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# The Disjunction Property of Intermediate Propositional Logics

**Abstract.** This paper is a survey of results concerning the disjunction property, Halldén-completeness, and other related properties of intermediate propositional logics and normal modal logics containing S4.

**0.0.** Practically immediately after the creation of the intuitionistic logic *Int* (Heyting, 1930) Gödel [1933] noted that it has *the disjunction property*: a formula $A \vee B$ is provable in *Int* iff at least one of the formulas $A$ or $B$ is provable in *Int*. (Gödel left his statement without a proof; later it was proved by Gentzen [1934–5] with the help of his famous cut elimination theorem, then by Wajsberg [1938], and next by McKinsey and Tarski [1948] who used algebraic methods.)

This property will really be sensible if we remember that the intuitionistic logic is an endeavour to describe effective logical connectives. Moreover, it is reasonable to require that every logic, claiming to be a formalization of effective, constructive way of reasoning, should have the disjunction property. On the contrary, the classical logic *Cl*, accepting the Law of the Excluded Middle $A \vee \neg A$, does not have the disjunction property.

In this connection a question naturally arises on existence (and description, if any) of logics with the disjunction property that are stronger than *Int*. If we take into account the relationship between the disjunction property and the existence property (see, for instance, Friedman [1975] and Friedman and Sheard [1989]) and an increasing interest in constructive proofs as an instrument for program development (we mean the concept of extraction of programs from constructive proofs) then this academic question may acquire a practical importance.

**0.1.** The problem of characterizing logics having the disjunction property turns out to be rather complicated even in the propositional case, and just for this case the most interesting results were obtained. So, in this survey we confine ourselves to considering only *propositional intermediate logics*, and the reader whose interests lie in the sphere of first-order logics is referred to Ono [1987].

The abbreviation DP (as well as others below) is used in two ways: for denoting the disjunction property itself and for denoting the class of intermediate logic having this property.

**0.2.** The question concerning the existence of intermediate logics having DP which are different from *Int* drew logicians' attention in the early fifties, when Łukasiewicz [1952] put forward the conjecture that this question has a negative solution. In those days only a countable set of extensions of *Int* was known: the logics that are obtained by adding to *Int* the formulas which Gödel used for proving that *Int* has no finite characteristic matrix, the logic of Kleene's realizability (see Kleene [1945], Rose [1953]), and maybe few others. Little was known about the realizability logic (not much more do we know about it today), in any case it was not clear whether it has DP. The logics axiomatizable by Gödel's formulas do not have DP because these formulas are disjunctions, with each disjunct being classically invalid. Thus, from the point of view of available information Łukasiewicz's conjecture seems to be justified.

**0.3.** However, in 1957 the Łukasiewicz conjecture was refuted. Kreisel and Putnam [1957] constructed a logic which is a proper extension of *int* and has DP, viz. the logic

$$KP = Int + (\neg p \supset q \vee r) \supset (\neg p \supset q) \vee (\neg p \supset r).$$

Now it is often referred to as the Kreisel-Putnam logic. A proof of the disjunction property and the finite model property of this logic using Kripke semantics can be found in Gabbay [1970].) Another example of a logic with the same properties, given in this paper, is the logic $SL$, which is due to Scott:

$$SL = Int + ((\neg\neg p \supset p) \supset p \vee \neg p) \supset \neg p \vee \neg\neg p.$$

Such were the first steps in the history of studies of the disjunction property.

**0.4.** In this paper we would like to present a picture of results, obtained up to date (and known to us), which concern the disjunction property of intermediate logics. Though this picture is not completed yet, since many important problems are waiting to be solved, nevertheless, in our opinion, there are rather large completed fragments, and we already have certain comprehension of the structure of the class DP.

**1.0.** After Kreisel and Putnam [1957] a number of various examples of intermediate logics with and without DP were constructed.

**1.1.** First, it turned out that the existing "constructive" logics, i.e. logics with constructive semantics in one sense or another, viz. the logic of Kleene's realizability and Medvedev's logic of finite problems, have DP (Varpakhovski [1965], Medvedev [1963]).

**1.2.** Gabbay and de Jongh [1969, 1974] constructed an infinite sequence of finitely axiomatizable intermediate logics having the finite model property and DP, viz. the logics

$$T_n = Int + \&_{i=0}^{n}((p_i \supset \bigvee_{j\neq i} p_j) \supset \bigvee_{j\neq i} p_j) \supset \bigvee_{i=0}^{n} p_i \ \ (n > 1)$$

of all finite $n$-ary trees (i.e. a formula is provable in $T_n$ iff it is valid in every Kripke frame which has the form of a finite $n$-ary tree). These logics are related by the proper inclusions

$$Int \subset \ldots \subset T_n \subset \ldots \subset T_3 \subset T_2,$$

and

$$Int = \bigcap_{n<\omega} T_n.$$

Note that *Int* is determined by the class of all (infinite) $n$-ary trees, for each $n \geq 2$, (Kirk [1979]).

Proving DP of $T_n$, Gabbay and de Jongh actually used the following semantic criterion: if a logic is determined by a class **K** of frames with least elements and, for each $\Phi_1, \Phi_2 \in$ **K**, there is a frame $\Phi \in$ **K** containing the disjoint union of the frames $\Phi_1$ and $\Phi_2$ as its generated subframe (see Fig. 1) then the logic has DP.

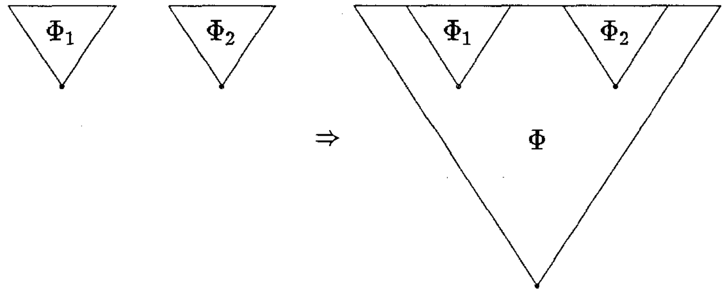


Figure 1

Indeed, if formulas $A$ and $B$ are refuted at the least elements of the frames $\Phi_1$ and $\Phi_2$, respectively, then, according to the heredity property of formula truth-values with respect to partial orderings in frames, both formulas $A$ and $B$ are refuted at the least element of $\Phi$, and so the disjunction $A \vee B$ is also refuted. This criterion may lead to the idea that $T_2$ is a maximal logic in the class DP (see Section 1.7).

**1.3.** Ono [1972] constructed another infinite sequence of finitely axiomatizable logics having DP:

$$B_n = Int + \&_{i=0}^{n}(\neg p_i \equiv \bigvee_{j\neq i} p_j) \supset \bigvee_{i=0}^{n} p_i.$$

The relationship between the Ono logics and the Gabbay and de Jongh logics can be represented as follows:

$$\begin{array}{ccccccccc} \ldots \subset & T_{n+1} & \subset & T_n & \subset & \ldots \subset & T_3 & \subset & T_2 \\ & \cup & & \cup & & & \cup & & \cup \\ \ldots \subset & B_{n+1} & \subset & B_n & \subset & \ldots \subset & B_3 & \subset & B_2 \end{array}$$

(all the inclusions here are proper).

**1.4.** We remind the reader of the definitions of logics of finite slice and finite width. An intermediate logic $L$ is said to be a *logic of n-th slice* (cf. Hosoi [1967]) if $L \vdash P_n$ where $P_0 = \bot$, $P_{m+1} = p_{m+1} \vee (p_{m+1} \supset P_m)$, i.e. any frame for $L$ has no chains with $n+1$ elements; $L$ is a *logic of width n* (cf. Fine [1974], Sobolev [1977]) if $L \vdash \bigvee_{i=1}^{n+1}(p_i \subset \bigvee_{j=1,j\neq i}^{n+1} p_j)$, i.e. any rooted frame for $L$ has no anti-chains with $n+1$ elements.

Ono [1972] noted that the logics of finite slices do not have DP. As Kuznetsov [1974] showed, there are a continuum of logics in $n$th slice, for each $n \geq 3$. It is easy to see that all logics of finite width do not possess DP either, and the cardinality of the family of such logics (even of width 2) is of continuum too.

**1.5.** Anderson [1972] studied DP of the intermediate logics which can be axiomatized by the formulas $F_k$ of the Nishimura [1960] sequence:

$F_\omega = p \supset p,\ F_0 = p\&\neg p,\ F_1 = \neg p,\ F_2 = p,$
$F_{2n+1} = F_{2n-1} \supset F_{2n},\ F_{2n+2} = F_{2n-1} \vee F_{2n}\ \ (n = 1, 2, \ldots).$

(Here we present a variant of these formulas which was suggested by Anderson.) Each formula containing one variable is equivalent in *Int* to some of $F_k$, whereas Nishimura's formulas themselves are pairwise non-equivalent in *Int*. Using a method due to Harrop [1956], Anderson showed that all logics of the form $Int + F_{2n+1}$, for $n \geq 4, n \neq 6$, have DP; moreover, it is evident that the logics $Int + F_{2n+2}$, for $n \geq 1$, $Int + F_5$, and $Int + F_7$ do not possess this property. The only formula which was not covered by Anderson [1972] is (of course) $F_{13}$. However, in spite of the unlucky number, Sasaki [1990], using a Gentzen-type technique (similar to that of Kreisel and Putnam [1957]), proved that $Int + F_{13}$ has DP.

Drugush [1978] and Bellissima [1989] considered a possibility of refuting DP with the help of formulas containing only one variable.

**1.6.** The question on the quantity of intermediate logics having DP was finally solved by Wroński [1973]. Using properties of the Jankov [1963] characteristic formulas, he proved that the cardinality of the class DP is of continuum.

**1.7.** Maksimova [1986] showed that the logic $T_2$ (of all finite binary trees; see Section 1.2) is not maximal in DP because it has a proper extension with DP, viz. the logic $L\Pi$ consisting of all formulas that are valid in all frames $\Pi_n$, for $n < \omega$, shown in Fig. 2. In the same paper the following new logics with DP were constructed:

$$ND_k = Int + (\neg p \supset \neg q_1 \vee \ldots \vee \neg q_k) \supset (\neg p \supset \neg q_1) \vee \ldots \vee (\neg p \supset \neg q_k) \quad (k \geq 2),$$

$$ND = \bigcup_{k \geq 2} ND_k.$$

It is clear that

$$ND_2 \subset ND_3 \subset \ldots \subset ND_n \subset \ldots \subset ND \subset KP.$$

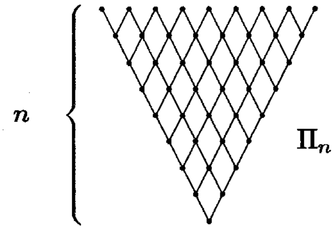


Figure 2

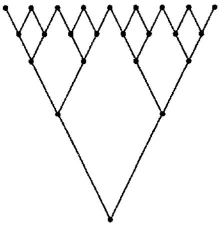

Figure 3

**2.0.** The results stated above show that both the class DP and its complement $\overline{\mathrm{DP}}$ are very large. Is it possible to describe in some way the structure of these classes?

**2.1.** In the survey by Hosoi and Ono [1973] it was noted that DP is not closed under intersections and unions (sums, to be more exact) of logics. As far as intersections are concerned, the following statement is evident: if a logic is represented as an intersection of two incomparable (by the inclusion) logics then it does not have DP. The fact that DP is not closed under unions follows, for instance, from Section 2.2.

The complement of DP is closed under intersections of logics. We do not know whether it is closed under unions. (We conjecture that it is not.)

**2.2.** To survey somehow the class DP one might try to find its boundaries. Using Zorn's lemma, it is not difficult to see that each intermediate logic with DP is contained in some maximal intermediate logic having DP. What is the set of these maximal logics? The most pleasant solution to this question would be, of course, the discovery of the unique maximal (that is, the greatest) logic in the class DP. Unfortunately, this proved to be not the case: Kirk [1982] showed that the greatest logic in DP does not exist; more exactly, there are at least two maximal logics having DP. So, how many maximal logics are there? Maksimova [1984] showed that the set of these logics is infinite, and afterwards Chagrov [1991] obtains the final result: there are a continuum of maximal logics in DP. In spite of this abundance, only one concrete example of maximal logic having DP is known — it is Medvedev's logic of finite problems (see also Section 2.2.1.). This fact was first proved by Levin [1969]; other proofs were later found by Maksimova [1986] (actually, she proved that Medvedev's logic is the greatest among those logics with DP which contain $ND$) and recently by Miglioli et al [1989].

In the last paper a general characterization of maximal logics in DP was given by means of *nonstandard maximal intermediate logics* with DP (in which only negated formulas are allowed to be substituted for variables when the rule of substitution is applied to the additional axioms), and these nonstandard logics are determined by maximal sets with DP of so called *negatively saturated formulas* (i.e. formulas in which all variables are within the scope of $\neg$).

Other pretenders to the role of maximal logics in DP, as one may suggest, are $L\Pi$ and the logic determined by all finite frames of the form shown in Fig. 3.

It is not known whether there exists a finitely axiomatizable maximal logic in DP. (This question was asked by Maksimova [1986]).

**2.2.1.** Quite recently Galanter [1990] has constructed a continuum of intermediate logics which are maximal in DP. Each of Galanter's logics is characterized by the class of frames of the form $\langle\{X : X \subseteq \{1, \dots, n\}, X \neq \emptyset, \overline{\overline{X}} \notin N\}, \supseteq\rangle$ where $n = 1, 2, \dots$ and $N$ is some fixed infinite set of natural numbers. Note that we obtain a semantic definition of Medvedev's logic by taking $N = \emptyset$.

**2.3.** How dense do the logics having DP lie in the class of intermediate logis? Near *Int* the picture is rather diverse: if $L \neq Int$ then between *Int* and $L$ there are continuum of logics with DP and as many without DP. Bellissima [1989] constructed a chain of logics $L_n$, $n < \omega$, such that $L_n$ does not have DP and, for any formula $A$ of at most $n$ variables, $L_n \vdash A$ implies $Int \vdash A$.

What about intervals near maximal logics in DP? In particular, how big intervals of logics having DP do there exist?

**2.4.** Maksimova [1986] noted that each logic with DP is an intersection of a decreasing sequence of logics and has no coverings in the lattice of all intermediate logics.

**3.0.** For the purpose of studying the disjunction property it is important to understand the structure of classes of frames or algebras which characterize logics having DP, i.e. to find a semantic equivalent of DP.

**3.1.** A sufficient condition, viz. the criterion of Gabbay and de Jongh, was presents in Section 1.2. Having reformulated it in algebraic terms (in order to escape the effect of incompleteness with respect to Kripke semantics), Maksimova [1986] proved that the resultant algebraic criterion is equivalent to DP. Thus, the problem from Section 3.0 may be considered as successfully solved.

**4.0.** The problem of syntactical characterization of intermediate logics having DP turned out to be much more difficult. Given the axiom of a logic, how to determine whether it has DP or not?

**4.1.** In the list of open problems in the survey by Hosoi and Ono [1973] a question was propounded whether the disjunctionless fragments of logics having DP are equal to the disjunctionless fragment of *Int*. Minari [1986] and Zakharyashchev [1987] independently gave a positive answer to this question. (Using Maksimova's algebraic criterion for DP (reformulated in terms of general frames) and a description of the structure of countermodels for disjunctionless formulas in Zakharyashchev [1983], Zakharyashchev [1990] found a briefer and more elegant proof than those in the papers mentioned above.) Earlier Szatkowski [1981] noted that the implicationless fragment of each intermediate logic with DP does not differ from the implicationless fragment of *Int* either. Thus, if a disjunctionless or implicationless formula is provable in a logic $L$ (in particular, is an axiom of $L$) and is not provable in *Int* then $L$ cannot have DP. Of course, the converse is not true (see Section 2.3). This is only a necessary syntactical condition for DP. The class of intermediate logics having the same disjunctionless fragment as *Int* we denote, following Minari [1986], by $D^0$.

**4.2.** Now, it is natural to try to supplement the necessary condition from Section 4.1 with sufficient syntactical conditions for DP. The problem here is that it is difficult to find any syntactical parameters which induce DP just as, for instance, the absence of the disjunction at least in one of additional axioms of a logic results in that the logic does not have DP. That is why the existing sufficient conditions are restricted only to formulas of some form connected in one way or another with frames or algebras. Wroński [1973] obtained a sufficient condition for DP in the case where the axioms of a logic are Jankov's characteristic formulas (the condition is imposed on the form of pseudo-Boolean algebras from which they are constructed). However, these formulas cannot axiomatize all intermediate logics.

**4.3.** The Jankov characteristic formulas are in effect a special case of the canonical formulas introduced by Zakharyashchev [1983, 1984, 1989]. They are defined as follows.

Let $\Phi$ be a finite rooted frame, with $a_0, \dots, a_n$ being its all distinct points and $a_0$ being the origin. A pair $\delta = (\overline{a}, \overline{b})$ of nonempty anti-chains in $\Phi$ is called a *disjunctive domain* (*d-domain*, for short) in $\Phi$ if

(i) $\overline{a}$ has at least two points;

(ii) $\forall a \in \overline{a}\, \forall b \in \overline{b}\ \neg a \leq b$;

(iii) $\forall c (\forall a \in \overline{a}\ c \leq a \Rightarrow \exists b \in \overline{b}\ c \leq b)$.

Now, take some (possibly empty) set $D$ of d-domains in $\Phi$. With $\Phi$ and $D$ we associate the following *canonical formula*

$$X(\Phi, D, \bot) = \&_{a_i \leq a_j} A_{ij} \& \&_{\delta \in D} B_\delta \& C \supset p_0$$

where

$$A_{ij} = (\&_{\neg a_j \leq a_k} p_k \supset p_j) \supset p_i,$$

$$C = \&_{i=0}^{n}(\&_{\neg a_i \leq a_k} p_k \supset p_i) \supset \bot$$

and if $\delta = (\overline{a}, \overline{b})$ then

$$B_\delta = \&_{a_i \in \overline{b}}(\&_{\neg a_i \leq a_k} p_k \supset p_i) \supset \bigvee_{a_i \in \overline{a}} p_i.$$

By $X(\Phi, D)$ we denote the *positive canonical formula* which is obtained from $X(\Phi, D, \bot)$ by deleting the conjunct $C$.

Zakharyashchev [1983, 1989] gave a necessary and sufficient condition for the refutability of canonical formulas in general frames and proved that there is an algorithm which, for any formula $A$, constructs canonical formulas $X(\Phi_1, D_1, \bot), \ldots, X(\Phi_n, D_n, \bot)$ such that

$$Int + A = Int + X(\Phi_1, D_1, \bot) + \ldots + X(\Phi_n, D_n, \bot).$$

For a positive $A$ (containing neither $\bot$ nor $\neg$) one can use only the positive canonical formulas. It is important that if $A$ has no disjunctions then $D_i = \emptyset$, for all $i = 1, \ldots, n$.

For a frame $\Phi$, we may in general define a number of various canonical formulas: $X(\Phi, \emptyset, \bot), \ldots, X(\Phi, D^*, \bot)$ where $D^*$ contains all d-domains in $\Phi$. It is worth noting that if $A$ is Jankov's formula associated with $\Phi$ then $Int + A = Int + X(\Phi, D^*, \bot)$.

According to the results in Minari [1986] and Zakharyaschev [1987] (see Section 4.1), if a logic $L$ has DP and $L \vdash X(\Phi, D, \bot)$ then $D \neq \emptyset$. Thus, to construct a logic with DP we have to choose canonical axioms having non-empty sets of d-domains.

**4.4** Though this necessary condition is not a sufficient one (see Section 2.3), the following question is open: is it true that $L = Int + X(\Phi, D, \bot)$ (with only one canonical axiom) has DP iff, for any $\Phi'$ and $D'$, $L \vdash X(\Phi', D', \bot)$ implies $D' \neq \emptyset$? Another question: is it true that $L = Int + X(\Phi, D, \bot)$ has an extension with DP iff $L$ has DP? In the case of positive solution to Minari's problem (see Section 6.1) these two questions are equivalent.

A point $a$ is said to be a *focus* for an anti-chain $a_1, \ldots, a_n$ $(n \geq 2)$ in a frame $\Phi$ if $\{a_1, \ldots, a_n\}$ is the set of all immediate successors of $a$ in $\Phi$. Bellissima [1989] calls $\Phi$ a *detailed frame* if each anti-chain in $\Phi$ has a focus. He gives a description of all detailed frames, shows (using formulas of one variable) that if $\Phi$ is detailed then $Int + X(\Phi, D^*, \bot)$ does not have DP and conjectures that the converse is also true. Since, for a detailed frame $\Phi$, $Int + X(\Phi, D^*, \bot) = Int + X(\Phi, \emptyset, \bot)$, the first question above is a generalization of Bellissima's conjecture.

**4.5** Chagrov and Zakharyashchev [1989, 1990a] found two sufficient conditions for DP which are imposed on the canonical axioms of intermediate logics. We give slightly simplified versions of these conditions.

(i) Let an intermediate logic $L$ be axiomatized by canonical formulas $X(\Phi, D, \perp)$ (or $X(\Phi, D)$)) such that the set $S$ of the immediate successors of the origin in $\Phi$ has at lest three points and $(\{a, b\}, \{c\}) \in D$, for all different $a, b, c \in S$. Then $L$ has DP.

(ii) Let $L$ be axiomatized by canonical formulas $X(\Phi, D, \perp)$ such that the "height" of $\Phi$ is greater than or equal to 3 and $D$ contains a d-domain $(\overline{a}, \overline{b})$ where $\overline{a}$ has no focus in $\Phi$ and consists of some maximal points in $\Phi$. Then $L$ has DP.

The first condition covers the Gabbay and de Jongh [1974] logics

$$T_n = Int + X(\Phi_n, D_n)$$

where $\Phi_n$ is the frame shown in Fig. 4 and $D_n$ contains all d-domains of the form $(\{a_i, a_j\}, \{a_k\})$. It also covers the Ono [1972] logics

$$B_n = Int + X(\Phi_n, D_n, \perp)$$

($\Phi_n$ and $D_n$ are the same as for $T_n$) and all logics constructed by Wroński [1973].

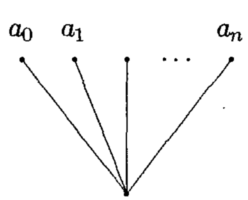


Figure 4

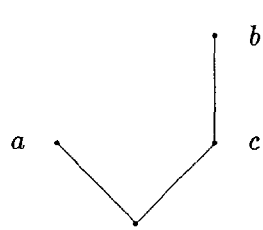


Figure 5

The second condition is clearly satisfied by the Scott logic

$$SL = Int + X(\Phi, D, \perp)$$

where $\Phi$ is depicted in Fig. 5 and $D$ has only one d-domain $(\{a, b\}, \{c\})$.

These conditions, of course, cannot be applied to all known logics with DP, for instance, to $KP$ and $ND_k$.

**4.6.** The difficulties, arising when the disjunction property of intermediate logics given by their axioms is investigated, turned out to be of principal nature. Chagrov and Zakharyashchev [1989, 1990a] proved that the disjunction property of intermediate calculi is algorithmically undecidable, i.e. there is no algorithm which is capable of deciding, given formulas $A_1, \ldots, A_n$, whether or not the logic $Int + A_1 + \ldots + A_n$ has DP. This result gives a solution to the problem 100 a) raised by Maksimova in "Logic Notebook" [1986].

Many related problems turn out to be undecidable too. For instance, Chagrov proved that there are no algorithms which are capable of deciding, given axioms of an intermediate calculus, whether it belongs to the class $D^0$) (see Section 4.1), whether it is axiomatizable by disjunctionless formulas or by canonical formulas satisfying the sufficient condition (ii) in Section 4.5 (a canonical axiomatization of a logic is not unique). The property "to have the same implicationless fragment as *Int*" is also undecidable. Some of these results can be found in Chagrov and Zakharyashchev [1989, 1990a].

**4.7.** The result in Anderson [1972] (see Section 1.5) makes it possible to prove the existence of a polynomial time algorithm deciding, given a formula $A$ containing one variable, whether the logic $Int + A$ has DP. Sasaki [1990] makes this existential statement constructive.

What about DP (the finite model property, decidability, and other properties) of intermediate logics with additional axioms containing two variables?

**5.0.** How is DP related to other standard properties of intermediate logics?

**5.1.** It follows from Sections 1.2 and 1.3 that the class DP contains decidable logics and logics having the finite model property. Wroński [1973] gave examples of logics with DP and without the finite model property, proved that there are a continuum of such logics and noted that there exist undecidable logics with DP (which are not finitely axiomatizable).

Using the sufficient condition (ii) mentioned in Section 4.5 and the existence of undecidable (positively axiomatizable) intermediate logics (see, for instance, Shekhtman [1978]), Zakharyashchev (in Chagrov and Zakharyashchev [1989]) gave a method for constructing undecidable finitely axiomatizable logics (i.e. calculi) having DP.

**5.2** Minari [1986] asked whether there exist incomplete (with respect to Kripke semantics) intermediate logics with DP. Using again the sufficient condition (ii) above and known incomplete (positively axiomatizable) logics (see Shekhtman [1977]), Zakharyashchev (in Chagrov and Zakharyashchev [1989]) constructed incomplete intermediate calculi having DP.

**5.3.** Among the logics without DP there are also logics (and even calculi) with the finite model property and without it, decidable and undecidable, complete and incomplete with respect to Kripke semantics.

**5.4.** Since DP $\subset D^0$ (see Section 4.1), no logic with DP has the polynomial finite model property, i.e. the number of elements in refutation Kripke frames for a logic cannot be bounded by a polynomial of the length of a refuted formula. This is a consequence of the fact that the disjunctionless fragment of *Int* does not have the polynomial finite model property (see Zakharyashchev and Popov [1980] and Chagrov [1985]).

It is worth noting that the Kreisel-Putnam, logic *KP* does not have the exponential finite model property, but has the double-exponential finite model property (see Chagrov and Zakharyashchev [1990b, 1990c]). Since the implicative fragment of *Int* is PSPACE-complete (Chagrov [1985]), each intermediate logic with DP is PSPACE-hard.

**5.5** Maksimova [1977] noted that *Int* is the unique intermediate logic with DP for which the Craig interpolation theorem holds (it states that if a formula $A \supset B$ is provable in $L$ then there is a formula $C$ such that both formulas $A \supset C$ and $C \supset B$ are provable in $L$ and all the variables in $C$ are common variables in $A$ and $B$). As Maksimova showed, there are seven intermediate logics having Craig's interpolation property and all of them except *Int* do not belong to $D^0$.

**6.0.** We say that a logic $L$ *admits* DP if $L$ is contained in some logic with DP. The property "to admit DP" and the class of logics having this property are denoted by ADP.

It follows from the definition that a logic had ADP iff it is contained in some maximal logic with DP.

**6.1.** According to Section 4.1, ADP $\subseteq D^0$. Does the inverse inclusion hold? In other words, is it true that ADP $= D^0$? This problem was raised by Minari [1986]. In case of its positive solution we would obtain a syntactical characterization of ADP. For the present no such characterization is known.

**6.2.** One can obtain a semantic characterization of ADP using Maksimova's algebraic criterion of DP (see Section 3.1).

**6.3** The property ADP as well as DP itself and as well as the property "to belong to $D^0$" (see Section 4.6) turns out to be algorithmically undecidable.

**6.4.** One can show (see Chagrov [1991]) that there are a continuum of maximal logics in $D^0$.

Galanter and Muravitski [1988] assert that Medvedev's logic of finite problems is maximal in the class $D^0$ (cf. Section 2.2). This gives some hope for the positive solution to Minari's problem in Section 6.1.

**6.5.** Galanter [1990] claims that all logics defined in Section 2.2.1 are maximal in $D^0$.

**7.0.** Halldén [1951] and McKinsey [1953] introduced a notion which is similar to DP though not related with any constructive interpretation of logical connectives. As a matter of fact, they proposed to consider "reasonable" or, as it is said nowadays, *Halldén-complete*, those logics in which, for any formulas $A$ and $B$ having no variables in common, from the provability of the disjunction $A \vee B$ it follows that at least one of its disjuncts $A$ or $B$ is provable. (In his original definition, Halldén called "unreasonable" only those logics $L$ for which there are formulas $A$ and $B$, each containing one variable, say, $p$ and $q$, such that $p \neq q$, $L \vdash A \vee B$, but neither $L \vdash A$ nor $L \vdash B$. In this case we say that $L$ is *strongly Halldén-incomplete*.)

For intermediate logics, DP implies Halldén-completeness; the converse is not true as the example of the classical logic $Cl$ shows. The property of Halldén-completeness and the class of logics having this property we will denote by HC. The existence of Halldén-incomplete intermediate logics was pointed out by Halldén [1951].

**7.1.** Lemmon [1966] showed that (intermediate or modal) logic is Halldén-incomplete iff it is represented as an intersection of two incomparable (by inclusion) logics (cf. Section 2.1). In particular, the class HC is not closed under intersections of logics. The fact that HC is not closed under unions was proved by Galanter [1988].

**7.2.** Wroński [1976] found an algebraic characterization of HC: an intermediate logic is Halldén-complete iff it is the logic of some subdirectly irreducible pseudo-Boolean algebra (i.e. pseudo-Boolean algebra with a second greatest element).

**7.3.** In parallel with the proof of the undecidability of DP (see Section 4.6) Chagrov and Zakharyashchev [1989, 1990a] proved the undecidability of Halldén-completeness for intermediate logics.

**7.4.** Minari's problem in Section 6.1 can be formulated as follows: do the maximal logics in the class $D^0$ have DP? On the way of solving this problem Galanter and Muravitski [1988] proved that each maximal logic in $D^0$ is Halldén-complete.

**7.5.** All intermediate logics having Craig's interpolation property (see Maksimova [1977]) are Halldén-complete. This fact can be established by a straightforward inspection of these logics. Another proof was found by Zachorowski [1978] (we are grateful to N.-Y. Suzuki for pointing out this paper).

**7.6.** Galanter [1988] showed that each Halldén-incomplete logic is contained in some maximal Halldén-incomplete logic and there are only two such maximal logics, viz. the logics of the pairs of the frames shown in Fig. 6 and Fig. 7, respectively.

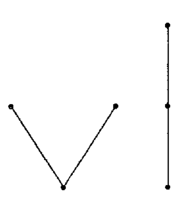

Figure 6

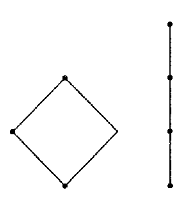

Figure 7

Though the notion of *Halldén-precompleteness* is useless for the proof of the decidability of HC (unlike, say, the notion of pretabularity, with the help of which the decidability of the tabularity is proved), since neither HC nor the absence of this property are hereditary, the description of Halldén-precomplete logics shows the typical situations of the origin of Halldén-incompleteness.

**7.7** *Conjecture*: All logics axiomatizable by formulas containing only one variable are Halldén-complete.

As a confirmation for this conjecture one may quote a result of Anderson (see Section 1.5) and the fact that each logic of the form $Int + F_n$, for

$0 \leq n \leq 10$, is Halldén-complete.

**7.8.** Galanter [1988] proved that there are a continuum of Halldén-incomplete intermediate logics (which are not strongly Halldén-incomplete) and as many Halldén-complete intermediate logics without DP.

**8.0.** Now, we note some facts on DP and HC of modal companions of intermediate logics, confining ourselves only to the normal extensions of the Lewis system $S4$. Recall that a modal logic $M$ is called *a modal companion* of an intermediate logic $L$ if a formula $A$ is provable in $L$ iff its Gödel's translation $T(A)$ (prefixing $\Box$ to all subformulas of $A$) is provable in $M$; in this case the logic $L$ is called *a superinuitionistic fragment* of $M$. The set of all modal companions of an intermediate logic $L = Int + \{A_i\}_{i \in I}$ is infinite and has the least and the greatest elements, viz. the logics $\tau L = S4 + \{T(A_i)\}_{i \in I}$ and $\sigma L = \tau L + \Box(\Box(p \supset \Box p) \supset p) \supset p$, respectively (see Maksimova and Rybakov [1974], Blok [1976] and Esakia [1979]). More information about modal companions of intermediate logics can be found in Chagrov and Zakharyashchev [1991].

We say that a modal logic $M$ has the (*modal*) *disjunction property* if from the provability of a disjunction $\Box A \vee \Box B$ in $M$ it follows that at least one of the formulas $\Box A$ or $\Box B$ is provable in $M$; the definition of Halldén-completeness remains the same.

**8.1.** It is evident that DP and HC are preserved while passing from a modal logic to its superintuitionistic fragment. Transferring in the opposite direction is more problematic as far as the preservation of these properties is concerned.

**8.2.** Gudovshchikov and Rybakov [1982] noted that DP is preserved while passing from an intermediate logic to its greatest modal companion. Using this fact and the undecidability of DP of intermediate logics (see Section 4.6), one can easily prove the undecidability of DP of modal logics.

Zakharyashchev [1989a] proved the preservation of DP (and some other properties as well) when passing to the least modal companion.

One can show that each intermediate logic with DP has infinitely many modal companions without DP; we conjecture that there are continuum of such companions.

**8.3.** A few results on DP of modal logics whose axioms are modal canonical formulas (see Zakharyashchev [1984, 1988]) were obtained in Zakharyashchev [1987].

The modal canonical formulas $Y(\Phi, D, \bot)$ are defined similarly to the intuitionistic canonical formulas: the only difference is that they are associated with quasi-ordered frames $\Phi = \langle W, R\rangle$:

$$Y(\Phi, D, \bot) = \&_{a_i R a_j} A_{ij} \& \&_{i=0}^{n} A_i \& \&_{\delta \in D} B_\delta \& C \supset p_0$$

where

$$A_{ij} = \Box(\Box p_j \supset p_i),$$
$$A_i = \Box((\&\Gamma_i \supset p_i) \supset p_i),$$
$$\Gamma_i = \{p_k, \Box p_l : k \neq i, \neg a_i R a_l\},$$
$$C = \Box(\&_{i=0}^{n} \Box p_i \supset \bot)$$

and if $\delta = (\bar{a}, \bar{b})$ then

$$B_\delta = \Box(\&_{a_i \in \bar{b}} \Box p_i \supset \bigvee_{a_i \in \bar{a}} \Box p_i).$$

Here $a_0, \ldots, a_n$ are all the distinct points in $W$ and $a_0$ is the origin in $\Phi$. By $Y(\Phi, D)$ we denote the *positive canonical formula* which is obtained from $Y(\Phi, D, \bot)$ by deleting the conjunct $C$.

Zakharyashchev [1984, 1988] gave a necessary and sufficient condition for the refutability of the modal canonical formulas in general frames and proved that each normal extension of $S4$ can be axiomatized by these formulas. A modal logic $M$ is a modal companion of an intermediate logic $L = Int + \{X(\Phi_i, D_i, \bot)\}_{i\in I}$ iff $M$ can be represented in the form

$$M = S4 + \{Y(\Phi_i, D_i, \bot)\}_{i\in I} + \{Y(\Phi_j, D_j, \bot)\}_{j\in J}$$

where each of the frames $\Phi_j$, for $j \in J$, contains a proper cluster;

$$\tau L = S4 + \{Y(\Phi_i, D_i, \bot)\}_{i\in I}$$

$$\sigma L = \tau L + Y(C_2, \emptyset)$$

where $C_2$ is the cluster with two elements.

In contrast to intermediate logics, modal logics of the form $S4+\{Y(\Phi_i, \emptyset, \bot)\}_{i\in I}$ may have DP, the witnesses are $S4Grz = S4 + Y(C_2, \emptyset)$ and $S4.1 = S4+Y(C_2, \emptyset, \bot)$.Zakharyashchev [1987] proved that a logic $S4+\{Y(\Phi_i, \emptyset, \bot)\}_{i\in I}$ with independent additional axioms does not have DP iff, for some $i \in I$, the first cluster and at least one of the last cluster in $\Phi_i$ are singletons. One can prove that this result gives an algorithm for recognizing DP of logics of the form $S4 + A_1 + \ldots A_n$ where $A_i$, for $i = 1, \ldots, n$, is a formula of Sahlqvist

[1975] or a subframe formula of Fine [1985].

**8.4.** Chagrov and Zakharyashchev [1990] showed that Halldén-completeness, unlike DP, may be not preserved even while passing to the least and to the greatest modal companions of an intermediate logic. As an example one can take the intermediate logic $L$ of the frame $\Phi$ shown in Fig. 8. It is evident that $L$ is Haalldén-complete. However, the formula $A \vee B$, where

$$A = \neg(\&_{i=1}^{3} \neg\Box\neg A_i \& \&_{i=1}^{3} \Box\neg(\&_{j=1, j\neq i}^{3} \neg\Box\neg A_j \& \Box\neg A_i)),$$

$$A_1 = \Box(p\&q), \quad A_2 = \Box(p\&\neg q), \quad A_3 = \Box(\neg p\&q),$$

$$B = \neg(r\&\neg\Box\neg(\Box\neg r\&s\&\neg\Box\neg\Box\neg s)),$$

is provable in every modal companion $M$ of $L$, but neither $A$ nor $B$ are provable in $M$, since $A$ is refuted at the point $a$ in the frame $\Phi$ (but not at $b$!) and $B$ is refuted at the point $b$.

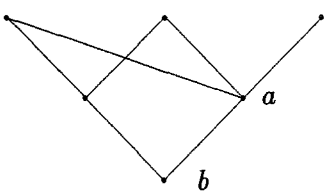


Figure 8

It is not difficult to show that there are continuum of Halldén-complete intermediate logics having no Halldén-complete modal companions (see Chagrov and Zakharyashchev [1990a], Chagrov [1991]).

Note also that the modal logic of the frame $\Phi$, being Halldén-incomplete, is not represented, nevertheless, as an intersection of two incomparable normal modal logics, though, according to Lemmon [1966], is represented as an intersection of two incomparable logics which are not normal.

*Conjecture*: The least modal companion of an intermediate logic is Halldén-complete iff the greatest companion is.

**8.5.** For modal logics, HC does not follow from DP. Moreover, Chagrov and Zakharyashchev [1990, 1990a] and Chagrov [1991] proved that there are continuum of modal logics in each of the classes $\mathrm{DP} \cap \mathrm{HC}$, $\overline{\mathrm{DP}} \cap \mathrm{HC}$, $\mathrm{DP} \cap \overline{\mathrm{HC}}$, $\overline{\mathrm{DP}} \cap \overline{\mathrm{HC}}$.

**8.6.** Halldén-completeness of modal logics, as was shown by Chagrov and Zakharyashchev [1990, 1990a], is undecidable. To prove this fact (which, according to Section 8.4, is not an immediate consequence of the undecidability of HC of intermediate logics), two syntactical sufficient conditions for HC were found. These conditions are of the same type as the sufficient conditions of DP in Section 4.5.

(i) Let a modal logic $M$ containing $S4Grz$ be axiomatized by canonical formulas $Y(\Phi, D, \bot)$ (or $Y(\Phi, D)$) such that the set $S$ of the immediate successors of the origin in $\Phi$ has at least three points and $(\{a, b\}, \{c\}) \in D$, for all different $a, b, c \in S$. Then $M$ is Halldén-complete.

(ii) Let a modal logic $M$ containing $S4Grz$ be Kripke complete and axiomatized by canonical formulas $Y(\Phi, D, \bot)$ (or $Y(\Phi, D)$) such that the origin in $\Phi$ has only one immediate successor. Then $M$ is Haldén-complete.

**8.7.** Van Benthem and Humberstone [1983] have the following semantic sufficient condition for HC which is satisfied by all known Haldén-complete modal logics. Lwt a modal logic $M$be determined by a class $\mathbf{K}$ of frames in which for any $\Phi_1, \Phi_2 \in \mathbf{K}$ and any points $w_1, w_2$ in $\Phi_1$ and $\Phi_2$, respectively, there exists a frame $\Phi \in \mathbf{K}$ with a point $w$ and two p-morphisms $f_1$ and $f_2$ from $\Phi$ to $\Phi_1$ and $\Phi_2$ such that $f_1(w) = w_1$, $f_2(w) = w_2$. Then $M$ is Haldén-complete.

For the present, it is not known whether this condition (reformulated in terms of general frames, of course) is necessary. However, for intermediate logics, the sufficient condition of van Benthem-Humberstone is not necessary, as the logic $L$ in Section 8.4 shows. Note that the condition is undecidable for both modal and intermediate logics.

**8.8.** Lemmon [1966] mentioned a question on the relationship between Halldén-incompleteness and strong Halldén-incompleteness. For intermediate logics, as we have seen in Section 7.8, the former does not imply the latter. For modal logics, the question is still open.

**8.9.** It is probably worth noting that the Gödel-Lob provability logic $GL$ ($= G$ of Solovay [1976]) and all its normal extensions, with the exception of the inconsistent and the maximal consistent, are Halldén-incomplete, since the formula $\Box\bot \vee \neg\Box\bot$ is provable in all of them but neither $\Box\bot$ nor $\neg\Box\bot$ are. On the contrary, Solovay's logic $S$ ($= G'$ obtained by adding to $GL$ the axiom $\Box p \supset p$ without taking the closure under the rule of necessitation) is Halldén-complete and this property in its extensions, as Chagrov [1990] shows, is undecidable.

**8.10.** The following property —so called *variable separation principle* — was considered by Maksimova [1976, 1979] for relevant and intermediate logics: if $L \vdash A\&B \supset C \vee D$, with $A \supset C$ and $B \supset D$ having no variables in common, then $L \vdash A \supset C$ or $L \vdash B \supset D$. This property is clearly related to Halldén-completeness, and we will call it *Maksimova-completeness* (MC).

For modal logics, MC is equivalent to HC. Fig. 9 illustrates the relationship between the classes DP, HC, and MC in the case of intermediate logics. Chagrov and Zakharyashchev [1990a] proved that the cardinality of each set of logics shown in Fig. 9 is of continuum. The intermediate logic of the frame depicted in Fig. 8 gives an example of a logic from the class $\mathrm{HC} \cap \overline{\mathrm{MC}}$. Indeed, let

$$A = (C_1 \supset C_2 \vee C_3)\&(C_2 \supset C_1 \vee C_3)\&(C_3 \supset C_1 \vee C_2),$$

$$B = \top, \quad C = C_1 \vee C_2 \vee C_3, \quad D = r_1 \vee (r_1 \supset r_2 \vee \neg r_2),$$

where $C_1 = \neg(p\&q)$, $C_2 = \neg(p\&\neg q)$, $C_3 = \neg(\neg p\&q)$. Then $L \vdash A\&B \supset C \vee D$, but neither $L \vdash A \supset C$ nor $L \vdash B \supset D$.

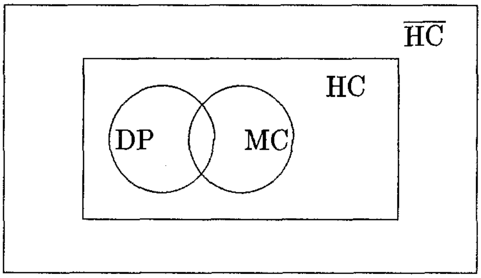


Figure 9

Chagrov and Zakharyashchev [1990a] noted that the sufficient conditions for Halldén-completeness from Section 8.6, reformulated in terms of the intuitionistic canonical formulas, are sufficient conditions for Maksimova-completeness of intermediate logics. They also proved the undecidability of MC.

*Conjecture*: For any intermediate logic $L$, $L$ is MC iff $\tau L$ is MC iff $\sigma L$ is MC.

The sufficient condition of van Benthem and Humberstone from Section 8.7 is a sufficient condition for MC of intermediate logics as well. Is it a necessary one?

**9.** The disjunction property, as we have seen, is not a characteristic property of *Int*. However, *Int* can be characterized by some properties that are similar to DP.

Kleene [1962] defined the notion $\Gamma \mid_L A$, for any sequence $\Gamma$ of formulas, any formula $A$, and any logic $L$, by the provability relation $\vdash_L$ in $L$:

$\Gamma \mid_L p$ if $\Gamma \vdash_L p$, for any variable $p$;
$\Gamma \mid_L A\&B$ if $\Gamma \mid_L A$ and $\Gamma \mid_L B$;
$\Gamma \mid_L A \vee B$ if $\Gamma \mid\!\vdash_L A$ or $\Gamma \mid\!\vdash_L B$;
$\Gamma \mid_L A \supset B$ if $\Gamma \mid\!\vdash_L A$ implies $\Gamma \mid_L B$;
$\Gamma \mid_L \neg A$ if $\Gamma \mid\!\vdash_L A$ implies $\Gamma \vdash_L A\&\neg A$,

where $\Gamma \mid\!\vdash_L A$ means "$\Gamma \mid_L A$ and $\Gamma \vdash_L A$".

Kleene proved that, for any $A, B, C$, if $A \mid_{Int} A$ and $\vdash_{Int} A \supset B \vee C$ then $\vdash_{Int} A \supset B$ or $\vdash_{Int} A \supset C$ and conjectured that this property is characteristic for *Int*. De Jongh [1968, 1970] confirmed the conjecture, having proved that if $L$ is an intermediate logic for which $A \mid_L A$ and $\vdash_L A \supset B \vee C$ imply $\vdash_L A \supset B$ or $\vdash_L A \supset C$ then $L = Int$. Note also that de Jongh pointed out another characteristic property of *Int*: if $A \mid_L A$ and $\vdash_L (A \supset B)\&(B \supset A)$ then $B \mid_L B$. (We are grateful to Prof. D. H. J. de Jongh for giving us his dissertation.)

One more characterization of *Int* was found by Skura [1989] who proved that *int* is the unique intermediate logic having the following *generalized disjunction property* (GDP): a logic $L$ has GDP if, for any $n \geq 2$ and any formulas $A_1, \ldots, A_n$, $B_1, \ldots, B_n$,

$L \vdash (A_1 \supset B_1)\& \ldots \&(A_n \supset B_n) \supset A_1 \vee \ldots \vee A_n$ implies
$L \vdash (A_1 \supset B_1)\& \ldots \&(A_n \supset B_n) \supset A_i$, for some $i$, $1 \leq i \leq n$.

**10.1.** Nakamura [1983] introduced for first-order logics the notion of *Harrop disjunction property*: a logic $L$ has this property if $L \vdash F \supset A \vee B$ implies $L \vdash F \supset A$ or $L \vdash F \supset B$, where $F$ is a Harrop formula, i.e. every occurrence of $\vee$ and $\exists$ in $F$ is either in the scope of a $\neg$ or in the antecedent of a $\supset$. Minari and Wroński [1988] proved that, for any intermediate logic $L$ and any Harrop formula $A$, if $L \vdash A \supset B \vee C$ then $L \vdash (A \supset B) \vee (A \supset C)$. It follows immediately that in the propositional case the Harrop disjunction property is equivalent to DP.

Minari and Wroński asked if the property "for any $L$, $B$, and $C$, if $L \vdash A \supset B \vee C$ then $L \vdash (A \supset B) \vee (A \supset C)$" implies that $A$ is equivalent in *Int* to some Harrop formula.

**10.2.** Komori [1978] noted that every intermediate logic $L$ has the property that, for any formulas $A$ and $B$ having no variables in common, $L \vdash A \supset B$ implies $L \vdash \neg A$ or $L \vdash B$. Suzuki [1990] considered this and other similar properties for intermediate predicate logics and showed that they are not so trivial as in the propositional case.

**10.3.** One more property related to DP naturally arises from an attempt to characterize the intermediate logics which have extensions with DP. One might suggest that a logic $L$ does not have such an extension (i.e. does not belong to the class ADP; see Section 6.0) if there are formulas $A$ and $B$ such that $L \vdash A \vee B$, but both $A$ and $B$ are classically invalid, that is, by the Glivenko Theorem, neither $L \vdash \neg\neg A$ nor $L \vdash \neg\neg B$. We say $L$ has DP* if, for any $A$ and $B$, $L \vdash A \vee B$ implies $L \vdash \neg\neg A$ or $L \vdash \neg\neg B$.

It is easy to prove that an intermediate logic $L$ has DP* iff not $L \vdash \neg p \vee \neg\neg p$ iff $L$ is contained in the logic $LV$ of the frame shown in Fig. 10. (Note that

$$LV = Int + p \vee (p \supset q \vee \neg q) + (p \supset q) \vee (q \supset p) \vee ((p \supset \neg q)\&(q \supset \neg p)).$$

It is one of the seven intermediate logics having the interpolation property; see Maksimova [1977].)

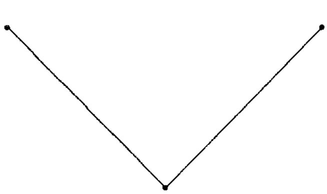

Figure 10

Thus, we obtain the following picture:

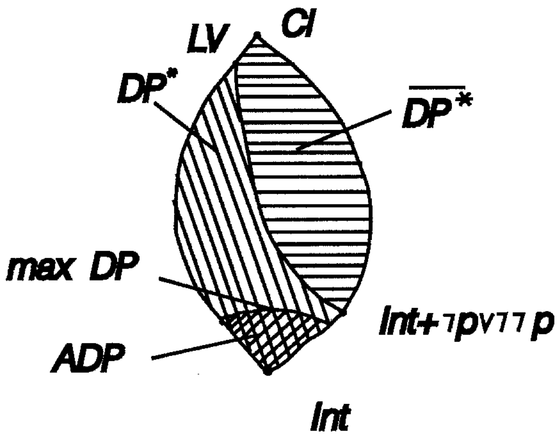


Fig. 11

The cardinality of all classes depicted in Fig. 11 is of continuum. The property DP* is readily decidable.

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