Linear switching systems
with slow growth of trajectories *

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Abstract

We prove the existence of positive linear switching systems (continuous time), whose trajectories grow to infinity, but slower than a given increasing function. This implies that, unlike the situation with linear ODE, the maximal growth of trajectories of linear systems may be arbitrarily slow. For systems generated by a finite set of matrices, this phenomenon is proved to be impossible in dimension 2, while in all bigger dimensions the sublinear growth may occur. The corresponding examples are provided and several open problems are formulated.

Keywords: dynamical systems, linear switching systems, growth of trajectories, stability, resonance

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1. Introduction

For a linear ordinary differential equation (ODE) with constant coefficients \( \dot{x}(t) = Ax(t) \), \( t \in \mathbb{R}_+ \), where \( x(t) \in \mathbb{R}^d \) and \( A \) is a real \( d \times d \) matrix, the fastest growth of trajectories is exponential with the parameter \( \sigma = \sigma(A) \) equal to the spectral abscissa, i.e., the biggest real part of eigenvalues of \( A \). In particular, the system is stable, i.e., all its trajectories converge to zero as \( t \to \infty \), precisely when \( \sigma < 0 \). In case \( \sigma = 0 \), the system is bounded, i.e., has bounded trajectories, apart from the case of resonance, when there are nontrivial Jordan blocks of eigenvalues with zero real part. In that case, the fastest growth is always polynomial with integer degree: \( \|x(t)\| \asymp t^{\ell-1} \), \( t \to \infty \), where \( \ell \) is the largest size of those Jordan blocks called the resonance degree of the system. In particular, every system is either bounded or has at least linear growth.

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The same situation occurs for discrete systems \( x(t+1) = Ax(t), \) \( t \in \mathbb{N} \cup \{0\} \). If the spectral radius \( \rho(A) \) is equal to one, then the trajectories are unbounded if and only if there are nontrivial Jordan blocks corresponding to the largest by modulus eigenvalues. The fastest growth is again \( t^{\ell-1} \), where \( \ell \) is the largest size of those blocks.

The resonance phenomenon have countless applications. Its analysis becomes much more difficult, when the matrix \( A \) may depend on time and take values from a given compact control set of matrices \( \mathcal{A} \). In this case, we obtain a dynamical system of the form

\[
\begin{cases}
\dot{x}(t) = A(t) x(t); \\
x(0) = x_0,
\end{cases}
\]

(1)

where \( A(\cdot) : [0,+\infty) \to \mathcal{A} \), is a measurable function called the switching law. This is a continuous linear switching system (LSS). A solution \( x : \mathbb{R}_+ \to \mathbb{R}^d \) of this system is called its trajectory corresponding to that switching law and to the initial condition \( x(0) = x_0 \). For a single-matrix set \( \mathcal{A} = \{A\} \), the LSS becomes a usual linear ODE. There is an extensive bibliography on the theory of LSS and many applications in control, dynamical systems, engineering, economics, biology, etc., see [1, 3, 4, 7, 12, 13, 16] and references therein.

The system is stable if \( x(t) \to 0 \) as \( t \to \infty \) for every switching law \( A(\cdot) \). Thus, the stability only depends on the compact family \( \mathcal{A} \). Denote \( F(t) = F_A(t) = \sup\{\|x(t)\| : \|x_0\| = 1\} \), where the supremum is computed over all switching laws \( A(\cdot) \). The system is stable precisely when \( F(t) \to 0 \) as \( t \to \infty \). The Lyapunov exponent \( \sigma(\mathcal{A}) \) is defined as

\[
\sigma(\mathcal{A}) = \limsup_{t \to \infty} \frac{1}{t} \ln F(t).
\]

For a single-matrix set \( \mathcal{A} \), this becomes the spectral abscissa. The system is stable if and only if \( \sigma(\mathcal{A}) < 0 \) [13]. If \( \sigma(\mathcal{A}) > 0 \), then there are unbounded trajectories with an exponential growth as \( t \to \infty \). In the boundary case \( \sigma(\mathcal{A}) = 0 \), the system is never stable, i.e., there is at least one trajectory that does not converge to the origin as \( t \to \infty \) [1]. We focus on the question whether the system is bounded in this case, i.e., all its trajectories are bounded.

**Definition 1** In the case \( \sigma(\mathcal{A}) = 0 \), system (1) is called marginally stable if it is bounded, otherwise it is called marginally unstable.

It is known that the trajectories of marginally unstable systems can grow at most polynomially, and, moreover, \( \|x(t)\| \leq C(1 + t^{d-1}), \) \( t \in \mathbb{R}_+, \) where \( d \) is the dimension. Similarly to the case of one matrix, a generic system with \( \sigma(\mathcal{A}) = 0 \) is bounded. It can be unbounded only if it is reducible, i.e., all matrices from \( \mathcal{A} \), in some common basis, have the block upper triangular form:

\[
A = \begin{pmatrix}
A^{(1)} & * & \cdots & * \\
0 & A^{(2)} & * & \vdots \\
& \vdots & \ddots & * \\
0 & \cdots & 0 & A^{(n)}
\end{pmatrix},
\]

(2)

with irreducible families \( \mathcal{A}^{(i)} = \{A^{(i)} : A \in \mathcal{A}\} \) in all the diagonal blocks, \( i = 1, \ldots, n \). It is known that \( \sigma(\mathcal{A}) = \max_{i=1,\ldots,n} \sigma(\mathcal{A}^{(i)}) \) [1]. The number \( r \) of blocks for which this maximum is attained is called the valency of the system. It was shown in [4] that \( F(t) \leq C(1 + t^{r-1}) \).
Similarly to (1), a discrete time switching system is the following difference equation:

\[
\begin{aligned}
\left\{
\begin{array}{l}
x(t+1) = A(t) x(t), \quad t \in \mathbb{N} \cup \{0\}; \\
x(0) = x_0,
\end{array}
\right.
\end{aligned}
\]

where the switching law \( A(t) \) is a sequence of elements from \( \mathcal{A} \). The notions of trajectory, stability, boundedness, the growth \( F(t) \), etc., are directly extended to discrete systems. A similar estimate of growth \( F(t) \leq C (1 + t^{r-1}) \) takes place for discrete systems [14].

1.1. Our results

There is a lot of similarity between linear ODE and general linear switching systems. In case \( \sigma = 0 \), the system is “typically” bounded. The existence of unbounded trajectories requires reducibility of all matrices to the form (2) and coincidence of Lyapunov exponents of several blocks (an analogue of resonance). The growth of trajectories in this case is at most polynomial with degree bounded above by the total number of blocks with the largest Lyapunov exponent (an analogue of the resonance degree). See [4] for sharpening those results, which revealed even more similarity with the single-matrix case. Our main problem is whether the growth is at least polynomial? We focus on continuous LSS and address the following questions:

Is the maximal growth of trajectories of a marginally unstable LSS always polynomial with integer degree? In particular, whether an unbounded system has at least linear growth, as in the single-matrix case? Can the function \( F(t) \) grow slower than polynomially, say, logarithmically?

We shall see that the answers depend on whether the family \( \mathcal{A} \) is finite (i.e., consists of finitely many matrices) or infinite. The finite case turns out to be more difficult and interesting. Note that the system with a control set \( \mathcal{A} \) has the same growth of trajectories as that with \( \text{co}(\mathcal{A}) \), where \( \text{co}(\cdot) \) is the convex hull [13]. Hence, the case of finite \( \mathcal{A} \) is essentially the same as the case of a polytope set \( \mathcal{A} \).

In Theorem 2, we show the existence of systems with arbitrarily slow growth. For every positive increasing to infinity function \( f(t) \), there is an unbounded LSS for which \( F(t) \) grows slower than \( f(t) \) as \( t \to \infty \). Such systems exist in all dimensions \( d \geq 2 \) and can be constructed positive (i.e., all their matrices are Metzler). However, in case \( d = 2 \), this phenomenon never occurs for finite systems. By Theorem 1 proved in Section 2, if the set of \( 2 \times 2 \) matrices \( \mathcal{A} \) is finite (polytope) or infinite but not containing zero, then a marginally unstable system always has linear growth: \( F(t) \asymp t \) as \( t \to \infty \). Already those results demonstrate a crucial difference between finite (polytope) and general cases. In view of Theorems 1 and 2, the slow growth phenomenon emerges because of matrices of small norms in \( \mathcal{A} \). A question arises if the slow growth possible for finite families in higher dimensions? Theorem 3 gives an affirmative answer. It provides an example of two matrices that generate an LSS in \( \mathbb{R}^3 \) with the maximal growth close to \( \sqrt{t} \). Thus, starting with the dimension 3, a sublinear growth of \( F(t) \) is possible even for finite families. The proof of Theorem 3 is surprisingly difficult and required some special technique (Section 5).
Theorems 1–3 answer two open questions formulated in [15]. Their possible generalizations (for a slower growth or for positive systems) are left as open problems in Section 6. To prove Theorem 1 we derive a criterion of marginal stability for two-dimensional finite systems (Proposition 1). An open problem is formulated in Section 6 on extensions of that result to higher dimensions.

1.2. Related works and known results

Resonance and marginal instability of linear switching systems have been analyzed in the literature in various contexts. In the study of wavelets, refinement functional equations, and affine fractal curves, marginal stability is responsible for Lipschitz continuity and for boundedness of variation of solutions [6, 8, 14]. It is important for trackability of autonomous agents in sensor networks [5], in classifications of finite semigroups of integer matrices [11], in the problem of asymptotic growth of some regular sequences [2], in the stability analysis of LSS [4, 16, 15], etc.

The maximal rate of growth of marginally unstable systems were estimated in succession in several works (see [4] for the discussion and references). Those results give only the upper bounds of the polynomial growth and necessary conditions for marginal instability. Criteria of marginal instability are known only in some favorable cases [4, 15, 16]. For discrete systems, possible rates of growth of trajectories were found in several special cases. It was proved to be polynomial with integer degree in case of integer nonnegative matrices [11], and then extended to all integer matrices [2]. The first examples of discrete systems with sublinear growth were presented in [10] for general LSS and in [15] for finite ones. For continuous LSS those constructions are not applicable and the answer was unknown [15, open problem 3].

1.3. Notation

In the sequel we consider only continuous LSS. We identify a system with its control set $A$ generating it. A matrix is called Metzler if all its off-diagonal elements are nonnegative. A system is positive if all matrices from $A$ are Metzler. See [7, 12] for results on positive systems.

We use the standard notation $g(t) = o(f(t))$ and $g(t) = O(f(t))$ as $t \to \infty$ meaning $\lim_{t \to \infty} \frac{|g(t)|}{|f(t)|} = 0$ and $\lim \sup_{t \to \infty} \frac{|g(t)|}{|f(t)|} < \infty$ respectively. We say that a system grows slower (not faster) than a positive function $f(t)$ if $F(t) = o(f(t))$ (respectively, $F(t) = O(f(t))$) as $t \to \infty$. Two values are asymptotically equivalent ($f(t) \approx F(t)$) if they grow not faster each other. We use bold letters to denote vectors, $\| \cdot \|$ is a Euclidean norm.

2. Two-dimensional finite systems:
the marginal instability means linear growth

We begin our analysis with two-dimensional LSS. In this case, any unbounded system has at least linear growth, provided the family $A$ does not contain a sequence that tends to zero.
Theorem 1 Suppose a family of $2 \times 2$ matrices is compact and does not contain zero matrix; then it is marginally unstable if and only if it has linear growth as $t \to \infty$.

Thus, under the assumptions of Theorem 1, the situation is the same as in the single-matrix case. Those assumptions, however, are essential and cannot be omitted. In the next section we shall see that if $\mathcal{A}$ contains a vanishing sequence, then the system may have arbitrarily slow growth.

Note that Theorem 1 holds for every finite family $\mathcal{A}$ of $2 \times 2$ matrices, even if $\mathcal{A}$ contains zero matrix. Indeed, removing zero matrix we do not change the maximal growth of trajectories. This argument, however, does not work for arbitrary compact families, because removal the zero matrix destroys compactness.

Theorem 1 follows from the next proposition that gives a criterion of marginal instability for two-dimensional systems.

Proposition 1 Let a compact family $\mathcal{A}$ of $2 \times 2$ matrices do not contain zero. Then it is marginally unstable if and only if there is a basis in $\mathbb{R}^2$ in which every matrix $A_i \in \mathcal{A}$ has the form

$$A_i = \begin{pmatrix} -a_i & s_i \\ 0 & b_i \end{pmatrix}, \quad i \in \mathcal{I},$$

with $a_i, b_i \geq 0$, and for at least one matrix from $\mathcal{A}$, we have $a_i = b_i = 0, s_i \neq 0$.

Thus, for $d = 2$, it is possible to decide marginal stability by an efficient criterion.

Proof of Theorem 1. The growth of marginally unstable system does not exceed $t^{r-1}$, where $r$ is its valency [4], hence in two-dimensional case, it does not exceed $t$. On the other hand, by Proposition 1, the set $\mathcal{A}$ contains a matrix $A_j$ with $a_j = b_j = 0, s_j \neq 0$. For the switching law $A(t) \equiv A_j$ the trajectories have linear growth.

Proof of Proposition 1. Sufficiency is obvious: if all matrices from $\mathcal{A}$ have the form (4), then $\sigma(\mathcal{A}) \leq 0$, and since there is a matrix $A_j$ with $a_j = b_j = 0, s_j \neq 0$, we see that $\sigma(\mathcal{A}) \geq \sigma(A_j) = 0$ and $\|e^{tA_i}\| \geq Ct, t \geq 1$, hence $\mathcal{A}$ is marginally unstable.

To prove the necessity we recall that a marginally unstable system must be reducible, which, in case of $2 \times 2$ matrices, means the form (4) with $a_i, b_i \geq 0$, because $\sigma(\mathcal{A}) \leq 0$. It remains to show the existence of a matrix with $a_i = b_i = 0, s_i \neq 0$. We assume the contrary and prove that the system is bounded in that case. The contrary means that there exists $\varepsilon > 0$ such that max $\{a_i, b_i\} > \varepsilon, A_i \in \mathcal{A}$. It suffices to prove the boundedness of the system after replacing all $s_i$ by $|s_i|$. Thus, we assume all matrices from $\mathcal{A}$ to have $s_i \geq 0$, i.e., to be Metzler, in which case the system is positive.

Let $\mathcal{A}_0 = \{ A_i \in \mathcal{A} \mid a_i > \varepsilon \}, \mathcal{A}_1 = \{ A_i \in \mathcal{A} \mid b_i > \varepsilon \}$. We have $\mathcal{A}_0 \cup \mathcal{A}_1 = \mathcal{A}$. Define two numbers:

$$p = \max_{A_i \in \mathcal{A}_0} \frac{s_i}{a_i}, \quad k = \min_{A_i \in \mathcal{A}_1} \frac{b_i}{s_i}.$$  (5)

On the plane $OXY$, we take a point $M = (p, 1)$ and draw two lines through it: a line $MP$ parallel to $OX$ ($P \in OY$) and a line $MN$ ($N \in OX$) such that $\tan \angle MNO = k$. We obtain
a rectangular trapezoid $MPON$. Let us show that for every point $x \in MN \cup MP$ and for each $A_i \in \mathcal{A}$, the vector $A_ix$ starting at $x$ is directed inside the trapezoid. Since the system is positive, this property implies stability [9, 13].

If a point $x = (x, y)$ is on the side $MP$, but does not coincide with its ends, then the vector $A_ix$ starting at $x$ is directed inside the trapezoid, because its second coordinate is negative: $(A_ix)_2 = -b_iy = -b_i \leq 0$. If $x \in MN$, then this vector is also directed inside the trapezoid, because it forms an angle $\alpha$ with the $OX$ axis bigger than $\angle MNO$. Indeed, $\tan \alpha = \frac{b_i y}{y(s_i + \frac{a_i}{k}) - a_ip - \frac{a_i}{k}}$. Expressing $x$ by the equation of the line $MN$: $x = p + \frac{1 - y}{k}$, we obtain

$$
\tan \alpha = \frac{b_i y}{y(s_i + \frac{a_i}{k}) - a_ip - \frac{a_i}{k}}. \tag{6}
$$

If $A_i \in \mathcal{A}_0$, then the tangent is negative and $\alpha \geq \pi/2$, which completes the proof. Indeed, since $y \leq 1$, we have $y(s_i + \frac{a_i}{k}) - a_ip - \frac{a_i}{k} \leq s_i + \frac{a_i}{k} - a_ip - \frac{a_i}{k} = s_i - a_ip \leq 0$. If $A_i \in \mathcal{A}_1$, then the denominator of (6) does not exceed $ys_i$, hence either $\tan \alpha \leq 0$, in which case $\alpha \geq \pi/2$, or $\tan \alpha \geq \frac{b_i}{s_i} \geq k = \tan \angle MNO$, which concludes the proof.

Remark 1 A different way to prove Proposition 1 is by applying Theorem 10 in [4] (necessary conditions for marginal instability).

3. Systems of arbitrarily slow growth

The following theorem answers one of the questions formulated in the Introduction. In contrast to linear ODE, linear switching systems may have arbitrarily slow growth of trajectories. This means that for every increasing unbounded function $f$, there is a system in $\mathbb{R}^d$ for which $F(t) = o(f(t))$. So, there are LSS growing slower than polynomially, logarithmically, etc. Moreover, there exist positive systems of that kind.

**Theorem 2** For every $d \geq 2$ and for an arbitrary increasing to infinity function $f : \mathbb{R}_+ \to \mathbb{R}_+$, there exists an unbounded positive system in $\mathbb{R}^d$ that grows slower than $f(t)$ as $t \to \infty$.

Note that it is sufficient to prove Theorem 2 only for $d = 2$. Then for arbitrary $d \geq 3$, one can consider a family $\mathcal{A}$ of block upper-triangular $d \times d$ matrices with two diagonal blocks of dimensions 2 and $d - 2$ respectively. The two-dimensional family $\mathcal{A}^{(1)}$ in the first diagonal block grows slower than $f(t)$, the second diagonal block contains an arbitrary family $\mathcal{A}^{(2)}$ with $\sigma(\mathcal{A}^{(2)}) < 0$. Then the family $\mathcal{A}$ has the same growth as $\mathcal{A}^{(1)}$ [4]. Further, we need the following two well-known facts:

**Fact 1.** If a function $f(t)$ is (strictly) concave on $\mathbb{R}_+$, then so is the function $\varphi(t) = tf\left(\frac{1}{t}\right)$.

**Fact 2.** For an arbitrary increasing to infinity function $f : \mathbb{R}_+ \to \mathbb{R}_+$, there exists a strictly concave increasing to infinity function that grows slower than $f(t)$ as $\to \infty$. 

6
For a concave function \( f : [1, +\infty) \to \mathbb{R}_+ \) such that \( f(t) = o(t) \) as \( t \to \infty \), we define \( \varphi(u) = uf\left(\frac{u}{2}\right) \), \( u \in (0, 1] \), \( \varphi(0) = 0 \), and consider the following compact family \( \mathcal{A} \) of \( 2 \times 2 \)-matrices:

\[
A_u = \begin{pmatrix} 0 & \varphi(u) \\ 0 & -u \end{pmatrix}, \quad u \in [0, 1].
\] (7)

**Proposition 2** If \( f \) is an increasing to infinity strictly concave function such that \( f(t) = o(t) \) as \( t \to \infty \), then the system defined by family (7) is unbounded and the growth of its trajectories does not exceed \( f(t) \) as \( t \to \infty \).

Note that the set of matrices in Proposition 2 must contain the zero matrix due to Theorem 1.

**Proof.** A trajectory \( \mathbf{x}(t) = (x(t), y(t)) \) corresponding to a switching law \( u(t) \) satisfies

\[
y(t) = -u(t) y(t); \quad \dot{x}(t) = \varphi\left(u(t)\right) y(t).
\]

Therefore,

\[
y(t) = y_0 e^{-\int_0^t u(\tau) d\tau}, \quad x(t) = x_0 + y_0 \int_0^t \varphi(u(\tau)) e^{-\int_0^\tau u(s) ds} d\tau.
\]

It suffices to consider the case \( x_0 = 0, y_0 = 1 \). Clearly \( y(t) \) is bounded above by one, so only \( x(t) \) can be unbounded. For arbitrary \( t > 0 \), we denote

\[
M(t) = \sup_{\int_0^t u(\tau) d\tau = 1} \int_0^t \varphi(u(\tau)) d\tau,
\]

where the supremum is over all measurable functions \( u : [0, t] \to [0, 1] \). Observe first that \( M(t) = f(t) \). Indeed, applying Jensen’s inequality to the concave function \( \varphi \), we obtain

\[
\int_0^t \varphi(u(\tau)) d\tau = \int_0^1 \varphi(u(ts)) t ds = t \int_0^1 \varphi(u(ts)) ds \leq t \varphi\left(\int_0^1 u(ts) ds\right) = t \varphi\left(\frac{1}{t}\right).
\]

Thus, \( \int_0^t \varphi(u(\tau)) d\tau \leq t \varphi\left(\frac{1}{t}\right) = f(t) \) and the equality is attained for \( u(\tau) \equiv \frac{1}{t}, \tau \in [0, t] \).

For an integer \( k \geq 0 \), we define \( t_k \) by the equality \( y(t_k) = e^{-k} \), provided such \( t_k \) exists. Note that \( t_0 = 0 \). For an arbitrary \( t \in [t_k, t_{k+1}] \), we have \( x(t) \leq x(t_k) + e^{-k} M(t-t_k) \). On the other hand, \( M(t-t_k) = f(t-t_k) \leq f(t) \), since \( f \) is an increasing function. Thus,

\[
x(t) \leq x(t_k) + e^{-k} f(t), \quad t \in [t_k, t_{k+1}].
\]

Iterating for all \( k \geq 0 \) and taking into account that \( x(t_0) = x(0) = 0 \), we obtain \( x(t) \leq f(t) \sum_{k=0}^{\infty} e^{-k} = \frac{e^{-1}}{e-1} f(t) \). Thus, for every trajectory, \( \|x(t)\| \leq Cf(t) \). On the other hand, taking \( t \) large enough we obtain the trajectory \( \mathbf{x}(\cdot) \) corresponding to the constant switching law \( u(\tau) \equiv \frac{1}{t}, \tau \in [0, t] \), for which \( x(t) \geq f(t) \). Since \( f(t) \) increases to infinity as \( t \to \infty \), it follows that the system is unbounded. \( \square \)
Proof of Theorem 2. By Fact 2, for an arbitrary increasing to infinity function there is a strictly concave increasing to infinity function with a slower growth. Applying Proposition 2 to that function, we conclude the proof.

4. Finite systems with a sublinear growth

In the proofs in Section 3 we see that the slow growth of trajectories is provided by switching laws $A(t)$ with $\lim \inf_{t \to \infty} \|A(t)\| \to 0$. This is unavoidable in view of Theorem 1. Nevertheless, formally, that theorem does not prohibit the slow growth for finite systems in dimensions $d \geq 3$. Constructing such finite LSS is a challenging problem, because now we do not have infinitesimal matrices in $A$, by which we could build a desired switching law. In this section, we present an example of such a system. It is three-dimensional, consists of two matrices and has $F(t) = O(\sqrt{t})$. Thus, finite systems do not inherit the linear growth property from ODE either.

We consider a three-dimensional system on the vector-function $x = (x, y, z)$. This system is generated by the control set $A = \{A_1, A_2\}$, where

$$
A_1 = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{pmatrix} ; \quad A_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
$$

These matrices have a block lower triangular form with two diagonal blocks. The first $2 \times 2$ block corresponds to the coordinates $x$ and $y$, the second block is one-dimensional, it corresponds to the coordinate $z$. In the second block, the “matrices” (actually, scalars) are both zeros. The matrices of the first block are

$$
\tilde{A}_1 = \begin{pmatrix}
0 & -1 \\
1 & 0 \\
\end{pmatrix} ; \quad \tilde{A}_2 = \begin{pmatrix}
0 & 0 \\
0 & -1 \\
\end{pmatrix}.
$$

They generate the following linear ODE's on the vector $\tilde{x} = (x, y)$:

$$
\begin{cases}
\dot{x} = -y \\
\dot{y} = x
\end{cases} ; \quad \begin{cases}
\dot{x} = 0 \\
\dot{y} = -y
\end{cases}
$$

The first ODE has circular trajectories centered at the origin: its solutions are

$$
(x(t), y(t)) = \left( r \cos(t + \varphi), r \sin(t + \varphi) \right), \quad t \in [0, +\infty),
$$

where $x(0) = r \cos \varphi$, $y(0) = r \sin \varphi$. The second ODE has straight vertical trajectories directed towards the $OX$ axis:

$$
(x(t), y(t)) = \left( x(0), e^{-t} y(0) \right), \quad t \in [0, +\infty).
$$

Note that both type of trajectories have non-increasing in time Euclidean norm of the solution. Thus, $\|\tilde{x}(t)\|$ is a non-increasing function for every switching law $\tilde{A}(t)$, and hence
$\sigma\{A_1, A_2\} = 0$. The second (one-dimensional) block is zero, and its Lyapunov exponent $\sigma$ is obviously zero. Therefore, $\sigma\{A_1, A_2\} = 0$. The third coordinate $z(t)$ behaves as follows. For the first system (corresponding to $A_1$), we have $\dot{z} = \dot{y}$, and therefore, $z(t) = z(0) + (y(t) - y(0))$. For the second system, $z(t) \equiv z(0)$. Thus, the second ODE in (10) does not change $z(t)$ at all, while the first one changes it precisely as it changes $y(t)$.

Consider a piecewise constant switching law $A(t)$ with the switching points $0 = t_0 < t_1 < t_2 < \ldots$. Note that the second system $A_2$ reduces the Euclidean norm of the vector $x(t)$.

So, it suffices to consider switching laws starting with $A_1$:

$$A(t) = \begin{cases} A_1, & t \in [t_{2j}, t_{2j+1}), \quad j \geq 0; \\ A_2, & t \in [t_{2j+1}, t_{2j+2}), \quad j \geq 0. \end{cases}$$

The corresponding trajectory $\tilde{x}(t)$ starting at a point $\tilde{x}(0)$ first goes along an arc of the circle $\{\tilde{x} \in \mathbb{R}^2 \mid \|\tilde{x}\| = \|\tilde{x}(0)\|\}$ to the point $\tilde{x}(t_1)$, then goes towards the abscissa $OX$ parallel to the axis $OY$, to a point $\tilde{x}(t_2)$, then goes along an arc of the the circle $\{\tilde{x} \in \mathbb{R}^2 \mid \|\tilde{x}\| = \|\tilde{x}(t_2)\|\}$ to a point $\tilde{x}(t_3)$, then again goes towards the abscissa to a point $\tilde{x}(t_4)$, etc.

We see that the trajectory consists of alternating arcs of circles centered at the origin and vertical segments.

**Theorem 3** Every trajectory $x(t)$ of the system $\{A_1, A_2\}$ such that $\|x(0)\| = 1$ satisfies

$$\|x(t)\| \leq 2 + \sqrt{\frac{t}{2}}, \text{ for all } t \geq 1.$$  

Moreover,

a) For arbitrary $\varepsilon > 0$ and $T > 0$, there is a trajectory such that $\|x(0)\| = 1$ and $\|x(t)\| > 2 + (1 - \varepsilon) \sqrt{\frac{t}{2}}$, for some $t > T$.

b) There exists a trajectory $x(t)$ such that $\limsup_{t \to \infty} \frac{\ln \|x(t)\|}{\ln t} = \frac{1}{2}$.

Thus, the fastest growth of trajectories of this system is equivalent to $\sqrt{t}$. Clearly, the dimension 3 is not a restriction and this example immediately provides examples in all dimensions $d \geq 3$ in a similar way as we did in Theorem 2. On the other hand, we are aware of neither possibility of any slower growth, nor existence of such examples with positive systems. Some open problems are formulated in Section 5.

In spite of simplicity of matrices (9), Theorem 3 has a highly nontrivial proof, which is put in the next section.

### 5. Proof of Theorem 3

The proof is based on one special inequality. For arbitrary positive vectors $a, b \in \mathbb{R}^n_+$, we denote

$$F = \sum_{k=1}^{n} (a_k - b_k); \quad L = \sum_{k=1}^{n} (a_k^2 - b_k^2); \quad T = \sum_{k=1}^{n} (\ln a_k - \ln b_k).$$  

(11)
Proposition 3 For arbitrary \( n \in \mathbb{N} \) and for arbitrary positive vectors \( a, b \in \mathbb{R}^n_+ \) such that \( a \geq b \), we have
\[
F^2 \leq \frac{1}{2} L T,
\]
and the constant \( \frac{1}{2} \) in this inequality is sharp.

In the proof we need the following simple lemma.

Lemma 1 For arbitrary positive numbers \( x, y \) such that \( x \geq y \), we have
\[
2(x - y) \leq (x + y) \left( \ln x - \ln y \right).
\]

Proof. Denote \( t = x / y \) and rewrite (13) in the form
\[
2 \frac{t - 1}{t + 1} \leq \ln t.
\]
This inequality holds for all \( t \geq 1 \) as established merely by differentiating both parts.

Proof\(^1\) of Proposition 3. Multiplying both sides of (13) by \( x - y \), taking their square roots, and substituting \( x = a_k, y = b_k \) we obtain \( \sqrt{2(a_k - b_k)} \leq \sqrt{a_k^2 - b_k^2} \sqrt{\ln a_k - \ln b_k} \). Summing over \( k = 1, \ldots, n \) and invoking the Cauchy-Schwarz inequality gives \( \sqrt{2} \sum_{k=1}^{n} \sqrt{a_k^2 - b_k^2} \sqrt{\ln a_k - \ln b_k} \leq \sqrt{LT} \). To see that the constant \( \frac{1}{2} \) in inequality (13) is sharp one can choose \( a_k = 1 + \delta, b_k = 1 \) for all \( k \) and take the limit as \( \delta \to 0 \).

We need an extension of inequality (13) to all numbers \( a_k, b_k \), possibly negative. For arbitrary vectors \( a, b \in \mathbb{R}^n \) with nonzero entries, we denote the values \( F \) and \( L \) as in (11) and \( T = \sum_{k=1}^{n} \ln \left| \frac{a_k}{b_k} \right| \).

Corollary 1 For each \( n \in \mathbb{N} \) and for arbitrary vectors \( a, b \in \mathbb{R}^n \) such that \( \frac{a_k}{b_k} \geq 1, k = 1, \ldots, n \), inequality (12) holds.

Proof. For each \( k \), the numbers \( a_k, b_k \), and \( a_k - b_k \) have the same sign. In particular, \( a_k - b_k \) is negative whenever \( a_k \) and \( b_k \) are. Hence, replacing all negative \( a_k \) and \( b_k \) by their moduli, we change the sign of all negative terms in the sum \( F = \sum (a_k - b_k) \). This sum increases, while \( L \) and \( T \) stay the same. Applying now Proposition 3 to the numbers \( |a_k|, |b_k|, k = 1, \ldots, n \), we conclude the proof.

Proof of Theorem 3 (Bound for the rate of growth). We use the following notation: \( r_k = \hat{x}(t_{2k}) \), \( k \geq 0 \), be the radius of the \( k \)th circle of the trajectory, \( a_k = y(t_{2k-1}) \), \( k \geq 1 \) and \( b_k = y(t_{2k}) \), \( k \geq 0 \). Thus, for the second coordinate of the point \( \hat{x}(t_j) \), we have in succession: \( y(t_0) = b_0, y(t_1) = a_1, y(t_2) = b_1, y(t_3) = a_2, \) etc. Note that the points \( \hat{x}(t_{2k-1}) \)

\(^1\)This elegant proof of Proposition 3 was suggested by the anonymous referee. My original proof was much longer.
and \( \hat{x}(t_{2k}) \) lie on one trajectory of the system \( \hat{A}_2 \). Hence, they belong to the same vertical line and are related as \( y(t_{2k}) = y(t_{2k-1})e^{t_{2k-1} - t_{2k}} \). Therefore,
\[
\frac{a_k}{b_k} = e^{t_{2k-1} - t_{2k}} > 1, \quad k \in \mathbb{N}.
\] (14)

Consequently, \( t_{2n} > \sum_{k=1}^{n}(t_{2k} - t_{2k-1}) = \sum_{k=1}^{n} \frac{a_k}{b_k} \). Thus,
\[
\sum_{k=1}^{n} \frac{a_k}{b_k} < t_{2n+1}, \quad k \in \mathbb{N}.
\] (15)

Furthermore, since \( \|\hat{x}(t_{2k-1})\| = r_{k-1} \) and \( \|\hat{x}(t_{2k})\| = r_k \), applying the Pythagoras theorem we obtain \( a_k^2 - b_k^2 = r_{k-1}^2 - r_k^2 \). Taking the sum over all \( k = 1, \ldots, n \), we obtain
\[
\sum_{k=1}^{n} (a_k^2 - b_k^2) = r_0^2 - r_n^2.
\] (16)

Finally, since the coordinate \( z(t) \) changes on the segments \([t_{2k-2}, t_{2k-1}]\) by the values \( y(t_{2k-1}) - y(t_{2k-2}) = a_k - b_{k-1}, \quad k \in \mathbb{N} \), and stays unchanged on the segments \([t_{2k-1}, t_{2k}]\), \( k \in \mathbb{N} \), we see that
\[
z(t_{2n+1}) = z(0) + \sum_{k=1}^{n+1} (a_k - b_{k-1}) = z(0) + a_{n+1} - b_0 + \sum_{k=1}^{n} (a_k - b_k).
\]

Thus,
\[
\sum_{k=1}^{n} (a_k - b_k) = z(t_{2n+1}) - z(0) + b_0 - a_{n+1}, \quad n \in \mathbb{N}.
\] (17)

We are now applying Corollary 1 to the numbers \( \{a_k, b_k\}_{k=1}^{n} \). Combining (15), (16), and (17) yields
\[
T < t_{2n+1}; \quad L = r_0^2 - r_n^2; \quad F = z(t_{2n+1}) - z(0) + b_0 - a_{n+1}.
\]

Since \( F \leq \sqrt{\frac{1}{2}LT} \), we obtain
\[
|z(t_{2n+1})| < \sqrt{(r_0^2 - r_n^2)/2 \sqrt{t_{2n+1}}} + |b_0| + |a_{n+1}| + |z(0)|
\]
\[
< \sqrt{(r_0^2 - r_n^2)/2t_{2n+1}} + r_0 + r_n + |z(0)|.
\]

On the other hand, \( z^2(0) = \|x(0)\|^2 - \|\hat{x}(0)\|^2 = 1 - r_0^2 \) and \( \|\hat{x}(t_{2n+1})\| = r_n \), hence \( \|x(t_{2n+1})\|^2 = r_n^2 + |z(t_{2n+1})|^2 \) and \( |z(0)| = \sqrt{1 - r_0^2} \). Thus,
\[
\|x(t_{2n+1})\|^2 < r_n^2 + \left( \sqrt{r_0^2 - r_n^2 \sqrt{t_{2n+1}}} + r_n + r_0 + \sqrt{1 - r_0^2} \right)^2.
\] (18)
To estimate this value we spot the expression $\sqrt{r_0^2 - r_n^2} \sqrt{\frac{t_{2n+1}}{2}} + r_n$, which by the Cauchy-Schwarz inequality does not exceed $\sqrt{(r_0^2 - r_n^2) + r_n^2} \sqrt{\frac{t_{2n+1}}{2} + 1} = r_0 \sqrt{\frac{t_{2n+1}}{2} + 1}$. Hence, the expression in the brackets in (18) does not exceed $r_0 \left( \sqrt{\frac{t_{2n+1}}{2} + 1} + 1 \right)$. This, by the same Cauchy-Schwarz inequality, is less than or equal to $\left( \sqrt{\frac{t_{2n+1}}{2} + 1} + 1 \right)^{1/2}$. Thus, we have

$$\|x(t_{2n+1})\|^2 < r_n^2 + \left( \sqrt{\frac{t_{2n+1}}{2} + 1} + 1 \right)^2 + 1.$$  

Since $r_n^2 \leq 1$, we easily conclude that the right hand side is smaller than $\left( \sqrt{\frac{t_{2n+1}}{2}} + 2 \right)^2$, whenever $t_{2n+1} \geq 1$. Thus, $\|x(t_{2n+1})\| < \sqrt{\frac{t_{2n+1}}{2}} + 2$ for all $t_{2n+1} \geq 1$.

(Existence of a trajectory growing as $\sqrt{t}$). We start with proving a) by presenting trajectories such that $\|x(t)\| > 2 + (1 - \varepsilon)\sqrt{1/2}$ for arbitrarily small $\varepsilon$ and arbitrarily large $t$. Then we use those trajectories to build a single trajectory with $t \to \infty$.

a). It is more convenient to construct a trajectory not for the pair $\{A_1, A_2\}$, but for its convex hull $A = \text{co} \{A_1, A_2\}$. It is well known [13] that any trajectory generated by the convex hull can be arbitrarily close approximated by a trajectory generated by the original set. We consider a straight horizontal trajectory of the two-dimensional system $A = \text{co} \{A_1, A_2\}$ going from a point $\tilde{x}(t_0) = (d, h)$ to a point $\tilde{x}(t_1) = (c, h)$. Both points are in the unit disc, $d > c$ and $h > 0$. As usual, $t_0 = 0$. We assume $d^2 + h^2 = 1$. Thus, $\|x(t_0)\| = \|\tilde{x}(t_0)\| = 1$, and therefore $z(t_0) = 0$. We are going to show that there is a trajectory $\tilde{x}(t)$ along the segment $[\tilde{x}(t_0), \tilde{x}(t_1)]$ and to find $t_1$ for that trajectory.

To keep the trajectory $\tilde{x}(t)$ on the horizontal line, one must have $\dot{y}(t) = 0$, for all $t \in [t_0, t_1]$. On the other hand, $\dot{y}$ is the second coordinate of the vector $A \tilde{x}$, where $A(t) \in \text{co} \{A_1, A_2\}$. Since $A_1 \tilde{x} = (-y, x)$ and $A_2 \tilde{x} = (0, -y)$, it follows that the convex combination of operators $A_1$ and $A_2$ with coefficients $\frac{x}{x+y}$ and $\frac{-y}{x+y}$ makes the second coordinate of the vector $A \tilde{x}$ zero. Thus,

$$\tilde{A}(t) = \frac{y(t)}{x(t) + y(t)} A_1 + \frac{x(t)}{x(t) + y(t)} A_2.$$  

Using the short notation $x = x(t)$ and taking into account that $y(t) = h$, we have

$$\tilde{A}(t) = \frac{h}{x + h} A_1 + \frac{x}{x + h} A_2.$$  

Therefore, for this trajectory, $\dot{x} = -\frac{h^2}{x + h}$ and $\dot{y} = 0$. Since $\dot{z}$ is equal to the second coordinate of the vector $A_1 \tilde{x}$, we see that

$$\dot{z} = \frac{h}{x + h} (A_1 \tilde{x})_2 = \frac{hx}{x + h}.$$  

12
Hence,
\[ \frac{dz}{dx} = \frac{\dot{z}}{\dot{x}} = \frac{hx}{x+h} - \frac{-h^2}{x+h} = -\frac{x}{h}. \]

Taking into account that \( z(t_0) = 0 \), we obtain
\[ z(t_1) = \int_c^d -\frac{x}{h} \, dx = \frac{d^2 - c^2}{2h}. \]  \hspace{1cm} (19)

On the other hand, since \( t_0 = 0 \), we have
\[ t_1 = \int_c^d \frac{dx}{\dot{x}} = \int_c^d \frac{x+h}{h^2} \, dx = \frac{d^2 - c^2}{2h^2} + \frac{d-c}{h}. \]  \hspace{1cm} (20)

Taking \( h \) small enough, we obtain \( t_1 \to \infty \) and \( \frac{z(t_1)}{\sqrt{h}} \to \frac{\sqrt{d^2-c^2}}{\sqrt{2}} \) as \( h \to 0 \). For \( c = 0 \) and \( d = \sqrt{1-h^2} \), this value becomes close to \( \frac{1}{\sqrt{2}} \).

Thus, the desired example is the trajectory along the horizontal segment from the point \((\sqrt{1-h^2}, h)\) to the point \((0, h)\), where \( h \) is small.

b) We consider the trajectory \( \hat{x}(t) \) along the broken line that consists of horizontal segment \([ (c_k, h_k), (d_k, h_k) ] \) with \( c_k < d_k \), \( c_k = d_{k+1} \), and \( h_{k+1} < h_k \), \( k \in \mathbb{N} \), connected subsequently by the vertical segments \([ (d_{k+1}, h_{k+1}), (c_k, h_k) ] \), \( k \in \mathbb{N} \). Denote by \( t_n \) the time by which the trajectory starts the \((n+1)\)st horizontal segment. Applying (19) and (20), we obtain
\[ z(t_n) = \sum_{k=1}^{n} \frac{d_k^2 - c_k^2}{2h_k^2}; \quad t_n = \sum_{k=1}^{n} \left( \frac{d_k^2 - c_k^2}{2h_k^2} + \frac{d_k - c_k}{h_k} + \ln \frac{h_k}{h_{k+1}} \right) \]
(the term \( \ln \frac{h_k}{h_{k+1}} \) is the time of going along the \( k \)th vertical segment reducing the altitude from \( h_k \) to \( h_{k+1} \)). It remains to choose concrete values of \( c_k, d_k \) and \( t_k \). We take \( c_k = 2^{-k-1} \), \( d_k = 2^{-k} \), and \( h_k = 2^{-k^2} \). After elementary simplifications we get
\[ z(t_n) = \frac{3}{16} \sum_{k=1}^{n} 2^{k-1} = \frac{3}{16} 2^{(n-1)^2} + o \left( 2^{(n-1)^2} \right), \quad n \to \infty. \]
\[ t_n = \sum_{k=1}^{n} \frac{3}{16} 2^{2+k-1} + 2^{k^2-k-1} + (2k+1) \ln 2 = \frac{3}{16} 2^{2+(n-1)^2} + o \left( 2^{2+(n-1)^2} \right), \quad n \to \infty. \]

Clearly, \( \lim_{n \to \infty} \frac{\ln \| \hat{x}(t_n) \|}{\ln t_n} \geq \lim_{n \to \infty} \frac{\ln z(t_n)}{\ln t_n} = \lim_{n \to \infty} \frac{n(\ln 2^{(n-1)^2})}{2^{n^2+(n-1)^2}} = \frac{1}{2} \), which completes the proof.
6. Open problems

In this section we formulate several open problems on the growth of linear switching systems. A progress in any of them would be, in our opinion, interesting and challenging.

**Problems 1. How slow can the growth of a finite system be?**

Theorem 3 gives an example of the growth $\sqrt{t}$ for a system of two $3 \times 3$ matrices. Is a slower growth possible for finite systems? For general compact systems, Theorem 2 gives an affirmative answer: the growth can be arbitrarily slow. However, the construction of such families $\mathcal{A}$ relies on special sequences of matrices from $\mathcal{A}$ that converge to zero, which is impossible if $\mathcal{A}$ is finite. For $d = 2$, the answer is negative: by Theorem 1, any finite system is either bounded or has a linear growth. So, we are interested in the case $d \geq 3$. Note that for discrete finite systems, there are examples of the growth $\sqrt{t}$ (see [15]).

**Problems 2. Are there examples of finite positive systems with a sublinear growth?**

Positive systems, which consist of Metzler matrices, possess many special properties. Their analysis is a separate chapter in the theory of LSS (see, [7] and references therein). The system from Theorem 3 is not positive: the matrix $A_1$ is not Metzler, because it has a negative off-diagonal element. Problem 2 addresses the question of sublinear growth for positive systems. Probably, the study of marginal stability of positive systems requires new ideas and a new technique.

**Problems 3. To find practically checkable sufficient conditions for marginal instability.**

For linear systems with one matrix, the marginal instability is easily checkable by the existence of a corresponding nontrivial Jordan block. For general linear switching systems, necessary and sufficient conditions for the marginal instability were derived in [4]. They are very important in the theoretical study of LSS. In practice, however, it is difficult to apply them for concrete systems, because it requires construction of Barabanov’s norm and checking the resonance condition. For discrete systems, the problem was partially solved in [15], where a practical criterion was presented. That criterion works under some mild assumptions on matrices, which is believed to be generic. For continuous LSS no practical conditions are known thus far.

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**References**


