric trace-free (STF) Cartesian tensors of rank \(l\) generate an irreducible representation of “weight” \(l\) and dimension \(2l+1\) of the group of proper rotation \(SO(3)\). The explicit expression of the STF part of a tensor \(A_{a_1\cdots a_l}\) is\(^27,30\)

\[
\{A_{a_1\cdots a_l}\}^{STF}=A_{<a_1\cdots a_l>} = \sum_{k=0}^{[l/2]} a^k_l \delta_{a_1 a_2} \cdots \delta_{a_{2k-1} a_{2k}} S_{a_{2k+1} \cdots a_l} b_k b_1 \cdots b_k b_l \tag{A1}
\]

where

\[
S_{a_1\cdots a_l}=A_{(a_1\cdots a_l)} \tag{A2}
\]

with numerical coefficients

\[
a^k_l=(-1)^k \frac{l!}{(l-2k)!(2k)!!} \frac{(2l-2k-1)!!}{(2l-1)!!}. \tag{A3}
\]

Here and there the round brackets around indices denote symmetrization, \([\frac{1}{2}p]\) is the integer part of \(\frac{1}{2}p\), \(\delta^{ik} = \delta^i_k = \delta_{ik} = \text{diag}(1,1,1)\) is the Kroneker’s symbol, and repeated indices indicate summation from 1 to 3. For example, \(T_{(ab)}=T_{(ab)}\) - \(\frac{1}{2} \delta_{ab} T_{kk}\); \(T_{(abc)}=T_{(abc)}\) - \(\frac{1}{3} \delta_{ab} T_{ckk}\) - \(\frac{1}{3} \delta_{bc} T_{akk}\) - \(\frac{1}{3} \delta_{ca} T_{bkk}\); etc.

The STF tensors are well understood. We refer the reader to Refs. 27, 30, 31, 34, and 35 for a detailed description of properties of these tensors and their usage in gravity theory. The symmetric transverse-traceless (STT) tensors are less well known.

APPENDIX B: SYMMETRIC TRANSVERSE-TRACELESS TENSORS

The symmetric transverse-traceless Cartesian tensors of rank \(l\) generate an irreducible representation of “weight” \(l\) and dimension \(2l\) of the group of proper rotation \(SO(2)\). The explicit expression of the STT part of a tensor \(A_{a_1\cdots a_l}\) is

\[
\{A_{a_1\cdots a_l}\}^{STT}=\hat{A}_{(a_1\cdots a_l)} = \sum_{k=0}^{[l/2]} b_n^k P_{(a_1\cdots a_k a_{k+1} \cdots a_l)} W_{a_{2k+1} \cdots a_l} b_k b_1 \cdots b_k b_l \tag{B1}
\]

\[
W_{a_1\cdots a_l}=[P_{a_1 b_1} \cdots P_{a_l b_l} A_{b_1 \cdots b_l}]^S, \tag{B2}
\]

where “\(S\)” denotes the full symmetrization over all free indices, \(P_{ij}\) is the projection operator onto the plane orthogonal to the vector \(k^i\) defined by (14), and the numerical coefficients \(b_n^k\) are [compare with (A3)]
We state here, without proofs, some formulas which were useful in our calculations. We practically useful formulas is as follows:

\[
b_n^l = \frac{(-1)^n}{(l-2n)! (2n)!} \frac{l!}{(l-2n)! (2l-2n+2)!} = \frac{(-1)^n}{2^{2n} n!} \frac{l!}{(l-n-1)! (l-2n)!}. \quad (B3)
\]

The list of practically useful formulas is as follows:

\[
P_{(a_1 a_2 \cdots a_{2k-1} a_{2k} n_{a_{2k+1}} \cdots n_{a_l})} = \frac{(2k)! (l-2k)!}{l!} P_{(a_1 a_2 \cdots a_{2k-1} a_{2k} n_{a_{2k+1}} \cdots n_{a_l})}, \quad (B4)
\]

\[
\hat{n}_{(a_1 \cdots a_l)} = \sum_{k=0}^{[l/2]} \frac{(-1)^k}{2^k} \frac{(l-k-1)!}{(l-1)!} P_{(a_1 a_2 \cdots a_{2k-1} a_{2k} n_{a_{2k+1}} \cdots n_{a_l})}, \quad (B5)
\]

\[
\hat{\alpha}_{(a_1 \cdots a_l)} = \sum_{k=0}^{[l/2]} \frac{(-1)^k}{2^k} \frac{(l-k-1)!}{(l-1)!} P_{(a_1 a_2 \cdots a_{2k-1} a_{2k} \hat{n}_{a_{2k+1}} \cdots \hat{\alpha}_{a_l})} \Delta^k, \quad (B6)
\]

\[
n \hat{n}_{(a_1 \cdots a_l)} = \hat{n}_{(a_1 \cdots a_l)} + \frac{1}{2} \hat{\alpha}_{(a_1 \cdots a_l)} \quad (B7)
\]

\[
n \hat{n}_{(a_1 \cdots a_l)} = \frac{1}{d} \left[n \hat{n}_{(a_1 \cdots a_l)} - 2 \hat{n}_{(a_1 \cdots a_l)} \right], \quad (B8)
\]

\[
\hat{n}_{(a_1 \cdots a_l)} \ln d = (-1)^l (2l-2)! \frac{\hat{n}_{(a_1 \cdots a_l)}}{d^l}, \quad (B11)
\]

\[
n \hat{n}_{(a_1 \cdots a_l)} \ln d = -\frac{1}{d} \hat{\alpha}_{(a_1 \cdots a_l)} \ln d, \quad l \gg 1. \quad (B12)
\]

The given formulas differ from the corresponding ones for STF tensors \(^30\) by numerical coefficients. This is explained by the fact that STT tensors reside in two-dimensional space only.

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29. Throughout the paper the repeated Latin indices imply the Einstein rule of summation.


