AN UNDECIDABLE PROBLEM IN CORRESPONDENCE THEORY

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§1. Introduction. In this paper we prove undecidability of first-order definability of propositional formulas. The main result is proved for intuitionistic formulas, but it remains valid for other kinds of propositional formulas by analogous arguments or with the help of various translations.

For general background on correspondence theory the reader is referred to van Bentham [1], [2] (see [3] for a survey of recent results).

The method for the proofs of undecidability in this paper will be to simulate calculations of a Minsky machine by intuitionistic formulas. §3 concerns this simulation. Effective procedures for the construction of simulating modal formulas can be found in the literature (cf. [4]).

The principal results of the paper are in §4. §5 gives some further undecidability results, that will be proved in another paper by modification of the method of this paper.

I am indebted to the referee for drawing my attention to some errors in an earlier version of this paper.

§2. First-order definable intuitionistic formulas. Two examples. Intuitionistic formulas are constructed in the usual way from $p_0, p_1, \ldots$ (sentence letters), $\neg$ (negation), $\supset$ (implication), $\&$ (conjunction) and $\lor$ (disjunction). A Kripke frame for intuitionistic propositional logic (a frame, for short) is a pair $\mathcal{F} = \langle W, \leq \rangle$, where $W$ is a nonempty set (whose elements $x, y, w, v, \ldots$ are called worlds) and $\leq \subseteq W \times W$ is a partial ordering. $\mathfrak{M} = \langle \mathcal{F}, V \rangle$ is a model (based on the frame $\mathcal{F}$) if $\mathcal{F}$ is a frame and $V$ is a function, called a valuation, that associates with each propositional variable $p$ a subset $V(p)$ of $W$ such that if $x \in V(p)$ and $x \leq y$ then $y \in V(p)$. Truth ($\models$) relative to a model $\mathfrak{M}$ is defined by

\[
    x \models p \quad \text{iff} \quad x \in V(p),
\]

\[
    x \models \neg A \quad \text{iff} \quad (\forall y \in W)(x \leq y \Rightarrow \neg(x \models A)),
\]

\[
    x \models A \& B \quad \text{iff} \quad x \models A \text{ and } x \models B,
\]

\[
    x \models A \lor B \quad \text{iff} \quad x \models A \text{ or } x \models B,
\]

\[
    x \models A \supset B \quad \text{iff} \quad (\forall y \in W)(x \leq y, y \models A \Rightarrow y \models B).
\]

Below, for not ($x \not\models A$) we write $x \not\models A$.

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\[ \mathcal{M} \models A \; (A \text{ is true in } \mathcal{M}) \text{ if } (\forall x \in W) \; (x \vdash A). \]
\[ \mathcal{F} \models A \; (A \text{ is valid in } \mathcal{F}) \text{ if } (\forall \mathcal{M} \text{ based on } \mathcal{F}) \; (\mathcal{M} \models A). \] Otherwise we write \[ \mathcal{F} \not\models A. \]

Let \( A \) be an implication \( B \supset C \). Then we write \( x \not\vdash A \) for \( x \vdash B \) and \( x \not\vdash C \). I shall assume some formal system of intuitionistic propositional logic, and write \( A \vdash B \) to mean that there exists a deduction of \( B \) from \( A \).

We say that intuitionistic formulas \( A \) and \( B \) are deductively equivalent iff \( A \vdash B \) and \( B \vdash A \).

An intuitionistic sentence \( A \) is first-order definable iff there is a first-order sentence \( A^* \) (\( A^* \) is a sentence of the first-order language (with equality) of a single binary predicate) such that, for any frame \( \mathcal{F}, \mathcal{F} \models A \) iff \( \mathcal{F} \models A^* \) in the classical first-order sense. If \( A^* \) is an \( \forall \)-formula (\( \forall \exists \)-formula), \( A \) is called \( \forall \)-definable (\( \forall \exists \)-definable). For example, an intuitionistic formula \( p \lor \neg p \) is \( \forall \)-definable (by the formula \( \forall x \forall y (x \leq y \lor x = y) \), and \( \neg p \lor \neg \neg p \) is \( \forall \exists \)-definable (by the formula \( \forall x \forall y \forall z \exists t(x \leq y \land x \leq z \supset y \leq t \land z \leq t) \)), but is not \( \forall \)-definable.

We need two examples of \( \forall \)-definable and \( \forall \exists \)-definable formulas—\( F_1 \) and \( F_2 \).

The formula \( F_1 \) is defined as follows:

\[
\begin{align*}
A_{-3}^1 &= s_8 \supset \bigvee_{i=1}^8 t_i, \\
B_{-3}^1 &= t_8 \supset \bigvee_{i=1}^8 s_i, \\
A_{-2}^1 &= s_7 \supset s_8 \lor A_{-3}^1, \\
B_{-2}^1 &= t_7 \supset t_8 \lor B_{-3}^1, \\
A_{-2}^2 &= s_6 \supset s_7 \lor A_{-1}^2, \\
B_{-2}^2 &= t_5 \supset t_6 \lor B_{-3}^2, \\
A_{-3}^2 &= s_4 \supset s_5 \lor A_{-2}^2, \\
B_{-3}^2 &= t_3 \supset t_4 \lor B_{-3}^3, \\
C_0 &= s_1 \land t_8 \supset s_2 \lor v_1, \\
D_0 &= t_1 \land s_8 \supset s_1 \lor s_2 \lor v_2, \\
C_1 &= s_2 \supset s_3 \lor A_{-2}^2, \\
D_1 &= t_2 \supset t_3 \lor B_{-2}^2, \\
C_2 &= s_1 \supset s_2 \lor C_1 \lor C_0, \\
D_2 &= t_1 \supset t_2 \lor D_1 \lor D_0, \\
C_3 &= D_0 \land (t_8 \supset s_8) \supset C_2 \lor s_1, \\
D_3 &= C_0 \lor (s_8 \supset t_8) \supset D_2 \lor v_1, \\
F_1 &= C_3 \lor D_3.
\end{align*}
\]

For the sake of readability, we may abbreviate a first-order sentence by its English translation with the help of pictures, in quotation marks.

**Lemma 1.** The formula \( F_1 \) is \( \forall \)-definable.

**Proof.** Clearly, for any frame \( \mathcal{F}, \mathcal{F} \not\models F_1 \) iff "there are \( x, y, z, \gamma_2, \gamma_1, \gamma_0, \alpha_{-2}, \alpha_{-3}, \alpha_{-2}, \alpha_{-3}, \alpha_{-1}, \delta_2, \delta_1, \delta_0, \beta_{-2}, \beta_{-3}, \beta_{-2}, \beta_{-3}, \beta_{-1}, \beta_{-1} \) in \( \mathcal{F} \) in the configuration shown in Figure 1". The sentence in quotation marks is an \( \exists \)-formula, and its negation, being an \( \forall \)-formula, is true precisely in those frames in which \( F_1 \) is valid.

The formula \( F_2 \) is obtained from \( F_1 \) by replacing the formulas \( A_{-1}^1 \) and \( B_{-1}^1 \) by the formulas \( s_8 \supset \bigvee_{i=1}^8 t_i \) and \( t_8 \supset \bigvee_{i=1}^8 s_i \), respectively.

**Lemma 2.** The formula \( F_2 \) is \( \forall \exists \)-definable, but not \( \forall \)-definable.

**Proof.** Clearly, for any frame \( \mathcal{F}, \mathcal{F} \not\models F_2 \) iff "there are \( x, y, z, \gamma_2, \gamma_1, \gamma_0, \alpha_{-2}, \alpha_{-3}, \alpha_{-2}, \alpha_{-3}, \alpha_{-1}, \delta_2, \delta_1, \delta_0, \beta_{-2}, \beta_{-3}, \beta_{-2}, \beta_{-3}, \beta_{-1}, \beta_{-1} \) in the configuration of Figure 1 such that \( \alpha_{-1}^2 \) and \( \beta_{-1}^2 \) have no successors in common". The sentence in quotation marks is an \( \exists \forall \)-formula, and its negation, being an \( \forall \exists \)-formula, is true precisely in those frames in which \( F_2 \) is valid.

To show that \( F_2 \) is not \( \forall \)-definable, let \( \mathcal{F} \) be the frame in Figure 1. Denote by \( \mathcal{F}" \)
the frame obtained from $\mathcal{F}'$ by adding a world $\gamma$ accessible from the worlds $\alpha_{1-3}$ and $\beta_{1-3}$. Clearly, $\mathcal{F}'' = F_2$ and $\mathcal{F}' \neq F_2$. But $\mathcal{F}'$ is a submodel of $\mathcal{F}''$ in the sense of classical model theory. Hence $F_2$ is not $\forall$-definable, by a well-known criterion.

It is clear that if $A$ and $B$ are deductively equivalent, then they are valid in the same frames, and so we obtain Lemma 3 by Lemmas 1 and 2.

**Lemma 3.** a) If an intutionistic formula is deductively equivalent to $F_1$, then it is $\forall$-definable.

b) If an intutionistic formula is deductively equivalent to $F_2$, then it is $\forall\exists$-definable, but not $\forall$-definable.

**Remark.** Lemmas 1 and 2 immediately follow from [6] or [7]. In [6] an algorithm is described which, given an intuitionistic formula without negative occurrences of disjunction ($F_2$ is such), constructs a first-order $\forall\exists$-equivalent, and which moreover is an $\forall$-formula if the given intuitionistic formula is $\forall$-definable. In particular, any formula without negative occurrences of disjunction and without occurrences of negation (such as $F_1$) is $\forall$-definable.

§3. Minsky machines and their simulation by intuitionistic formulas. A Minsky machine is a two-tape machine operating on two integers $s_1$ and $s_2$. A Minsky machine program is a finite set of instructions $I$ of the forms:

1. $q_x \rightarrow q_\beta T_1 T_0$ — in state $q_x$, add 1 to $s_1$, go to $q_\beta$;
2. $q_x \rightarrow q_\beta T_0 T_1$ — in state $q_x$, add 1 to $s_2$, go to $q_\beta$;
3. $q_x \rightarrow q_\beta T_{-1} T_0 (q_\gamma T_0 T_0)$ — in state $q_x$, subtract 1 from $s_1$, if $s_1 \neq 0$, and go to $q_\beta$, otherwise go to $q_\gamma$;
(4) \( q_a \rightarrow q_{\beta}T_0T_{-1}(q_{\gamma}T_0T_0) \) — in state \( q_a \), subtract 1 from \( s_2 \), if \( s_2 \neq 0 \), and go to \( q_{\beta} \), otherwise go to \( q_{\gamma} \).

A Minsky machine configuration is an ordered triple \((i, j, k)\) of natural numbers, where \( i \) is a state number, \( j = s_1 \), and \( k = s_2 \).

We write \( P: (\alpha, m, n) \rightarrow (\beta, k, l) \) if the program \( P \), starting at configuration \((\alpha, m, n)\), can reach a configuration \((\beta, k, l)\), otherwise we write \( P: (\alpha, m, n) \leftrightarrow (\beta, k, l) \). Define

\[ P(i, j, k) = \{(l, m, n) \mid P: (i, j, k) \rightarrow (l, m, n)\}. \]

The basic unsolvable problem to be used is to recognize, given a Minsky machine \( P \) and two configurations \((\alpha, m, n)\) and \((\beta, k, l)\), whether \((\beta, k, l) \in P(\alpha, m, n) [15]\).

We introduce the following formulas:

\[ A_{n+1}^{1} = B_{n}^{1} \supseteq A_{n}^{1} \lor B_{n-1}^{1}, \quad B_{n+1}^{1} = A_{n}^{1} \supseteq B_{n}^{1} \lor A_{n-1}^{1} \quad (n \geq -2), \]
\[ Q_{-2} = r, \quad Q'_{-2} = s, \quad Q_{-1} = p, \quad Q'_{-1} = q. \]
\[ Q_{n+1} = A_{3}^{3} \land B_{3}^{3} \land Q' \supseteq Q_{n} \lor Q'_{n-1} \lor A_{2}^{3} \lor B_{2}^{3}, \]
\[ Q'_{n+1} = A_{3}^{3} \land B_{3}^{3} \land Q \supseteq Q_{n} \lor Q_{n-1} \lor A_{2}^{3} \lor B_{2}^{3}, \quad (n \geq -1), \]
\[ A_{n+1}^{2} = A_{3}^{3} \land B_{3}^{3} \land Q \supseteq A_{n}^{2} \lor B_{n+1}^{2} \lor A_{3}^{3} \lor B_{3}^{3}, \]
\[ B_{n+1}^{2} = A_{3}^{3} \land B_{3}^{3} \land A \supseteq B_{n+1}^{2} \lor A_{3}^{3} \lor B_{3}^{3}, \quad (n \geq -2), \]
\[ R_{-2} = r', \quad R'_{-2} = s', \quad R_{-1} = p', \quad R'_{-1} = q'. \]
\[ R_{n+1} = C_{1} \land D_{1} \land R_{n} \supseteq R_{n} \lor R'_{n-1} \lor A_{3}^{3} \lor B_{3}^{3}, \]
\[ R'_{n+1} = C_{1} \land D_{1} \land R_{n} \supseteq R'_{n} \lor R_{n-1} \lor A_{3}^{3} \lor B_{3}^{3}, \quad (n \geq -1), \]
\[ A_{n+1}^{3} = C_{1} \land D_{1} \land B_{3}^{3} \supseteq A_{n}^{3} \lor B_{n+1}^{3} \lor A_{3}^{3} \lor B_{3}^{3}, \]
\[ B_{n+1}^{3} = C_{1} \land D_{1} \land A \supseteq B_{n+1}^{3} \lor A_{3}^{3} \lor B_{3}^{3}, \quad (n \geq -2), \]
\[ T(n, Q_{1}, R_{1}) = A_{n+1}^{1} \land B_{n+1}^{1} \land Q_{i+1} \land Q'_{i+1} \land R_{j+1} \land R'_{j+1} \]
\[ \supseteq A_{n}^{1} \lor B_{n}^{1} \lor Q_{i} \lor Q'_{i} \lor R_{j} \lor R'_{j} \quad (n, i, j \geq 0). \]

The content of these formulas is contained in the fact that \( x \not \models A_{1} \land A_{2} \land \cdots \land A_{n} \supseteq B_{1} \lor B_{2} \lor \cdots \lor B_{m} \) if \( x \models A_{1}, x \models A_{2}, \ldots, x \models A_{n} \) and \( x \not \models B_{1}, x \not \models B_{2}, \ldots, x \not \models B_{m}, \) i.e. not \((x \models y_{i})\), for any \( y_{i} \) such that \( y_{i} \not \models A_{i} (1 \leq i \leq n) \), and \( x \leq z_{j} \), for some \( z_{j} \) such that \( z_{j} \not \models B_{j} (1 \leq j \leq m) \).

Because the formulas \( A_{1}^{3}, B_{1}^{3}, A_{i}^{3}, B_{1}^{3} (i \geq 0) \) are obtained from the formulas \( Q_{1}, Q'_{1}, R_{1}, R'_{1} \) by replacing \( r, s, p, q \) by the formulas \( A_{i-3}^{3}, B_{i-3}^{3}, A_{i-2}^{3}, B_{i-2}^{3} \) and \( r', s', p', q' \) by the formulas \( A_{i-3}^{3}, B_{i-3}^{3}, A_{i-2}^{3}, B_{i-2}^{3} \) respectively, we write

\[ T(n, A_{1}^{3}, \Phi_{1}) \quad \text{for} \quad T(n, Q_{1}, \Phi_{1})(A_{i-3}^{3}/r, B_{i-3}^{3}/s, A_{i-2}^{3}/p, B_{i-2}^{3}/q), \]
\[ T(n, A_{1}^{3}, \Phi_{2}) \quad \text{for} \quad T(n, Q_{1}, \Phi_{2})(A_{i-3}^{3}/r', B_{i-3}^{3}/s', A_{i-2}^{3}/p', B_{i-2}^{3}/q'), \]

where \( \Phi_{1} \) does not contain \( r, s, p, q \), and \( \Phi_{2} \) does not contain \( r', s', p', q' \).

We are now in a position to define the set of formulas \( AxI_{j} \) which corresponds to the instruction set of the program \( P \). For each instruction \( I_{j} \) of \( P \) the formula \( AxI_{j} \) is defined as follows:

(1) If \( I_{j} \) is of form (1), then \( AxI_{j} \) will be the formula

\[ AxI_{j} = T(\beta, Q_{2}, R_{1}) \supseteq T(\alpha, Q_{1}, R_{1}) \lor F. \]
(2) If $I_j$ is of form (2), then $AxI_j$ will be the formula

$$AxI_j = T(\beta, Q_1, R_2) \Rightarrow T(x, Q_1, R_1) \vee F.$$  

(3) If $I_j$ is of form (3), then $AxI_j$ will be the formula

$$AxI_j = (T(\beta, Q_1, R_1) \Rightarrow T(x, Q_2, R_1) \vee F) \& (T(\gamma, A_0^2, R_1) \Rightarrow T(x, A_0^2, R_1) \vee F).$$

(4) If $I_j$ is of form (4), then $AxI_j$ will be the formula

$$AxI_j = (T(\beta, Q_1, R_1) \Rightarrow T(x, Q_1, R_2) \vee F) \& (T(\gamma, Q_1, A_0^3) \Rightarrow T(x, Q_1, A_0^3) \vee F).$$

Here $F$ is either $F_1$ or $F_2$. The difference between $F_1$ and $F_2$ is almost never essential; when we do need a particular one we will say so explicitly.

Define the axiom $AxP$ as follows:

$$AxP = \bigwedge_{i \in P} AxI_i.$$  

**Lemma 4.** If $(x, y, z) \in P(i, j, k)$, then

$$AxP \vdash T(x, A^2_y, A^3_z) \Rightarrow T(i, A^2_j, A^3_k) \vee F.$$  

**Proof.** We proceed by induction on the number of steps from $(i, j, k)$ to $(x, y, z)$. When this number is 0, it is obvious that $T(x, A^2_y, A^3_z) \Rightarrow T(i, A^2_j, A^3_k) \vee F$ is provable.

Now suppose that the lemma holds for computations $r$ steps long, and let $(\xi, \eta, \zeta)$ be the configuration after the first $r$ steps of an $(r + 1)$-step computation from $(i, j, k)$. By the induction hypothesis, $T(\xi, A^2_\eta, A^3_\zeta) \Rightarrow T(i, A^2_j, A^3_k) \vee F$ is provable. Now the next step of the computation is to apply $I_i$. We shall treat the case when $I_i$ is of the form: "in $q_t$, subtract 1 from $s_1$, if $s_1 \neq 0$, and go to $q_x$, otherwise go to $q_y$". The other cases are similar.

First consider the possibility that $\eta = 0$. Then, at the $(r + 1)$st step, after the application of the instruction, the configuration will be $(\gamma, 0, \zeta)$, so we must show that $T(\gamma, A^2_0, A^3_\zeta) \Rightarrow T(i, A^2_j, A^3_k) \vee F$ is provable. But the formula $AxI_i$ corresponding to $I_i$ contains a conjunct

$$T(\gamma, A^2_0, R_1) \Rightarrow T(x, A^2_0, R_1) \vee F,$$

by substitution. Taken with

$$T(\gamma, A^2_0, A^3_\zeta) \Rightarrow T(x, A^2_0, A^3_\zeta) \vee F$$

this leads to the desired result.

If $\eta \neq 0$, then at the $(r + 1)$st step, after the application of the instruction, the configuration will be $(x, \eta - 1, \zeta)$, so we must show that $T(x, A^2_\eta, A^3_\zeta) \Rightarrow T(x, A^2_{\eta + 1}, A^3_\zeta) \vee F$ is provable. Now we can use the conjunct $T(x, Q_1, R_1) \Rightarrow T(x, Q_2, R_1) \vee F$ of $AxI_i$ and proceed as above.

**Remark.** Lemmas 10 and 11 imply the converse of Lemma 4. Thus an undecidable calculus $Int + AxP$ can be constructed by an appropriate choice of a program $P$.  

§4. Proofs of the principal results. Define the following formulas:

\[ G = ((C_1 \lor D_1 \supset p) \supset (C_2 \land D_2 \supset C_1 \lor D_1)), \]

\[ E = (p \land D_1 \supset s_2 \lor C_1) \land (p \land C_1 \supset t_2 \lor D_1) \supset (C_2 \land D_2 \supset C_1 \lor D_1), \]

\[ H = G \land E \supset F, \]

\[ B(P, (\alpha, m, n), (i, j, k)) = \exists x P \land ((T(\alpha, A_m^2, A_n^3) \supset T(i, A_j^2, A_k^3) \lor F) \supset F) \land (F \lor H). \]

**Lemma 5.** If \((\alpha, m, n) \in P(i, j, k)\), then \(B(P, (\alpha, m, n), (i, j, k))\) is first-order definable. Beside, if \(F\) is \(F_1\), then \(B(P, (\alpha, m, n), (i, j, k))\) is \(\forall\)-definable, and if \(F\) is \(F_2\), then \(B(P, (\alpha, m, n), (i, j, k))\) is \(\forall\exists\)-definable but not \(\forall\)-definable.

**Proof.** We use Lemma 3. It is enough for us to show that if \((\alpha, m, n) \in P(i, j, k)\), then \(B(P, (\alpha, m, n), (i, j, k))\) is deductively equivalent to \(F\).

By Lemma 4 we obtain that

\[ B(P, (\alpha, m, n), (i, j, k)) \vdash T(\alpha, A_m^2, A_n^3) \supset T(i, A_j^2, A_k^3) \lor F, \]

and using the second conjunct of \(B(P, (\alpha, m, n), (i, j, k))\) and modus ponens we have

\[ B(P, (\alpha, m, n), (i, j, k)) \vdash F. \]

Now we note that all conjuncts of \(B(P, (\alpha, m, n), (i, j, k))\) have one of the forms

\[ A \lor F \text{ or } A \supset B \lor F \text{ or } A \supset F, \]

and so \(F \vdash B(P, (\alpha, m, n), (i, j, k)).\)

**Lemma 6.** If \((\alpha, m, n) \notin P(i, j, k)\), then \((B(P, (\alpha, m, n), (i, j, k))\) is not first-order definable.

**Proof.** Let \((\alpha, m, n) \notin P(i, j, k)\). We shall write \(B\) instead of \(B(P, (\alpha, m, n), (i, j, k))\). For the proof we construct an uncountable elementary subframe \(\mathcal{F}\) in which \(B\) is valid, such that in some countable elementary subframe \(\mathcal{F}^\ast\) the formula \(B\) is false (cf. [1]).

Define the frame \(\mathcal{F}\) as follows. The set of elements \(W\) of \(\mathcal{F}\) is to contain:

\[ f, c_1, d_1, c_2, d_2, c_0, d_0, \tau, \sigma, \]

\[ a_m^n, b_m^n \text{ for each } n \in \{1, 2, 3\}, -3 \leq m < \omega, \]

\[ t(p, q, r), \text{ where } (p, q, r) \in P(i, j, k), \]

\[ \alpha_n, \alpha_{n, i} \text{ for each } n \in \omega, i \in \{0, 1\}, \]

\[ \beta_\phi, \text{ where } \phi \in 2^\omega. \]

All of these elements of \(W\) are to be distinct from one another. Now we define the relation \(\leq\) on \(W\) as the least partial ordering containing the following binary relation \(R:\)

\[ xRy \iff x = f \lor (x = a_1^t \land y = a_p^s \land t \geq p) \lor (x = b_1^t \land y = b_p^s \land t \geq p) \]

\[ \lor (x = a_1^t \land y = b_p^s \land p \leq t - 2) \lor (x = b_1^t \land y = a_p^s \land p \leq t - 2) \]

\[ \lor (x = a_2^3 \land y = a_1^{1-2}) \lor (x = a_3^1 \land y = a_2^{2-2}) \lor (x = c_1 \land y = a_2^{3-2}) \]

\[ \lor (x = b_3^3 \land y = b_2^{2-2}) \lor (x = b_3^2 \land y = b_1^{2-2}) \lor (x = d_1 \land y = b_1^{3-2}) \]

\[ \lor (x = c_2 \land y = c_1) \lor (x = d_2 \land y = d_1) \lor (x = c_2 \land y = c_0) \]

\[ \lor (x = d_2 \land y = d_0) \land (x = t(p, q, r) \land \exists (a_p^1, b_p^1, a_q^2, b_q^2, a_r^3, b_r^3)) \lor \]

\[ (x = t(p, q, r) \land \exists (a_p^1, b_p^1, a_q^2, b_q^2, a_r^3, b_r^3)). \]
\[ \forall (x = \tau \& y = c_2) \lor (x = \sigma \& y = d_2) \lor (x = \alpha_n \& y = \alpha_n), \]
\[ \forall (x = \alpha_{n,0} \& y = c_1) \lor (x = \alpha_{n,1} \& y = d_1), \]
\[ \forall (x = \beta_0 \& ((y = \alpha_{n,0} \& n \notin \varphi) \lor (y = \alpha_{n,1} \& n \in \varphi) \lor y \in \{c_1, d_1\}), \]

where each disjunctive member is the disjunction on all possible meanings of undefined indices. The frame \( \mathcal{F} \) is sketched in Figure 2.

**Lemma 7.** For any \( x \in W, x \not\models F_1 \leftrightarrow F_2 \) iff \( x = f \) and either

\[
\begin{align*}
\{ x \mid x \not\models A^1_{-3} \} &= \{ a^1_{-3} \}, & \left\{ x \mid x \not\models s_8 \& \neg t_8 \supset \bigvee_{i=1}^{8} t_i \right\} &= \{ a^1_{-3} \}, \\
\{ x \mid x \not\models B^1_{-3} \} &= \{ b^1_{-3} \}, & \left\{ x \mid x \not\models t_8 \& \neg s_8 \supset \bigvee_{i=1}^{8} s_i \right\} &= \{ b^1_{-3} \}, \\
\{ x \mid x \not\models A^2_{-2} \} &= \{ a^2_{-2} \}, & \{ x \mid x \not\models B^2_{-2} \} &= \{ b^2_{-2} \}, \\
\{ x \mid x \not\models A^2_{-3} \} &= \{ a^2_{-3} \}, & \{ x \mid x \not\models B^2_{-3} \} &= \{ b^2_{-3} \}, \\
\{ x \mid x \not\models A^3_{-2} \} &= \{ a^3_{-2} \}, & \{ x \mid x \not\models B^3_{-2} \} &= \{ b^3_{-2} \}, \\
\{ x \mid x \not\models A^3_{-3} \} &= \{ a^3_{-3} \}, & \{ x \mid x \not\models B^3_{-3} \} &= \{ b^3_{-3} \}, \\
\{ x \mid x \not\models C_0 \} &= \{ c_0 \}, & \{ x \mid x \not\models D_0 \} &= \{ d_0 \}, \\
\{ x \mid x \not\models C_1 \} &= \{ c_1 \}, & \{ x \mid x \not\models D_1 \} &= \{ d_1 \}, \\
\{ x \mid x \not\models C_2 \} &= \{ c_2 \}, & \{ x \mid x \not\models D_2 \} &= \{ d_2 \}, \\
\{ x \mid x \not\models C_3 \} &= \{ \tau \}, & \{ x \mid x \not\models D_3 \} &= \{ \sigma \},
\end{align*}
\]

or

\[
\begin{align*}
\{ x \mid x \not\models A^1_{-3} \} &= \{ b^1_{-3} \}, & \left\{ x \mid x \not\models s_8 \& \neg t_8 \supset \bigvee_{i=1}^{8} t_i \right\} &= \{ b^1_{-3} \}, \\
\{ x \mid x \not\models B^1_{-3} \} &= \{ a^1_{-3} \}, & \left\{ x \mid x \not\models t_8 \& \neg s_8 \supset \bigvee_{i=1}^{8} s_i \right\} &= \{ a^1_{-3} \}, \\
\{ x \mid x \not\models A^2_{-2} \} &= \{ b^2_{-2} \}, & \{ x \mid x \not\models B^2_{-2} \} &= \{ a^2_{-2} \}, \\
\{ x \mid x \not\models A^2_{-3} \} &= \{ b^2_{-3} \}, & \{ x \mid x \not\models B^2_{-3} \} &= \{ a^2_{-3} \}, \\
\{ x \mid x \not\models A^3_{-2} \} &= \{ b^3_{-2} \}, & \{ x \mid x \not\models B^3_{-2} \} &= \{ a^3_{-2} \}, \\
\{ x \mid x \not\models A^3_{-3} \} &= \{ b^3_{-3} \}, & \{ x \mid x \not\models B^3_{-3} \} &= \{ a^3_{-3} \}, \\
\{ x \mid x \not\models C_0 \} &= \{ d_0 \}, & \{ x \mid x \not\models D_0 \} &= \{ c_0 \}, \\
\{ x \mid x \not\models C_1 \} &= \{ d_1 \}, & \{ x \mid x \not\models D_1 \} &= \{ c_1 \}, \\
\{ x \mid x \not\models C_2 \} &= \{ d_2 \}, & \{ x \mid x \not\models D_2 \} &= \{ c_2 \}, \\
\{ x \mid x \not\models C_3 \} &= \{ \sigma \}, & \{ x \mid x \not\models D_3 \} &= \{ \tau \}.
\end{align*}
\]

**Proof.** The statement follows from Lemmas 1 and 2, respectively. In the sequel we shall suppose that the set of conditions (**) is satisfied. The case when (***) holds is similar.
LEMMA 8. If $\mathcal{F} \not\models F$, then, for any $x \in W$ and any numbers $n \geq -3$ and $s \in \{1, 2, 3\}$,
   
a) $x \not\models A^s_n$ iff $x = a^s_n$,
   
b) $x \not\models B^s_n$ iff $x = b^s_n$

   (when $F = F_2$ we have that $x \models s_8 \& \neg t_8$ and $x \not\models \sqrt[n]{i=1}^{s_8} t_i$ iff $x = a^1_{-3}$, while $x \models t_8 \& \neg s_8$ and $x \not\models \sqrt[n]{i=1}^{s_8} s_i$ iff $x = b^1_{-3}$).

   PROOF. We use induction on $n$. The cases for $n = -2$ and $n = -3$ hold by Lemma 7.

   Now suppose that $x \not\models A^k_k (k \geq -1)$. This means that $x \models B^k_{k-1}, x \not\models A^k_{k-1}$, and $x \not\models B^k_{k-2}$. By the induction hypothesis, $x \not\models b^k_{k-1}, x \models a^k_{k-1}$, and $x \not\models b^k_{k-2}$. Thus $x = a^k_k$.

   Similarly $y \not\models B^k_k$ implies $y = b^k_k$.

   For the converse we use the fact that if $x = a^k_k$ and $y = b^k_k$, then $x \not\models A^k_k$ and $y \not\models B^k_k$ by the induction hypothesis, because $a^k_k \leq a^k_{k-1}$, $a^k_k \leq b^k_{k-1}$, $a^k_k \not\leq b^k_{k-1}$, and $b^k_k \not\leq a^k_{k-1}$.

LEMMA 9. If $\mathcal{F} \not\models F$ and, for any $\alpha \in \mathcal{F}$, $x \geq 0$, and $\epsilon, \delta \in \{1, 2\}, \alpha \not\models T(x, Q_\epsilon R_\delta),$
then $\alpha = t(x, y, z)$ for some triple $(x, y, z) \in P(i, j, k)$, and

   a) either $a^2_\epsilon \not\models Q_\epsilon, b^2_\delta \not\models Q_\epsilon, a^2_{\epsilon+1} \not\models Q_{\epsilon+1}, b^2_{\delta+1} \not\models Q_{\epsilon+1},$
   
   or $a^2_\epsilon \not\models Q_\epsilon, b^2_\delta \not\models Q_\epsilon, a^2_{\epsilon+1} \not\models Q_{\epsilon+1}, b^2_{\delta+1} \not\models Q_{\epsilon+1};$

   b) either $a^2_\epsilon \not\models R_\delta, b^2_\delta \not\models R_\delta, a^2_{\epsilon+1} \not\models R_{\delta+1}, b^2_{\delta+1} \not\models R_{\delta+1},$
   
   or $a^2_\epsilon \not\models R_\delta, b^2_\delta \not\models R_\delta, a^2_{\epsilon+1} \not\models R_{\delta+1}, b^2_{\delta+1} \not\models R_{\delta+1}.$

   PROOF. If $\alpha \not\models T(x, Q_\epsilon R_\delta)$, then by Lemma 8, $\alpha \not\models a^1_{\epsilon+1}, \alpha \not\models b^1_{\delta+1}, \alpha \leq a^2_\epsilon$, and $\alpha \leq b^2_\delta$; therefore $\alpha = t(x, y, z)$ for some $y \geq 0$ and $z \geq 0$. There are $\beta_1$ and $\beta_2$ such that $t(x, y, z) \models \beta_1, t(x, y, z) \models \beta_2$ and $\beta_1 \not\models Q_\epsilon, \beta_2 \not\models Q_{\epsilon+1}$, which implies $\beta_1, \beta_2 \in \{a^2_\epsilon, b^2_\delta | u, v \geq -1\}$ and $\beta_1 \not\models \beta_2, \beta_2 \not\models \beta_1$ by the construction of $Q_\epsilon, Q_{\epsilon+1}$ and $\mathcal{F}$.

   If $\beta_1 = a^2_\epsilon$ for some $u \geq -1$, then there are $\gamma_1$ and $\gamma_2$ such that $a^2_\epsilon \leq \gamma_1, \beta_2 \leq \gamma_1, \gamma_1 \not\models Q_{\epsilon-1}$, and $\gamma_2 \not\models Q_{\epsilon+1}$, which implies $\gamma_1, \gamma_2 \in \{a^2_\epsilon, b^2_\delta | r, s \geq -1\}$ and $\gamma_1 \not\models \gamma_2, \gamma_2 \not\models \gamma_1$; therefore $\gamma_1 = a^2_{\epsilon-1}, \beta_2 = b^2_\delta$, and $\gamma_2 = b^2_{\epsilon - 1}$.

   Then $\gamma_1 \not\models Q_{\epsilon+1} + 1$ and $b^2_{\epsilon + 1} \not\models Q_{\epsilon+1}$, which implies $t(x, y, z) \not\models a^2_{\epsilon+1} + 1$ and $t(x, y, z) \not\models b^2_{\epsilon+1}$. Therefore $u = y$.

   If $\beta_1 = b^2_\delta$, then $\beta_2 = a^2_\epsilon$ and $u = y$.

   Clause b) is similar.

LEMMA 10. If $V$ is a valuation on $\mathcal{F}$ such that $\mathcal{F} \not\models F$, then, for any $\alpha \in \mathcal{F}$ and $x, y, z \geq 0$,

   $\alpha \not\models T(x, A^2_\epsilon, A^2_\delta)$ iff $\alpha = t(x, y, z)$.

   PROOF. Clearly $t(x, y, z) \not\models T(x, A^2_\epsilon, A^2_\delta)$, by the definition of $\mathcal{F}$. The converse is proved similarly to Lemma 9.

LEMMA 11. $\mathcal{F} \models AxP$.

   PROOF. We must show that the formulas $AxI_j$ corresponding to the instructions $I_j$ hold in $\mathcal{F}$.

   We consider the formulas which arise from rules of form (1). So suppose instruction $I_j$ is in $q_{a_j}$, add 1 to $s_1$; go to $q_{b_j}$. To show that $AxI_j$ holds we first assume that $w \not\models T(x, Q_{a_j} R_{a_j}) \lor F$. Then $w \not\models F$, so, by Lemma 7, $w = f$. Then $f \not\models T(x, Q_{a_j} R_{a_j})$ i.e. there is an $x$ such that $f \leq x$ and $x \not\models T(x, Q_{a_j} R_{a_j})$. By Lemma 9, $x = t(x, y, z)$, and $(x, y, z) \in P(i, j, k)$. Now, in such a configuration, the machine will
proceed, using \( I \), to a configuration \((\beta, y + 1, z)\), so the definition of \( \leq \) provides that \( f \leq t(\beta, y + 1, z) \). To show that the formula holds, we must prove that \( f \not\models T(\beta, Q_2, R_1) \). But by Lemmas 8 and 9 we see that \( t(\beta, y + 1, z) \not\models T(\beta, Q_2, R_1) \) and hence \( f \not\models T(\beta, Q_2, R_1) \). Then we have

\[
\mathcal{F} \models T(\beta, Q_2, R_1) \supset T(\alpha, Q_1, R_1) \lor F.
\]

The other cases are similar.

**Lemma 12.** If \((\alpha, m, n) \not\in P(i, j, k)\), then

\[
\mathcal{F} \models (T(\alpha, A_m^2, A_n^3) \supset T(i, A_j^2, A_k^3) \lor F) \supset F.
\]

**Proof.** If \( \mathcal{F} \not\models F \), then, by Lemma 7,

\[
(1) \quad f \not\models F.
\]

By Lemma 10,

\[
(2) \quad f \models T(\alpha, A_m^2, A_n^3),
\]

\[
(3) \quad f \not\models T(i, A_j^2, A_k^3).
\]

By conditions (1)–(3),

\[
f \not\models T(\alpha, A_m^2, A_n^3) \supset T(i, A_j^2, A_k^3) \lor F.
\]

**Lemma 13.** If \( \mathcal{F} \not\models F \), for some valuation \( V \), then, by this valuation, besides \((*)\) of Lemma 7 the following conditions hold:

a) \( x \not\models C_2 \& D_2 \supset C_1 \lor D_1 \) iff \( x = \alpha_n \) or \( x = \beta_\varphi \), for some \( n \in \omega \) and \( \varphi \in 2^\omega \).

b) \( x \not\models D_1 \supset s_2 \lor C_1 \) implies \( x = \alpha_{n,0} \), for some \( n \in \omega \), or \( x = \tau \) or \( x = c_2 \).

c) \( x \not\models C_1 \supset t_2 \lor D_1 \) implies \( x = \alpha_{n,1} \), for some \( n \in \omega \), or \( x = \sigma \) or \( x = c_2 \).

**Proof.** a) \( x \not\models C_2 \& D_2 \supset C_1 \lor D_1 \) iff (by Lemma 7) \( x \not\in c_2 \), \( x \not\in d_2 \), \( x \leq c_1 \), \( x \leq d_1 \) iff \( x = \alpha_n \) or \( x = \beta_\varphi \).

b) \( x \not\models D_1 \supset s_2 \lor C_1 \) implies (by Lemma 7) \( x \not\in d_1 \) and \( x \leq c_1 \), which implies \( x = \alpha_{n,0} \), for some \( n \in \omega \), or \( x = \tau \) or \( x = c_2 \).

c) is proved similarly to b).

**Lemma 14.** \( \mathcal{F} \models H \).

**Proof.** If \( x \not\models F \), by some valuation, then, by Lemma 7, \( x = f \). If \( x \models E \), then, by Lemma 13a), for each \( n \in \omega \),

\[
\alpha_n \not\models p \& C_1 \supset t_2 \lor D_1 \quad \text{or} \quad \alpha_n \not\models p \& D_1 \supset s_2 \lor C_1.
\]

Hence, by Lemma 13b), c), \( \alpha_{n,0} \models p \) or \( \alpha_{n,1} \models p \). Choose \( \varphi \in 2^\omega \) such that \( \alpha_{n,1} \models p \) iff \( n \in \varphi \). If \( n \notin \varphi \), then \( \alpha_{n,1} \not\models p \); therefore \( \alpha_{n,0} \not\models p \). Thus, clearly, \( \beta_\varphi \not\models G \), so \( x \not\models G \).

**Lemma 15.** \( \mathcal{F} \models B \).

**Proof.** Immediate from Lemmas 11, 12 and 14.

**Lemma 16.** There is an elementary subframe \( \mathcal{F}^* \) of the frame \( \mathcal{F} \) such that \( \mathcal{F}^* \models F \lor H \) and so \( \mathcal{F}^* \not\models B \).

**Proof.** Let \( \mathcal{F}^* \) be some countable elementary subframe of \( \mathcal{F} \) whose domain contains \( f, a_n^m \) and \( b_n^m \), for all \( n \in \{1, 2, 3\} \) and \( -3 \leq m < \omega \); \( c_1, d_1, c_2, d_2, \tau, \sigma \);
t(p, q, r) for all triples \((p, q, r) \in P(i, j, k)\); and \(\alpha_n, \alpha_{n,1}\) and \(\alpha_{n,0}\) for all \(n \in \omega\). There must be some \(\varphi \in \omega^\omega\) such that \(\beta_\varphi \in W \setminus W^*\), because \(W\) is uncountable. Define \(V\) on \(\mathcal{F}^*\) by the following conditions: \(V(p) = \{\alpha_{n,1} \mid n \in \varphi\} \cup \{\alpha_{n,0} \mid n \notin \varphi\} \cup \{x \mid c_1 \leq x \lor d_1 \leq x\}\) and \(V(r)\) for \(r \in \{s_0, s_1, \ldots, s_8, t_0, t_1, \ldots, t_8\}\) is the same as in Lemma 7(\(\ast\)). Then we have 1) \(f \not\models F, f \models E\), and 2) \(f \models G\). Here 2) follows from the fact that \(\alpha_{n} \models G\) for \(n \in \omega\) and \(\beta_\varphi \models G\) for all \(\beta_\varphi \in W^*\) (because \(\psi \neq \varphi\)). To see that 1) holds, first note that, for each \(n \in \omega\),

\[
\alpha_n \not\models p \& C_1 \supset t_2 \lor D_1 \quad \text{or} \quad \alpha_n \not\models p \& D_1 \supset s_2 \lor C_1.
\]

Moreover, for each \(\beta_\varphi \in W^*\), there is some \(n \in \omega\) such that \(n \in \psi \cap \varphi\) or \(n \in \omega \setminus (\psi \cup \varphi)\). (If \(\psi = \omega \setminus \varphi\) then \(\beta_\varphi\) would be in \(W\) since the existence of "complementary" worlds \(\beta_\varphi\) is elementarily expressible.) Because of this fact, \(\beta_\varphi \not\models p \& C_1 \supset t_2 \lor D_1\) or \(\beta_\varphi \not\models p \& D_1 \supset s_2 \lor C_1\). In other words, \((G \& E \supset F) \lor F\) has been shown to fail (at \(f\)) in \(\mathcal{F}^*\).

Lemma 6 is finally proved.

Thus, since the problem "\((\alpha, m, n) \in P(i, j, k)\)?" is undecidable, and the formula \(B(P(\alpha, m, n), (i, j, k))\), given \(P, (\alpha, m, n)\) and \(i, j, k\), is constructed effectively, from Lemmas 5 and 6 the following theorems are obtained.

**Theorem 1.** The problem of first-order definability of intuitionistic formulas is algorithmically undecidable.

**Theorem 2.** The problem of \(\forall\)-definability of intuitionistic formulas is algorithmically undecidable.

**Theorem 3.** The problem of \(\forall \exists\)-definability of intuitionistic formulas is algorithmically undecidable.

**Theorem 4.** There is no algorithm which recognizes, given an intuitionistic formula, whether it is \(\forall \exists\)-definable and not \(\forall\)-definable.

**Remark.** If we consider the formula \(F\) as the formula \(F_1\), then the formula \(B(P(\alpha, m, n), (i, j, k))\) does not contain negation; and conjunction, as we know, can be eliminated from any formula. Thus, in Theorems 1 and 2 we can consider intuitionistic formulas constructed from sentence letters, implication and disjunction only. Further simplification of formulas in this direction is impossible, because all disjunctionless and all implicationless intuitionistic formulas are first-order definable (cf. [7] or [8]).

**§5. Further results.** In a forthcoming paper we intend to present some other results on undecidability in the correspondence theory. We mention two of them here.

**Theorem.** The problem of first-order definability of intuitionistic formulas in the class of countable frames is undecidable.

**Theorem.** The set of intuitionistic formulas that are first-order definable in the class of countable frames but not in the class of all frames is undecidable.

The proofs of these theorems use variants of the formula \(A \times P\) that are first-order definable, and the proof of their first-order definability is quite bulky. The variant of the proof of Theorem 1 that we have given here is obtained with the help of an idea of A. V. Chagrov from [9] that is used in the proof of Lemmas 5 and 12 (cf. [10]).
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