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FRACTIONAL LEVY MOTION WITH DEPENDENT INCREMENTS AND ITS APPLICATION TO NETWORK TRAFFIC MODELING

C. De Nikola[1], Y. S. Khokhlov[2], M. Pagano[3], and O. I. Sidorova[4]

Abstract: Since the beginning of the 1990s, accurate traffic measurements carried out in different network scenarios highlighted that Internet traffic exhibits strong irregularities (*burstiness*) both in terms of extreme variability and long-term correlations. These features, which cannot be captured in a parsimonious way by traditional Markovian models, have a deep impact on the network performance and lead to the introduction of α-stable distribution and self-similar processes into the network traffic modeling. In this paper, a generalization of fractional Brownian motion (fBm), which is able to capture both above-mentioned features of the real traffic, is considered.

Keywords: fractional Brownian motion; α-stable subordinator; self-similar processes; buffer overflow probability

1 Introduction

The application of probabilistic methods in the modeling and the analysis of telecommunication systems has a long history. Namely, the first researches in this framework date back to the beginning of the last century when A. K. Erlang (1878–1929), as a scientific collaborator and the head of the newly-established physico-technical laboratory of the Copenhagen Telephone Company, studied the issues related to loss and waiting time in automatic telephone exchanges. In the 1930s, the interests for these topics grew from a practical as well as theoretical point of view. Indeed, Erlang's results were soon used by telephone companies in several countries and gave birth to a new branch in the framework of probability theory, known as queueing theory, which attracted the interests of well-known probabilists such as Palm, Pollachek, Lindly, Khincine, Gnedenko, to name just a few.

In the 1920–1930s, many empirical works showed that, in case of telephone traffic, a suitable model is represented by the Poisson process. At the same time, Poisson flows have many "useful" mathematical properties:

- the superposition of independent Poisson processes is still a Poisson process;
- it has independent and stationary increments; and
- under some mild regularity conditions, the superposition of independent flows converges to a Poisson flow, if the number of flows grows, but the individual rates become infinitesimal so that the overall rate stays constant.

Because of the last property, in many works it has been proposed that the amount of traffic in global telecommunication backbones can be modelled as a Poisson process. For several decades, such model has been used without any further experimental validation and applied to new network scenarios, such as packet-switching networks.

At the beginning of the 1990s, a lot of empirical studies have been conducted in order to better understand the statistical features of packet traffic in global networks, such as Internet, as well as in local area networks inside research institutes, university campuses, and corporates [1–3]. Statistical studies of the collected data highlighted their radical differences with respect to the ubiquitous Poisson process and other traditional (typically Markovian, for the sake of analytical tractability) models. For instance, it is enough to visually check the behavior of real traffic data under different level of aggregations [4]. It is easy to see that at all the aggregation levels (in the range from milliseconds to hours) the data keep a random behavior, which appears to be almost the

[1]University of Salerno, denicola@diima.unisa.it
[2]People's Friendship University of Russia, yskhokhlov@yandex.ru
[3]University of Pisa, m.pagano@iet.unipi.it
[4]Tver State University, Oksana.I.Sidorova@yandex.ru

same at all the different scales (apart from a normalization factor, related to the length of the observation window).

More accurate mathematical analyses [1] pointed out that real data presents **fractal** properties, i. e., they can be interpreted as trajectories of so-called **automodel** or **self-similar processes**. Moreover, it was showed that traffic flows, unlike the Poisson model, presents **long range dependence**, which has a huge impact on queuing performance. The third important characteristic of traffic data is that the distribution of many different traffic features (such as file length, duration of on and off periods of single sources) presents **heavy tails**.

These properties of actual traffic flows pointed out the necessity of new traffic models, able to captures them in a parsimonious way. It is worth mentioning that similar models were already known in the field of probability theory since they have been successfully applied in different frameworks, such as turbulence modeling and statistical physics.

The rationale behind the fractal nature of traffics and the links among the above-mentioned characteristics of measured traces have been widely investigated [5]. In particular, it has been shown that if locally the traffic load presents heavy tails, then under a sufficiently high level of aggregation it converges to a self similar process (for a precise formulation of the problem and the related scaling conditions (see [6–9]). According to the considered aggregation regime, two different models might arise: fBm and α-stable Levy motion, which, as will be clarified in the following, present "opposite" features. In more detail, fBm presents long range dependence, but the tails of its marginal distribution decay fast (by definition, according to Gaussian law!). On the contrary, α-stable Levy motion is characterised by independent increments (i. e., no long memory at all!), but has heavy tailed distribution (i. e., its tails decay as a power law).

The goal is to build a model, able to take into account both these features of real traffic. Moreover, using such model as input to a queeing system, it would be also interesting to determine relevant queueing parameters, such as the probability of buffer overflow, which gives an upper bound for the loss probability in finite buffer queues.

From the historical point of view, the first attempt to apply the fractional concept to traffic modelling was to use fBm $B_H(t)$ instead of traditional Poisson-based models. Compared to standard Brownian motion (BM), fBm has one extra parameter, the Hurst parameter H, which quantifies the strength of the fractional scaling. It is said usually, that fBm is self-similar, or fractional, with Hurst parameter H. In [10], Norros has proposed the following model for cumulative traffic

$$A(t) = mt + (\sigma m)^{1/2} B_H(t)$$

where $m > 0$ is the mean input rate, σ is the scale factor. This model has been widely studied and have been proposed asymptotic lower bounds [10] as well as exact asymptotics in the case of large buffers [11, 12].

It is important to point out that in this case, one has a long-range correlation, but not heavy tails of marginal a distributions.

To deal with this issue, several papers extended Norros model by modelling the input traffic as α-stable Levy motion [13, 14] or, to take into account also the long range correlations, fractional α-stable Levy motion (see [15, 16]).

In the paper, a new variant of fractional Levy motion is suggeated and, following the approach proposed in [10], an asymptotic lower bound for the overflow probability is determined.

2 Stable Distributions and Processes

Levy processes have been popular in modeling the teletraffic. Below, some definitions are given and some properties of such processes are considered.

Definition 1. *A stochastic process* $Y = (Y(t), t \geq 0)$ *is a Levy process if*

(1) $Y(0) = 0$ *almost surely;*

(2) Y *has independent increments; and*

(3) Y *has stationary increments.*

Usually, for the sake of regularity, the following property is required: with probability one all trajectories of Y are right-continuous and have finite limits from the left.

The distributions of the process Y is defined uniquely by the distribution of random variable $Y(1)$, which is infinitely divisible.

The most familiar example of Levy process is the BM (Weiner process).

Definition 2. *A Levy process* $B = (B(t), t \geq 0)$ *is called Brownian Motion if for any* $t \geq 0, h > 0$ *the increment* $B(t+h) - B(t)$ *has Gaussian distribution with zero meaning and variance* $\sigma^2 h$.

If $\sigma^2 = 1$, one has a standard BM. It is easily seen that

$$K(t, s) = \mathrm{Cov}\,(Y(t), Y(s)) = \sigma^2 \min(t, s)\,.$$

By definition, BM has Gaussian distributions. Such distributions have been got for normalized sums of independent identically distributed random variables with finite variance. In the case of infinite variance, the so-called stable distributions are considered.

Definition 3. *A random variable* Y *is said to have an* α*-stable distribution if its characteristic function has the following form:*

$$\varphi(\omega) := E\left[e^{j\omega X}\right] = \exp\{j\mu\omega - \sigma|\omega|^\alpha[1 - j\beta\,\mathrm{sgn}\,(\omega)\,\theta(\omega,\alpha)]\}$$

where $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, $\mu \in R^1$, *and*

$$\theta(\omega,\alpha) = \begin{cases} \tan\left(\dfrac{\alpha\pi}{2}\right), & \alpha \neq 1; \\ -\dfrac{2}{\pi}\ln|\omega|, & \alpha = 1. \end{cases}$$

Parameter α is called *characteristic exponent* and specifies the level of burstiness in distribution, i. e., it specifies the weight of the tails of the distribution. σ and μ are called *scale* and *location parameters*. β is called *skewness parameter*. If $\beta = 0$ then X is symmetrically distributed around μ. If $0 < \alpha < 1$, $\mu = 0$ and $\beta = 1$ then X has positive values with probability 1. In what follows, a random variable Y is said to have standard α-stable distribution if $\mu = 0$ and $\sigma = 1$.

The α-stable distribution is infinitely divisible. So, it generates some Levy process.

Definition 4. *A stochastic process* $L_\alpha = (L_\alpha(t), t \geq 0)$ *is said to be an* α*-stable Levy motion if it is a Levy process such that* $L_\alpha(1)$ *has a given* α*-stable distribution.*

If the distribution of $L_\alpha(1)$ is totally positive skewed ($0 < \alpha < 1$, $\beta = 1$), then all trajectories of the process L_α are nondecreasing and nonnegative. Such process is called α-*stable subordinator*.

If $\alpha = 2$, $\mu = 0$, one has again BM B.

There exists very interesting relation between α-stable Levy motions with different α.

Theorem 1. *If* $(L_{\alpha_1}(t), t \geq 0)$, $0 < \alpha_1 \leq 2$, *is a* α_1*-stable Levy motion with symmetric distributions,* $(L_{\alpha_2}(t), t \geq 0)$, $0 < \alpha_2 < 1$, *is a* α_2*-stable subordinator, then stochastic process* $Y = (Y(t) := L_{\alpha_1}(L_{\alpha_2}(t)), t \geq 0)$ *is* $\alpha_1\alpha_2$*-stable Levy motion with symmetric distributions.*

This theorem is a corollary of the following result by Zolotarev [17, theorem 3.3.1].

Theorem 2. *If* Y_1 *has symmetric* α_1*-stable distribution,* $0 < \alpha_1 \leq 2$, Y_2 *has one-sided* α_2*-stable distribution,* $0 < \alpha_2 < 1$, *then random variable* $Y = Y_1 Y_2^{1/\alpha_1}$ *has symmetric* $\alpha_1\alpha_2$*-stable distribution.*

In particular, for $\alpha_1 = 2$ and $0 < \alpha_2 = \alpha/2 < 1$, one gets the following

Theorem 3. *If* $B = (B(t), t \geq 0)$ *is the Brownian motion,* $L_{\alpha/2} = (L_{\alpha/2}(t), t \geq 0)$ *is a* $\alpha/2$*-stable subordinator, then* $L_\alpha = (L_\alpha(t) := B(L_{\alpha/2}(t)), t \geq 0)$, $0 < \alpha < 2$, *is an* α*-stable Levy motion with symmetric distributions.*

3 Self-Similar Processes

Definition 5. *A process* $Y = (Y(t), t \geq 0)$ *is self-similar, with Hurst parameter* $H \geq 0$, *if it satisfies the condition*

$$Y(t) \stackrel{d}{=} c^{-H} Y(ct), \ \forall t \geq 0, \ \forall c > 0,$$

where the equality is the sense of finite-dimensional distributions.

Two of the most popular examples of self-similar processes are fBm and α-stable Levy motion.

Definition 6. *The fractional Brownian motion with Hurst parameter* H *is a Gaussian process* $(B_H(t), t \geq 0)$ *with zero mean and correlation function*

$$K_H(t,s) = \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right].$$

The definition of α-stable Levy motion see above.

More information about stable and self-similar processes can be found in [18, 19].

4 New Variant of Fractional Levy Motion

Above, it was shown how to get symmetric α-stable Levy motion using BM and $\alpha/2$-stable subordinator. Below, it is proposed to use the same construction to get fractional Levy motion from fBm B_H and $\alpha/2$-stable subordinator $L_{\alpha/2}$.

Let $(B_H(t), t \geq 0)$ be the fBm with Hurst parameter H, $(L^1_\alpha(t), t \geq 0)$, $(L^2_\alpha(t), t \geq 0)$ be standard α-stable subordinators, $0 < \alpha < 1$, and B_H, L^1_α and L^2_α are independent. Consider the new process

$$X(t) := \begin{cases} B_H(L^1_\alpha(t)), & t \geq 0; \\ -B_H(L^2_\alpha(t)), & t < 0. \end{cases}$$

Theorem 4. *The above process* X *is self-similar with Hurst parameter* $H_1 = H/\alpha$.

P r o o f. The processes $(L^k_\alpha(t), t \geq 0)$, $k = 1, 2$, are α-stable and self-similar with Hurst parameter $1/\alpha$. So, one has

$$(L^k_\alpha(ct), t \geq 0) \stackrel{d}{=} (c^{1/\alpha} L^k_\alpha(t), t \geq 0).$$

Then,

$$(X(ct), t \in R^1) = \pm B_H(L^k_\alpha(c|t|), t \in R^1) \stackrel{d}{=} (\pm B_H(c^{1/\alpha} L^k_\alpha(|t|), t \in R^1).$$

Using self-similarity of B_H for fixed $\tau = L^k_\alpha(|t|)$, for any $a > 0$, one has

$$(B_H(a\tau), \tau \geq 0) \stackrel{d}{=} (a^H B_H(\tau), \tau \geq 0)$$

or

$$(\pm B_H(c^{1/\alpha}\tau), \tau \geq 0) \stackrel{d}{=} (\pm c^{H/\alpha} B_H(\tau), \tau \geq 0)\,.$$

Due to the complete probability formula, the result is obtained.

Corollary 1. *For any $t > 0$,*

$$X(t) \stackrel{d}{=} (L^1_\alpha(t))^H Y$$

where Y has standard normal distribution and $L^1_\alpha(t)$ and Y are independent.

Remark. Hurst parameter H_1 for above process X can be any positive number. But for traffic applications, it is more interesting the case where $1/2 < H_1 < 1$. So, it is assumed in what follows.

Theorem 5. *The above process X has stationary increments.*

Proof. Fractional Brownian motion B_H has stationary increments. So for any $t_1 < t_2$

$$B_H(t_2) - B_H(t_1) \stackrel{d}{=} B_H(t_2 - t_1)\,.$$

Then, for any $t \geq 0$, $h > 0$ and fixed $L^k_\alpha(t+h) = t_2$, $L^k_\alpha(t) = t_1$, one has

$$B_H(L^k_\alpha(t+h)) - B_H(L^k_\alpha(t)) \stackrel{d}{=} B_H(L^k_\alpha(t+h) - L^k_\alpha(t))\,.$$

Due to the complete probability formula, one has the same for random moments of time. The process $L^k_\alpha(t)$ has stationary increments too. So, one gets

$$B_H(L^k_\alpha(t+h) - L^k_\alpha(t)) \stackrel{d}{=} B_H(L^k_\alpha(h))\,.$$

5 Application to Traffic Modeling

Define the cumulative traffic (or arrival) process $A(t)$, i. e., the total amount of load produced by a source in the time interval $[0,t]$, $t > 0$, by

$$A(t) := mt + (\sigma m)^{1/\beta} X(t)\,,$$

where $m > 0$ is the mean input rate, σ is the scale factor, $\beta = \alpha/H = 1/H_1$, X is the process defined above.

Consider a single server queue with constant service rate $r > 0$ and infinite buffer space, where input is the stable self-similar process defined above ($r > m$ for stability). The buffer occupancy $Q(t,r)$ at time $t \in R^1$ (queue size or queue length) can be written as

$$Q(t,r) = \sup_{s \leq t}(A(t) - A(s) - r(t-s))\,.$$

Due to theorem 2, the process $Q = (Q(t,r),\ t \in R^1)$, is stationary. So, the most interesting is the following probability of overflow:

$$\varepsilon(b) = P(Q(0,r) > b) = P\left(\sup_{\tau \geq 0}(A(\tau) - r\tau) > b\right)\,.$$

Using the technique elaborated in papers [1, 13], one can get the lower bound for the probability of buffer overflow for large b.

It is easily seen that

$$\begin{aligned}
\varepsilon(b) &\geq \sup_{\tau \geq 0} P((A(\tau) - r\tau) > b) \\
&= \sup_{\tau \geq 0} P(m\tau + (\sigma m)^{1\beta} X(\tau) - r\tau > b) \\
&= \sup_{\tau \geq 0} P\left(X(\tau) > \frac{b + (r-m)\tau}{(\sigma m)^{1/\beta}}\right) \\
&= \sup_{\tau \geq 0} P\left(\tau^{1/\beta} X(1) > \frac{b + (r-m)\tau}{(\sigma m)^{1/\beta}}\right) \\
&= \sup_{\tau \geq 0} P\left(X(1) > \frac{b + (r-m)\tau}{(\sigma m \tau)^{1/\beta}}\right)\,.
\end{aligned}$$

Last probability under supremum is a decreasing function of the value

$$f(\tau) = \frac{b + (r-m)\tau}{(\sigma m \tau)^{1/\beta}}\,.$$

Elementary calculations give us that the minimal value of this function is achieved at the point

$$\tau_0 = \frac{b}{\beta(1 - 1/\beta)(r-m)} = \frac{bH_1}{(1-H_1)(r-m)}\,.$$

It follows

$$\varepsilon(b) \geq P(X(1) > f(\tau_0) = b_1)$$

where

$$b_1 = \frac{(r-m)^{H_1}(1-H_1)^{-(1-H_1)}}{(\sigma m H_1)^{H_1}}\, b^{1-H_1}\,.$$

Using corollary 1, one gets

$$\begin{aligned}
P(X(1) > b_1) &= P((L^1_\alpha(1))^H Y > b_1) \\
&\geq P((L^1_\alpha(1))^H Y > b_1, Y > 1) \\
&\geq P((L^1_\alpha(1))^H > b_1, Y > 1) \\
&= P(L^1_\alpha(1) > (b_1)^{1/H}) P(Y > 1)\,.
\end{aligned}$$

For large $x > 0$ (see [20, theorem 2.4.1]), one has

$$P(L^1_\alpha(1) > x) \sim C(\alpha) x^{-\alpha}$$

where

$$C(\alpha) = \frac{\sin(\pi\alpha)}{\pi}\Gamma(\alpha)\,.$$

It follows for large b

$$\begin{aligned}
\varepsilon(b) &\geq C(\alpha)(b_1)^{-1/H_1} P(Y > 1) \\
&= C_1(\alpha, H_1)\sigma \frac{m}{r-m}\, b^{-(1-H_1)/H_1}\,.
\end{aligned}$$

Finally, one has the following

Theorem 6. *An asymptotic lower bound for the overflow probability is given by*

$$\varepsilon(b) \geq C_1(\alpha, H_1)\sigma \frac{m}{r-m} b^{-(1-H_1)/H_1}, \quad b \to \infty .$$

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ДРОБНОЕ ДВИЖЕНИЕ ЛЕВИ С ЗАВИСИМЫМИ ПРИРАЩЕНИЯМИ И ЕГО ПРИЛОЖЕНИЕ К МОДЕЛИРОВАНИЮ СЕТЕВОГО ТРАФИКА

К. Де Никола[1], Ю. С. Хохлов[2], М. Пагано[3], О. И. Сидорова[4]

[1]Университет г. Салерно, denicola@diima.unisa.it

[2]Российский университет дружбы народов, yskhokhlov@yandex.ru

[3]Университет г. Пиза, m.pagano@iet.unipi.it

[4]Тверской государственный университет, Oksana.I.Sidorova@yandex.ru

Аннотация: С начала 1990-х гг. были проведены многочисленные высокоточные измерения для различных сетевых сценариев, которые показали, что трафик Интернет проявляет сильную иррегулярность, выраженную в чрезвычайной вариабельности, а также в наличии долговременной зависимости. Эти новые особенности, которые не удается описать экономным образом с помощью традиционных марковских моделей, имеют сильное влияние на поведение сети, и это привело к необходимости введения в моделирование сетевого трафика α-устойчивых распределений и самоподобных процессов. В настоящей работе рассматривается некоторое обобщение дробного броуновского движения, которое позволяет охватить одновременно обе отмеченные выше особенности реального трафика.

Ключевые слова: дробное броуновское движение; α-устойчивый субординатор; самоподобные процессы; вероятность переполнения буфера