MOTION OF AN ELECTRON IN A HOMOGENEOUS MAGNETIC FIELD, WITH THE REACTION OF THE EMISSION INCLUDED

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The article gives a solution of the classical Dirac equation for a point-like electron in a homogeneous magnetic field. The solution is obtained in the form of a power series of a characteristic small parameter. The solution is compared with the results of other scientists.

The problem of electron motion in a magnetic field was previously considered with proper regard for the emission-induced reaction [1, 2]. The solution of the classical Dirac equation in a homogeneous magnetic field was determined in [1], but, as indicated by the author, the approximation is valid only for sufficiently small eigentimes

$$\tau \ll \left(\frac{mc^2}{E_0}\right)^2 T,$$

where $T \sim m^3c^5/e^4H^2$ and E_0 denotes the initial energy of the particle.

An approximating equation (see [3], p.274) was used from the very beginning in [2]. Obviously, when this approach is used, the accuracy of the solution is necessarily limited. It is therefore interesting to consider the accurate equation and to use perturbation theory for obtaining a solution which has any desired degree of accuracy.

We base our considerations on the classical Dirac equation for a point-like electron (see, for example, [4]):

$$m\dot{u}_\mu = \frac{e}{c} F_{\mu\nu} u_\nu + \gamma m \left(\ddot{u}_\mu - \frac{1}{c^2} a^2 u_\mu\right), \tag{1}$$

where $\gamma = (2/3)(e^2/mc^3)$; $F_{\mu\nu}$ denotes the tensor of the external electromagnetic field; $u_\mu = \dot{x}_\mu$ denotes the four-dimensional velocity; and $a^2 = \dot{u}_\mu \dot{u}_\mu$ denotes the square of the four-dimensional acceleration. Throughout the present article, a dot is used to denote a derivative with respect to the eigentime. A metric with an imaginary force component of four-dimensional vectors is used in the present article.

Assume the magnetic field to be parallel to the z-axis. Thus, when Eq. (1) is resolved into components, we obtain:

$$\begin{gathered} \dot{u}_x = -\omega_0 u_y + \gamma\left(\ddot{u}_x - \frac{1}{c^2} a^2 u_x\right), \quad \dot{u}_y = \omega_0 u_x + \gamma\left(\ddot{u}_y - \frac{1}{c^2} a^2 u_y\right), \\ \dot{u}_z = \gamma\left(\ddot{u}_z - \frac{1}{c^2} a^2 u_z\right), \quad \dot{\varepsilon} = \gamma\left(\ddot{\varepsilon} - \frac{1}{c^2} a^2 \varepsilon\right), \end{gathered} \tag{2}$$

where $\omega_0 = e_0 H/mc$ ($e = -e_0 < 0$ denotes the charge of the electron); and ε denotes the particle energy expressed in multiples of mc^2. It follows from the last two equations of (2) that the z component of the three-dimensional velocity is constant: $v_z = c\beta_{z0}$ and, hence, we have for the four-dimensional velocity:

$$u_z = c\beta_{z0}\varepsilon. \tag{3}$$

We look for a solution for $u_x(\tau)$ and $u_y(\tau)$ in the form*

*The results of [1, 2] point to a solution of this type; this solution follows also from the physics involved.

Lomonosov State University. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii Fizika, No. 3, pp. 87-90, March, 1972. Original article submitted December 15, 1970.

$$u_x(\tau) = u_\perp(\tau)\cos\varphi(\tau), \quad u_y(\tau) = u_\perp(\tau)\sin\varphi(\tau), \tag{4}$$

where

$$u_\perp(\tau) = c\beta_{\perp 0} e^{-\psi(\tau)}\left[1 - \beta_{z0}^2 - \beta_{\perp 0}^2 e^{-2\psi(\tau)}\right]^{-\frac{1}{2}},$$

with $\beta_{\perp 0} = \sqrt{\beta_x^2(0) + \beta_y^2(0)}$ denoting the initial value of the velocity component which is perpendicular to the magnetic field. We obtain the following equation system for the functions ψ and φ:

$$\dot\psi = \gamma(\dot\varphi^2 + \ddot\psi - \dot\psi^2 \operatorname{cth}(\psi + a)), \quad \dot\varphi = \omega_0 + \gamma[\ddot\varphi - \dot\varphi\dot\psi(1 + \operatorname{cth}(\psi + a))] \tag{5}$$

with the initial conditions $\psi(0) = 0$, $\varphi(0) = 0$, where a is given by the formula

$$\operatorname{cth} a = 2(1 - \beta_{z0}^2)\varepsilon_0^2 - 1, \tag{6}$$

and ε_0 denotes the initial energy. We introduce new functions ξ and η which are related to ψ and φ by the formulas

$$\dot\psi = \delta\xi, \quad \dot\varphi = \omega_0\eta, \tag{7}$$

where $\delta = \gamma\omega_0^2$. By switching to the new independent variable $s = \operatorname{cth}(\psi + a)$, we obtain for $\xi(s)$ and $\eta(s)$

$$\xi = \eta^2 + \lambda\xi[(1 - s^2)\xi' - s\xi], \quad \eta = 1 + \lambda\xi[(1 - s^2)\eta' - (1 + s)\eta], \tag{8}$$

where $\lambda = \gamma^2\omega_0^2$. Obviously, λ is a small parameter. As a matter of fact, even at fields of the order of the critical field [4], we have $\lambda \sim 10^{-6}$. It is therefore logical to determine a solution of equation system (8) in the form of an expansion in powers of λ:

$$\xi(s) = \sum_{n=0}^{\infty}\lambda^n P_n(s), \quad \eta(s) = \sum_{n=0}^{\infty}\lambda^n Q_n(s). \tag{9}$$

By substituting this expansion into Eq. (8), we obtain recursion relations for the functions P_n and Q_n:

$$\begin{gathered} P_n = Q_n + \sum_{\kappa=0}^{n-1}\{Q_{n-1-\kappa}Q_\kappa + P_{n-1-\kappa}[(1 - s^2)P_\kappa' - sP_\kappa]\}, \\ Q_n = \sum_{\kappa=0}^{n-1}P_{n-1-\kappa}[(1 - s^2)Q_\kappa' - (1 + s)Q_\kappa], \quad P_0(s) \equiv Q_0(s) \equiv 1. \end{gathered} \tag{10}$$

It is easy to infer from these formulas that P_n and Q_n are polynomials of order n in s. The following are formulas for the first two polynomials:

$$\begin{gathered} P_1(s) = -(2 + 3s), \quad Q_1(s) = -(1 + s), \quad P_2(s) = 2(1 + 10s + 10s^2), \\ Q_2(s) = 2 + 7s + 5s^2. \end{gathered} \tag{11}$$

We obtain the following equations (see Eqs. (7) and (9)) for the initial functions ψ and φ:

$$\psi = \delta\left(\tau + \sum_{n=1}^{\infty}\lambda^n\int_0^\tau P_n(s)\,d\tau\right), \quad \varphi = \omega_0\left(\tau + \sum_{n=1}^{\infty}\lambda^n\int_0^\tau Q_n(s)\,d\tau\right). \tag{12}$$

The solution of these equations is represented in the form of a power expansion in λ:

$$\psi = \delta\left(\tau + \sum_{n=1}^{\infty}\lambda^n f_n(\tau)\right), \quad \varphi = \omega_0\left(\tau + \sum_{n=1}^{\infty}\lambda^n g_n(\tau)\right). \tag{13}$$

By substituting these expansions into Eq. (12), we obtain formulas for the functions f_n and g_n. The following are expressions for the first three functions:

$$\begin{gathered} f_1 = \int_0^\tau P_1(s_0)\,d\tau, \quad f_2 = \int_0^\tau [f_1\dot P_1(s_0) + P_2(s_0)]\,d\tau, \\ f_3 = \int_0^\tau \left[f_2\dot P_1(s_0) + \frac{1}{2!}f_1^2\ddot P_1(s_0) + f_1\dot P_2(s_0) + P_3(s_0)\right]d\tau, \end{gathered} \tag{14}$$

where $s_0 = \operatorname{cth}(\delta\tau + a)$.

Similar formulas for g_n are obtained when $P_n(s_0)$ is replaced by $Q_n(s_0)$ (as above, the functions f_n remain in the expressions under the integral signs).

When we use Eqs. (11), (13), and (14), we can calculate first and second approximations:

$$f_1(\tau) = -\left(2\tau + 3\delta^{-1}\ln\frac{\operatorname{sh}(\delta\tau + a)}{\operatorname{sh} a}\right),$$

$$f_2(\tau) = 13\tau - 11\delta^{-1}[\operatorname{cth}(\delta\tau + a) - \operatorname{cth} a] + 6\tau\operatorname{cth}(\delta\tau + a) + \delta^{-1}[14 + 9\operatorname{cth}(\delta\tau + a)]\ln\frac{\operatorname{sh}(\delta\tau + a)}{\operatorname{sh} a}. \qquad (15)$$

Similarly, we obtain

$$g_1(\tau) = -\left(\tau + \delta^{-1}\ln\frac{\operatorname{sh}(\delta\tau + a)}{\operatorname{sh} a}\right),$$

$$g_2(\tau) = 4\tau - 2\delta^{-1}[\operatorname{cth}(\delta\tau + a) - \operatorname{cth} a] + 2\tau\operatorname{cth}(\delta\tau + a) + \delta^{-1}[5 + 3\operatorname{cth}(\delta\tau + a)]\ln\frac{\operatorname{sh}(\delta\tau + a)}{\operatorname{sh} a}. \qquad (16)$$

In the zeroth approximation in λ, we obtain the well-known solution of [2]. Let us consider the limit cases of small and large τ. We obtain from Eqs. (13), (15), and (16) for $\delta\tau \ll a$

$$\psi(\tau) = \delta\tau[1 - \lambda(2 + 3\operatorname{cth} a) + 2\lambda^2(1 + 10\operatorname{cth} a + 10\operatorname{cth}^2 a) + \cdots],$$
$$\varphi(\tau) = \omega_0\tau[1 - \lambda(1 + \operatorname{cth} a) + \lambda^2(2 + 7\operatorname{cth} a + 5\operatorname{cth}^2 a) + \cdots]. \qquad (17)$$

In the case $\delta\tau \gg a$, we obtain

$$\psi(\tau) = \delta\tau(1 - 5\lambda + 42\lambda^2 + \cdots), \quad \varphi(\tau) = \omega_0\tau(1 - 2\lambda + 14\lambda^2 + \cdots). \qquad (18)$$

It follows from Eq. (17) that this expansion is useful if

$$\lambda\operatorname{cth} a \lesssim 1.$$

The second term of the expansion of Eq. (17) is then smaller than the first term.

In the case of a two-dimensional motion ($\beta_{z0} = 0$) of an ultra-relativistic particle ($\varepsilon_0 \gg 1$), the latter condition assumes the form $\varepsilon_0 \lessdot \lambda^{-1/2}$ in accordance with Eq. (6). For fields $H \sim 10^4$ Oe this inequality is valid to energies $E_0 = \varepsilon_0 mc^2 \lessdot 10^{-7}$ GeV. The classical description is limited by the energy $E_0 \lessdot 10^5$ GeV for a given H.

Let us consider the nonrelativistic limit when $\operatorname{cth} a \simeq 1$. We have in this case

$$\psi(t) = \delta t(1 - 5\lambda + 42\lambda^2 + \cdots), \quad \varphi(t) = \omega_0 t(1 - 2\lambda + 14\lambda^2 + \cdots), \qquad (19)$$

which coincides with Eq. (18). This is not surprising because for $\delta\tau \gg a$, the particle looses a large portion of its energy and becomes nonrelativistic. Equation (19) is an expansion of the accurate solution of the nonrelativistic equation, taking into account the emission-induced reaction (solution obtained in [1]). The solution can be written in our notation as follows:

$$\psi(t) = \frac{t}{2\gamma}\left\{\left[\frac{1}{2} + \frac{1}{2}(1 + 16\lambda)^{\frac{1}{2}}\right]^{\frac{1}{2}} - 1\right\},$$

$$\varphi(t) = \frac{t}{2\gamma}\left[-\frac{1}{2} + \frac{1}{2}(1 + 16\lambda)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

An expansion of this solution in powers of λ results in Eq. (19).

In conclusion, the authors thank professor A. A. Sokolov for formulating the problem and for discussing the results.

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