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Simplified Kripke-Style Semantics for Some Normal Modal Logics

Abstract. Pietruszczak (Bull Sect Log 38(3/4):163–171, 2009. https://doi.org/10.12775/LLP.2009.013) proved that the normal logics K45, KB4 (= KB5), KD45 are determined by suitable classes of simplified Kripke frames of the form $\langle W, A\rangle$, where $A \subseteq W$. In this paper, we extend this result. Firstly, we show that a modal logic is determined by a class composed of simplified frames if and only if it is a normal extension of K45. Furthermore, a modal logic is a normal extension of K45 (resp. KD45; KB4; S5) if and only if it is determined by a set consisting of finite simplified frames (resp. such frames with $A \neq \varnothing$; such frames with $A = W$ or $A = \varnothing$; such frames with $A = W$). Secondly, for all normal extensions of K45, KB4, KD45 and S5, in particular for extensions obtained by adding the so-called "verum" axiom, Segerberg's formulas and/or their T-versions, we prove certain versions of Nagle's Fact (J Symbol Log 46(2):319–328, 1981. https://doi.org/10.2307/2273624) (which concerned normal extensions of K5). Thirdly, we show that these extensions are determined by certain classes of finite simplified frames generated by finite subsets of the set $\mathbb{N}$ of natural numbers. In the case of extensions with Segerberg's formulas and/or their T-versions these classes are generated by certain finite subsets of $\mathbb{N}$.

Keywords: Simplified Kripke-style semantics, Semi-universal frames, Normal modal logics.

Introduction

Semi-universal frames introduced in [5] for some normal logics are Kripke frames of the form $\langle W, R\rangle$, where W is a non-empty set of possible worlds and R is an accessibility relation such that $R = W \times A$, for some subset A of W (so A is a set of common alternatives for all worlds).[1] Instead of semi-universal frames we can use *simplified frames* of the form $\langle W, A\rangle$, where W and A are as above. In [5] it is proved that the logics K45, KB4 (= KB5) and KD45 are determined, respectively by (a) the class of all simplified frames;

[1] For $A = W$ we have *universal frames*, i.e., $R = W \times W$. For $A = \varnothing$ we have *empty frames*, i.e., $R = \varnothing$ (since $W \times \varnothing = \varnothing$). Instead of empty frames we can use the single empty frame $\mathfrak{F}_\varnothing := \langle \{1\}, \varnothing\rangle$ $(= \langle \{1\}, \{1\} \times \varnothing\rangle)$.

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(b) the class of simplified frames such that $A = \varnothing$ or $A = W$; (c) the class of simplified frames with $A \neq \varnothing$.[2] In this paper, we focus on extensions of these logics by means of Segerberg's formulas ($\mathtt{Alt}_n$) and their T-versions ($\mathtt{Talt}_n$), for any $n > 0$.

The structure of the paper is as follows. In Section 1 we introduce a modal language, basic notions and facts about normal logics, consider Kripke semantics, and recall determination theorems for normal logics.

In Section 2 we develop the notion of *semi-universal* frames and so *simplified* Kripke-style semantics for the logics K45, KD45, KB4, S5 and their normal extensions by formulas selected from ($\mathtt{Alt}_n$) and ($\mathtt{Talt}_m$). For such logics, we introduce a special type of frames generated by certain finite subsets of $\mathbb{N}$.

In Section 3 we present Nagle's Fact from [2] and its versions for K45, KB4, KD45, S5 and for these logics with additional axioms selected from ($\mathtt{Alt}_n$) and ($\mathtt{Talt}_n$). We obtain that a modal logic is a normal extension of K45 if and only if it is determined by a subclass of the class of finite semi-universal frames.[3] Furthermore, we obtain that a modal logic is a normal extension of KD45 (resp. KB4; S5) if and only if it is determined by a set consisting of finite semi-universal frames with $A \neq \varnothing$ (resp. with $A = W$ or $A = \varnothing$; $A = W$). For the logics with an additional axiom ($\mathtt{Alt}_n$) and/or ($\mathtt{Talt}_m$) we obtain that a modal logic is a normal extension of one of these logics if and only if it is determined by a suitable class of semi-universal frames whose cardinalities are suitable limited using numbers n and/or m. Also in this case, we will use frames generated by certain finite subsets of $\mathbb{N}$.

1. Preliminaries

1.1. Normal Modal Logics

Let $\mathtt{At}$ be the set of all atoms (or propositional letters): 'p_1', 'q_1', 'p_2', 'q_2', 'p_3', 'q_3', ... (for 'p_1' and 'q_1' we use 'p' and 'q', respectively). The set $\mathtt{For}$ of all formulas for (propositional) modal logics is standardly formed from atoms, brackets, truth-value operators: '$\neg$', '$\vee$', '$\wedge$', '$\supset$', and '$\equiv$' (connectives of negation, disjunction, conjunction, material implication, and material equivalence, respectively), and the modal operator '$\Box$' (the necessity sign; the possibility sign '$\Diamond$' is the abbreviation of '$\neg\Box\neg$'). For any $k > 0$

[2] It is well known that the logic S5 is determined by the class of universal frames; and so by the class of simplified frames of the form $\langle W, W\rangle$.

[3] A frame (model) is said to be *finite* just in case the number of members of W is finite.

and any formula of the form $\ulcorner\varphi_1 \vee \cdots \vee \varphi_k\urcorner$ we write $\bigvee_{i=1}^{k} \varphi_i$. Finally, we put $\top :=$ '$p_2 \supset p_2$' and $\bot :=$ '$p_2 \wedge \neg p_2$'.

Let Taut be the set of *classical tautologies*, i.e., all truth-functional tautologies. Moreover, let PL be the set of formulas which are instances of classical tautologies. A subset Λ of $\mathtt{For}$ is a *modal logic* iff Taut $\subseteq \Lambda$ and Λ is closed under two rules: detachment for material implication (modus ponens) and uniform substitution. Thus, by uniform substitution, all modal logics include the set PL. Moreover, this set is the smallest modal logic.

A modal logic Λ is *normal* iff Λ contains the following formula:

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \tag{K}$$

and is closed under the *necessity rule*:

$$\text{if } \varphi \in \Lambda \text{ then } \ulcorner\Box\varphi\urcorner \in \Lambda. \tag{RN}$$

Any normal logic Λ is closed under the *monotonicity* and *regularity* rules:

$$\text{if } \ulcorner\varphi \supset \psi\urcorner \in \Lambda \text{ then } \ulcorner\Box\varphi \supset \Box\psi\urcorner \in \Lambda. \tag{RM}$$

$$\text{if } \ulcorner\varphi_1 \supset (\varphi_2 \supset \psi)\urcorner \in \Lambda \text{ then } \ulcorner\Box\varphi_1 \supset (\Box\varphi_2 \supset \Box\psi)\urcorner \in \Lambda. \tag{RR}$$

Thus, for any normal logic Λ and any $k \geqslant 0$ we obtain:

$$\text{if } \ulcorner(\varphi_1 \wedge \cdots \wedge \varphi_k) \supset \psi\urcorner \in \Lambda \text{ then } \ulcorner(\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_k) \supset \Box\psi\urcorner \in \Lambda.$$

We recall that K is the smallest normal modal logic. To simplify the naming of normal logics, for any formulas $(\mathtt{X}_1), \ldots, (\mathtt{X}_k)$, the smallest normal logic including all of these formulas will be denoted by $\mathrm{KX}_1 \ldots \mathrm{X}_k$, i.e., $\mathrm{KX}_1 \ldots \mathrm{X}_k := \mathrm{K} \oplus \{\mathtt{X}_1, \ldots, \mathtt{X}_k\}$.

In order to define other logics we will use the following formulas:[4]

$$\Box q \tag{Q}$$

$$\Box p \supset p \tag{T}$$

$$p \supset \Box p \tag{$\mathtt{T_c}$}$$

$$(\Box p \supset p) \vee \Box q \tag{$\mathtt{T_q}$}$$

$$\Box p \supset \Diamond p \tag{D}$$

$$\Diamond p \supset \Box p \tag{$\mathtt{D_c}$}$$

$$p \supset \Box\Diamond p \tag{B}$$

$$\Box p \supset \Box\Box p \tag{4}$$

[4]The name '$(\mathtt{T_q})$' is an abbreviation for 'quasi-$\mathtt{T}$', because quasi-reflexive frames are adequate for $\mathrm{K} \oplus (\mathtt{T_q})$, while $\mathrm{K} \oplus (\mathtt{T})$ is determined by the class of reflexive frames.

$$\Diamond p \supset \Box\Diamond p \tag{5}$$

$$\Box\Diamond p \supset \Diamond p \tag{5_c}$$

It is known that (5_c) $\in$ KD4, (D) $\in$ K5_c and (4) $\in$ K55_c [see, e.g., 1,3, 4]. Hence KD4 $=$ K45_c and KD45 $=$ K55_c. Moreover, notice that for an arbitrary modal logic Λ: $\ulcorner \mathtt{T} \supset \mathtt{T_q} \urcorner$ and $\ulcorner \mathtt{Q} \supset \mathtt{T_q} \urcorner$ belong to Λ. So ($\mathtt{T_q}$) $\in \Lambda \oplus$ (T) and ($\mathtt{T_q}$) $\in \Lambda \oplus$ (Q), for any normal logic Λ.

We put T := KT, S4 := KT4 and S5 := KT5. We have KT $=$ KD $\oplus$ ($\mathtt{T_q}$) and KB4 $=$ KB5 $=$ KB45 $=$ K5 $\oplus$ ($\mathtt{T_q}$) $=$ K45 $\oplus$ ($\mathtt{T_q}$). So also KB4 $=$ KB4 $\oplus$ ($\mathtt{T_q}$). Moreover, S5 := KT5 $=$ KD5 $\oplus$ ($\mathtt{T_q}$) $=$ KTB4 $=$ KDB5 $=$ KDB4 $=$ KD45 $\oplus$ ($\mathtt{T_q}$) [see, e.g., 1,6,7]. For the semantic proof of these facts see Remark 1.3.

Let Ver ("Verum") be the smallest logic containing (Q) (in [8] it is the logic Abs, called the "Absurd System"). We have Ver $=$ K$\oplus$(Q). All formulas of the form $\ulcorner \varphi \supset \Box\psi \urcorner$ and $\ulcorner \varphi \vee \Box\psi \urcorner$ belong to Ver. So (B), (4), (5), and ($\mathtt{T_q}$) belong to Ver, and so K4 $\oplus$ (Q) $=$ K5 $\oplus$ (Q) $=$ KB $\oplus$ (Q) $=$ Ver.

Any logic Λ which contains both (Q) and (D) is inconsistent, i.e., if (D), (Q) $\in \Lambda$ then $\Lambda = \mathtt{For}$. So Ver $\oplus$ (D) $= \mathtt{For} =$ Ver $\oplus$ (T), as well S5 $\oplus$ (Q) $=$ KD $\oplus$ (Q) $=$ KD45 $\oplus$ (Q) $= \mathtt{For}$.

Let Triv be the smallest logic containing (T) and ($\mathtt{T_c}$). The logic Triv is normal and it contains ($\mathtt{T_q}$), (D), (B), (4) and (5). So S5 $\subsetneq$ Triv.

We also use Segerberg's formulas and their T-versions for any $n > 0$:

$$\Box q_1 \vee \Box(q_1 \supset q_2) \vee \cdots \vee \Box((q_1 \wedge \cdots \wedge q_n) \supset q_{n+1}) \tag{$\mathtt{Alt}_n$}$$

$$(\Box p \supset p) \vee (\mathtt{Alt}_n) \tag{$\mathtt{Talt}_n$}$$

Note that

- K $\oplus$ ($\mathtt{Alt}_1$) $=$ K $\oplus$ ($\mathtt{D_c}$).

For an arbitrary modal logic Λ and $n > 0$ we have: $\ulcorner \mathtt{T_q} \supset \mathtt{Talt}_n \urcorner$, $\ulcorner \mathtt{Q} \supset \mathtt{Alt}_n \urcorner$, $\ulcorner \mathtt{Alt}_n \supset \mathtt{Talt}_n \urcorner$, $\ulcorner \mathtt{Alt}_n \supset \mathtt{Alt}_{n+1} \urcorner$ and $\ulcorner \mathtt{Talt}_n \supset \mathtt{Talt}_{n+1} \urcorner$ belong to Λ. Hence we get:

- ($\mathtt{Talt}_n$) $\in \Lambda \oplus$ ($\mathtt{T_q}$);
- because ($\mathtt{T_q}$) $\in$ KB4, if Λ is a normal extension of KB4, then $\Lambda = \Lambda \oplus (\mathtt{T_q}) = \Lambda \oplus (\mathtt{Talt}_n)$ and $\Lambda \oplus (\mathtt{Alt}_n) = \Lambda \oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$, for any $m > 0$;
- ($\mathtt{Alt}_{n+1}$) $\in \Lambda \oplus$ ($\mathtt{Alt}_n$);
- ($\mathtt{Talt}_{n+1}$) $\in \Lambda \oplus$ ($\mathtt{Talt}_n$);
- if $m \geqslant n$ then ($\mathtt{Talt}_m$) $\in \Lambda \oplus (\mathtt{Alt}_n)$; so K$\oplus$($\mathtt{Alt}_n$) $=$ K$\oplus\{\mathtt{Alt}_n, \mathtt{Talt}_m\}$.

Note that for any $n > 0$, ($\mathtt{Alt}_n$) belongs to Ver, and ($\mathtt{Talt}_n$) belongs to S5 $\subsetneq$ Triv. It is known that:

- S5 $\subsetneq \cdots \subsetneq$ S5 $\oplus$ ($\mathtt{Alt}_n$) $\subsetneq \cdots \subsetneq$ S5 $\oplus$ ($\mathtt{Alt}_1$) = Triv; and this sequence comprise all the normal extensions of S5 [see 8, p. 122];
- K4 $\subsetneq \cdots \subsetneq$ K4 $\oplus$ ($\mathtt{Alt}_n$) $\subsetneq \cdots \subsetneq$ K4 $\oplus$ ($\mathtt{Alt}_1$) $\subsetneq$ K4 $\oplus$ ($\mathtt{Q}$) = Ver;
- K5 $\subsetneq \cdots \subsetneq$ K5 $\oplus$ ($\mathtt{Alt}_n$) $\subsetneq \cdots \subsetneq$ K5 $\oplus$ ($\mathtt{Alt}_1$) $\subsetneq$ K5 $\oplus$ ($\mathtt{Q}$) = Ver;
- KB $\subsetneq \cdots \subsetneq$ KB $\oplus$ ($\mathtt{Alt}_n$) $\subsetneq \cdots \subsetneq$ KB $\oplus$ ($\mathtt{Alt}_1$) $\subsetneq$ KB $\oplus$ ($\mathtt{Q}$) = Ver;
- K45 $\subsetneq \cdots \subsetneq$ K45 $\oplus$ ($\mathtt{Alt}_n$) $\subsetneq \cdots \subsetneq$ K45 $\oplus$ ($\mathtt{Alt}_1$) $\subsetneq$ K45 $\oplus$ ($\mathtt{Q}$) = Ver;
- KB4 $\subsetneq \cdots \subsetneq$ KB4 $\oplus$ ($\mathtt{Alt}_n$) $\subsetneq \cdots \subsetneq$ KB4 $\oplus$ ($\mathtt{Alt}_1$) $\subsetneq$ KB4 $\oplus$ ($\mathtt{Q}$) = Ver;
- K5 $\subsetneq \cdots \subsetneq$ K5 $\oplus$ ($\mathtt{Talt}_n$) $\subsetneq \cdots \subsetneq$ K5 $\oplus$ ($\mathtt{Talt}_1$) $\subsetneq$ K5 $\oplus$ ($\mathtt{T_q}$) = KB5 = KB4;
- K45 $\subsetneq \cdots \subsetneq$ K45 $\oplus$ ($\mathtt{Talt}_n$) $\subsetneq \cdots \subsetneq$ K45 $\oplus$ ($\mathtt{Talt}_1$) $\subsetneq$ K45 $\oplus$ ($\mathtt{T_q}$) = KB4;
- KD5 $\subsetneq \cdots \subsetneq$ KD5 $\oplus$ ($\mathtt{Talt}_n$) $\subsetneq \cdots \subsetneq$ KD5 $\oplus$ ($\mathtt{Talt}_1$) $\subsetneq \cdots \subsetneq$ KD45 $\oplus$ ($\mathtt{Talt}_1$) $\subsetneq$ S5;
- KD45 $\subsetneq \cdots \subsetneq$ KD45 $\oplus$ ($\mathtt{Talt}_n$) $\subsetneq \cdots \subsetneq$ KD45 $\oplus$ ($\mathtt{Talt}_1$) $\subsetneq$ S5.

Cf., e.g., [8, p. 127], Theorems 1.3, 1.2, 3.3, [6, p. 120] and [7, p. 207]. Notice that KD5 $\oplus$ ($\mathtt{Talt}_1$) = KD45 $\oplus$ ($\mathtt{T_q}$) = S5.

Remark 1.1. The formulas ($\mathtt{Q}$), ($\mathtt{Alt}_n$), ($\mathtt{T_q}$) and ($\mathtt{Talt}_n$) are connected with the following formulas for any $n \geqslant 0$:[5]

$$p \vee \Box(p \supset q_1) \vee \Box((p \wedge q_1) \supset q_2) \vee \cdots \vee \Box((p \wedge q_1 \wedge \cdots \wedge q_n) \supset q_{n+1}) \quad (\mathtt{C}_n)$$

We will prove that K $\oplus$ $\{\mathtt{Alt}_1, \mathtt{T_q}\}$ = K $\oplus$ ($\mathtt{C}_0$) and K $\oplus$ $\{\mathtt{Alt}_{n+1}, \mathtt{Talt}_n\}$ = K $\oplus$ ($\mathtt{C}_n$), for any $n > 0$ [see Remark 1.3(2)]. Therefore, it is unnecessary to consider formulas ($\mathtt{C}_n$).

1.2. Kripke Semantics for Normal Logics

For the semantical analysis of normal logics we may use standard *frames* of the form $\langle W, R\rangle$, where W is a non-empty set of *worlds* and R is a binary *accessibility* relation on W. For any frame $\langle W, R\rangle$, a *model* is any triple $\langle W, R, V\rangle$, where V is a function which for any pair consists of a formula and a world assigns a truth-value with respect to R. More precisely, $V\colon \mathtt{For} \times W \longrightarrow \{0, 1\}$ preserves classical conditions for truth-value operators and for any $\varphi \in \mathtt{For}$ and $x \in W$ we have:

[5] One of the anonymous reviewers of this paper drew our attention to the formulas ($\mathtt{C}_n$).

$$(\mathrm{V}^R_\Box) \qquad V(\Box\varphi, x) = 1 \quad \text{iff} \quad \forall_{y\in R[x]}\, V(\varphi, y) = 1,$$

where for any $x \in W$ we put $R[x] := \{y \in W : x\, R\, y\}$.

As usual, we say that a formula φ is *true in a world* x of a model $\langle W, R, V\rangle$ iff $V(\varphi, x) = 1$. We say that a formula is *true in a model* iff it is true in all worlds of this model. Next we say that a formula is *true in a frame* iff it is true in every model which is based on this frame. A formula is *valid in a class of frames* (resp. *models*) iff it is true in all frames (resp. models) from this class. Moreover, for any modal logic Λ and any class $\boldsymbol{C}$ of frames (resp. models) we say that: Λ is *sound wrt* $\boldsymbol{C}$ iff all formulas from Λ are valid in $\boldsymbol{C}$; Λ is *complete wrt* $\boldsymbol{C}$ iff all valid formulas in $\boldsymbol{C}$ are members of Λ; Λ is *determined* by $\boldsymbol{C}$ iff Λ is sound and complete wrt $\boldsymbol{C}$.[6]

A binary relation R on W is called, respectively: (i) *empty* iff $R = \varnothing$; (ii) *universal* iff $R = W \times W$; (iii) *reflexive* iff $\forall_{x\in W}\; xRx$; (iv) *quasi-reflexive* iff $\forall_{x\in W}(\exists_{y\in W}\; x\,R\,y \Rightarrow x\,R\,x)$ iff $\forall_{x\in W}(x\,R\,x$ or $R[x] = \varnothing)$; (v) *serial* iff $\forall_{x\in W}\exists_{y\in W}\; x\,R\,y$; (vi) *symmetric* iff $\forall_{x,y\in W}(x\,R\,y \Rightarrow y\,R\,x)$; (vii) *transitive* iff $\forall_{x,y,z\in W}(x\,R\,y \;\&\; y\,R\,z \Rightarrow x\,R\,z)$; (viii) *Euclidean* iff $\forall_{x,y,z\in W}(x\,R\,y \;\&\; x\,R\,z \Rightarrow y\,R\,z)$; (ix) *vacant* iff $\forall_{x,y\in W}(x\,R\,y \Rightarrow x = y)$; (x) *identity* iff $R = \{\langle x, x\rangle : x \in W\}$. We will transfer this terminology for properties of accessibility relations to the frames with those relations.

Notice that for any binary relation R we have:

($\star$) R is reflexive iff R is serial and quasi-reflexive.

($\star\star$) R is symmetric and transitive iff R is symmetric and Euclidean iff R is Euclidean and quasi-reflexive.

Additionally, for any $n \geqslant 0$ we will consider three classes of relations satisfying the following conditions: $(\mathrm{xi})_n$ $\forall_{x\in W}\; \mathrm{Card}R[x] \leqslant n$; $(\mathrm{xii})_n$ $\forall_{x\in W}(xRx$ or $\mathrm{Card}R[x] \leqslant n)$; and $(\mathrm{xiii})_n$ $\forall_{x\in W}\; \mathrm{Card}(R[x]\setminus\{x\}) \leqslant n$. Of course, $(\mathrm{xi})_0 =$ (i) and $(\mathrm{xii})_0 =$ (iv). Moreover, we have:

($\dagger$) for all $n \geqslant 0$ and $x \in W$: $\mathrm{Card}(R[x]\setminus\{x\}) \leqslant n$ iff $\mathrm{Card}R[x] \leqslant n + 1$ and either $x\,R\,x$ or $\mathrm{Card}R[x] \leqslant n$.

Hence:

($\ddagger$) R satisfies $(\mathrm{xiii})_0$ iff R satisfies $(\mathrm{xi})_1$ and (iv); and for any $n > 0$: R satisfies $(\mathrm{xiii})_n$ iff R satisfies $(\mathrm{xi})_{n+1}$ and $(\mathrm{xii})_n$.

[6] Later in this paper for any class $\boldsymbol{C}$ of frames, $\boldsymbol{C}_{\mathrm{fin}}$ (resp. $\boldsymbol{C}^+$, $\boldsymbol{C}^+_{\mathrm{fin}}$) will be the subclass of finite (resp. non-empty, non-empty finite) frames from $\boldsymbol{C}$.

1.3. Determination Theorems for Some Normal Logics

We can assign appropriate kinds of frames to individual formulas. We have the following pairs: emptiness to ($\mathtt{Q}$); reflexivity to ($\mathtt{T}$); quasi-reflexivity to ($\mathtt{T_q}$); vacuity to ($\mathtt{T_c}$); seriality to ($\mathtt{D}$); symmetry to ($\mathtt{B}$); transitivity to ($\mathtt{4}$); Euclideanness to ($\mathtt{5}$); the condition $(\mathrm{xi})_n$ to ($\mathtt{Alt}_n$); the condition $(\mathrm{xii})_n$ to ($\mathtt{Talt}_n$), for any $n > 0$; the condition $(\mathrm{xiii})_n$ to ($\mathtt{C}_n$), for any $n \geqslant 0$.

Determination theorems for the logic K and its normal extensions by some of the formulas ($\mathtt{T}$), ($\mathtt{T_c}$), ($\mathtt{D}$), ($\mathtt{B}$), ($\mathtt{4}$) and ($\mathtt{5}$) are standard [cf., e.g., 1,8,9]. For normal extensions of $\mathrm{K} \oplus (\mathtt{Alt}_n)$, see [8, pp. 52–53]. Moreover, for normal extensions of $\mathrm{K} \oplus (\mathtt{T_q})$, $\mathrm{K} \oplus (\mathtt{Talt}_n)$ or $\mathrm{K} \oplus (\mathtt{C}_n)$ we will adopt Segerberg's proof of Lemma 5.3 given for normal extensions of $\mathrm{K} \oplus (\mathtt{Alt}_n)$.

Lemma 1.1. 1. [cf. 8, Lemma 5.3] *Let Λ be a normal extension of $\mathrm{K}\oplus(\mathtt{Alt}_n)$, where $n > 0$ and $\mathfrak{M}_\Lambda = \langle W_\Lambda, R_\Lambda, V_\Lambda \rangle$ be a canonical model for Λ. Then for any $x \in W_\Lambda$, $\mathrm{Card} R_\Lambda[x] \leqslant n$.*

2. *Let Λ be a normal extension of $\mathrm{K} \oplus (\mathtt{Talt}_n)$, where $n > 0$ (resp. of $\mathrm{K} \oplus (\mathtt{T_q})$). Let $\mathfrak{M}_\Lambda = \langle W_\Lambda, R_\Lambda, V_\Lambda \rangle$ be a canonical model for Λ. Then for any $x \in W_\Lambda$ either $x \, R_\Lambda \, x$ or $\mathrm{Card} R_\Lambda[x] \leqslant n$ (resp. either $x \, R_\Lambda \, x$ or $\mathrm{Card} R_\Lambda[x] = 0$).*

3. *Let Λ be a normal extension of $\mathrm{K} \oplus (\mathtt{C}_n)$, where $n \geqslant 0$, and $\mathfrak{M}_\Lambda = \langle W_\Lambda, R_\Lambda, V_\Lambda \rangle$ be a canonical model for Λ. Then for any $x \in W_\Lambda$ we have $\mathrm{Card}(R_\Lambda[x] \backslash \{x\}) \leqslant n$.*

Proof. *Ad* (2): Assume for a contradiction that there are pairwise different $x_0, x_1, \ldots, x_{n+1}$ from W_Λ such that $x_0 R_\Lambda x_1, \ldots, x_0 R_\Lambda x_{n+1}$ and it is not the case that $x_0 \, R_\Lambda \, x_0$ (for $\mathrm{K} \oplus (\mathtt{T_q})$ we use the case where $n = 0$). For arbitrary different $i, j \in \{0, \ldots, n+1\}$ there is a formula $\varphi_{i,j}$ such that $\varphi_{i,j} \notin x_i$ and $\varphi_{i,j} \in x_j$. Now for any $i \in \{0, \ldots, n+1\}$ we put $\psi_i := \bigvee_{j=0}^{n+1} \varphi_{i,j}$. So for all $i, j \in \{0, \ldots, n+1\}$ we have: $\psi_i \in x_j$ iff $i \neq j$. Hence the formula $\ulcorner(\Box\psi_0 \supset \psi_0) \vee \Box\psi_1 \vee \Box(\psi_1 \supset \psi_2) \vee \cdots \vee \Box((\psi_1 \wedge \cdots \wedge \psi_n) \supset \psi_{n+1})\urcorner$ does not belong to x_0. This contradicts the facts that, by ($\mathtt{Talt}_n$) (resp. ($\mathtt{T_q}$), if $n = 0$), this formula belongs to all members of W_Λ.

Ad (3): Assume for a contradiction that there are pairwise different x_0, $x_1, \ldots, x_{n+1}$ from W_Λ such that $x_0 \, R_\Lambda \, x_1, \ldots, x_0 \, R_\Lambda \, x_{n+1}$. As above for any $i \in \{0, \ldots, n+1\}$, we define a formula ψ_i such that for all $i, j \in \{0, \ldots, n+1\}$ we have: $\psi_i \in x_j$ iff $i \neq j$. Furthermore, for any $i \in \{1, \ldots, n+1\}$ there is a formula χ_i such that $\chi_i \notin x_0$ and $\chi_i \in x_i$. Now we put $\pi := \bigvee_{i=1}^{n+1} \chi_i$. So for all $i \in \{0, \ldots, n+1\}$ we have: $\pi \in x_i$ iff $i \neq 0$. Hence the formula $\ulcorner\pi \vee \Box(\pi \supset \psi_1) \vee \Box((\pi \wedge \psi_1) \supset \psi_2) \vee \cdots \vee \Box((\pi \wedge \psi_1 \wedge \cdots \wedge \psi_n) \supset \psi_{n+1})\urcorner$

does not belong to x_0. This contradicts the facts that, by ($\mathtt{C}_n$), this formula belongs to all members of W_Λ. ■

THEOREM 1.2 (**8, Theorem 5.4, p. 52**). *The following logics are determined by the following conditions on frames* $\langle W, R\rangle$:

1. $\mathrm{K} \oplus (\mathtt{Alt}_n)$*—for any* $x \in W$, $\mathrm{Card} R[x] \leqslant n$.
2. $\mathrm{KD} \oplus (\mathtt{Alt}_n)$*—*$R$ *is serial and for any* $x \in W$, $\mathrm{Card} R[x] \leqslant n$.
3. $\mathrm{K4} \oplus (\mathtt{Alt}_n)$*—*$R$ *is transitive and* $\mathrm{Card} W \leqslant n + 1$.
4. $\mathrm{KD4} \oplus (\mathtt{Alt}_n)$*—*$R$ *is serial and transitive, and* $\mathrm{Card} W \leqslant n + 1$.
5. $\mathrm{K45} \oplus (\mathtt{Alt}_n)$*—*$R$ *is transitive and Euclidean, and* $\mathrm{Card} W \leqslant n + 1$.
6. $\mathrm{KD45} \oplus (\mathtt{Alt}_n)$*—*$R$ *is serial transitive Euclidean and* $\mathrm{Card} W \leqslant n + 1$.
7. $\mathrm{S4} \oplus (\mathtt{Alt}_n)$*—*$R$ *is reflexive and transitive, and* $\mathrm{Card} W \leqslant n$.
8. $\mathrm{S5} \oplus (\mathtt{Alt}_n)$*—*$R$ *is universal and* $\mathrm{Card} W \leqslant n$.

In the cases 1, 4 and 6–8 the sign '$\leqslant$' can be replaced by '$=$'.

In the standard way, we get:

THEOREM 1.3. 1. K *is determined by the class of all frames.*

2. S5 *is determined by the class* $\mathbf{U}$ *of all universal frames, as well by the class* $\mathbf{U}_{\mathrm{fin}}$ *of all finite universal frames.*
3. Triv *is determined by the class of frames with* $R = \{\langle x, x\rangle : x \in W\}$, *as well by the single universal frame* $\mathfrak{F}_1 := \langle\{1\}, \{\langle 1, 1\rangle\}\rangle$.
4. Ver *is determined by the class of empty frames, as well by the single empty frame* $\mathfrak{F}_\varnothing := \langle\{1\}, \varnothing\rangle$.
5. *Let* ($\mathtt{X}_1$), ..., ($\mathtt{X}_k$) *be any formulas from among the following ones*: ($\mathtt{Q}$), ($\mathtt{T}$), ($\mathtt{T_q}$), ($\mathtt{T_c}$), ($\mathtt{D}$), ($\mathtt{B}$), ($\mathtt{4}$), ($\mathtt{5}$), ($\mathtt{Alt}_n$), ($\mathtt{Talt}_m$), ($\mathtt{C}_k$), *for all* $n, m > 0$, $k \geqslant 0$. *Then the logic* $\mathrm{K} \oplus \{\mathtt{X}_1, \dots, \mathtt{X}_k\}$ *is determined by the class of all frames which satisfy all conditions for formulas* ($\mathtt{X}_1$), ..., ($\mathtt{X}_k$).[7]

Remark 1.2. For the pair $\{\mathtt{Alt}_n, \mathtt{Talt}_m\}$ we obtain the following condition:

- $\forall_{x \in W}(\mathrm{Card} R[x] \leqslant n$ and either $x\,R\,x$ or $\mathrm{Card} R[x] \leqslant m)$.

Therefore if $m \geqslant n$ we get the condition $(\mathrm{xi})_n$.

Remark 1.3. 1. From Theorem 1.3(5) and the fact ($\star$) we have $\mathrm{KT} = \mathrm{KD} \oplus (\mathtt{T_q})$. Moreover, by using the fact ($\star\star$) we obtain: $\mathrm{KB4} = \mathrm{KB5} = \mathrm{K5} \oplus (\mathtt{T_q})$; so

[7] If $\mathrm{K} \oplus \{\mathtt{X}_1 \dots \mathtt{X}_k\} = \mathtt{For}$ then $\mathrm{K} \oplus \{\mathtt{X}_1 \dots \mathtt{X}_k\}$ is determined by the empty class of frames.

also $\mathrm{KB4} = \mathrm{KB45} = \mathrm{K45} \oplus (\mathtt{T_q})$. Thus, we obtain $\mathrm{S5} := \mathrm{KT5} = \mathrm{KD5} \oplus (\mathtt{T_q}) = \mathrm{KTB4} = \mathrm{KDB5} = \mathrm{KDB4} = \mathrm{KD45} \oplus (\mathtt{T_q})$.

2. From Theorem 1.3(5) and the fact (‡) we have $\mathrm{K} \oplus \{\mathtt{Alt}_1, \mathtt{T_q}\} = \mathrm{K} \oplus (\mathtt{C}_0)$ and $\mathrm{K} \oplus \{\mathtt{Alt}_{n+1}, \mathtt{Talt}_n\} = \mathrm{K} \oplus (\mathtt{C}_n)$, for any $n > 0$.

2. Simplified Semantics for Normal Extension of K45

2.1. Semi-Universal Frames

We say that a relation R in a frame $\langle W, R\rangle$ is *semi-universal* (and we call the frame *semi-universal*) iff $R = W \times A$, where A is a subset of W. Furthermore, if $A \subsetneq W$, then R and $\langle W, R\rangle$ we call *properly semi-universal*. Let **sU** and **psU** be the classes of all semi-universal and properly semi-universal frames, respectively. Note that all empty frames belong to $\mathbf{psU}_{\mathrm{fin}}$.

Lemma 2.1 **(5, Lemma 2.2)**. *For any semi-universal frame $\langle W, R\rangle$:*

1. *R is transitive and Euclidean.*
2. *R is reflexive iff R is universal, i.e. $R = W \times W$.*
3. *R is symmetric iff R is universal or empty, i.e. $R = W \times W$ or $R = \varnothing$.*
4. *R is serial iff R is non-empty, i.e. $R \neq \varnothing$.*

Moreover, for any $n \geqslant 0$, if $R = W \times A$:

5. *If $\mathrm{Card}A = n$ then $\forall_{x\in W}\ \mathrm{Card}R[x] = n$. So $\forall_{x\in W}\ \mathrm{Card}R[x] \leqslant n$ iff $\mathrm{Card}A \leqslant n$.*
6. *$\forall_{x\in W}(x\,R\,x$ or $\mathrm{Card}R[x] \leqslant n)$ iff either $W = A$ or $\mathrm{Card}A \leqslant n$.*
   $\forall_{x\in W}(x\,R\,x$ or $\mathrm{Card}R[x] = n)$ iff either $W = A$ or $\mathrm{Card}A = n$.

Proof. Points 1–4 are obvious. *Ad* 5: Because $R[x] = A$, for any $x \in W$.

Ad 6: $\forall_{x\in W}(x\,R\,x$ or $\mathrm{Card}R[x] \leqslant n)$ iff $\forall_{x\in W}(x\,R\,x$ or $\mathrm{Card}A \leqslant n)$ iff $\forall_{x\in W}\, x\,R\,x$ or $\mathrm{Card}A \leqslant n$ iff either R is reflexive or $\mathrm{Card}A \leqslant n$ iff either R is universal or $\mathrm{Card}A \leqslant n$. Similarly for '='. ■

In [5] the following was proved [cf. 5, Lemma 2.1]:

Lemma 2.2. *Let $\langle W, R\rangle$ be a frame. Firstly, for arbitrary $x, y \in W$ we put $x\,R^1\,y := x\,R\,y$ and for any $n > 1$ let: $x\,R^n\,y$ iff there are $y_1, \ldots, y_{n-1} \in W$ such that $x\,R\,y_1,\ \ldots,\ y_{n-1}\,R\,y$. Secondly, let*

$$A^x := \{y \in W \mid x\,R^n\,y \text{ for some } n > 0\},$$
$$W^x := \{x\} \cup A^x, \qquad R^x := R \cap (W^x \times W^x).$$

Then for any $x \in W$:

1. *If $x\,R\,x$ then $A^x = W^x$.*
2. *If there is a $y \in W$ such that $x\,R\,y$ then $A^x \neq \varnothing$.*
3. *If R is transitive then $A^x = R[x]$.*
4. *If R is symmetric then either $A^x = W^x$ or $A^x = \varnothing$.*
5. *Let R be Euclidean. Then if $x\,R\,x$, then $R^x = W^x \times W^x$; otherwise $(W^x \backslash \{x\}) \times (W^x \backslash \{x\}) \subseteq R^x \subseteq W^x \times (W^x \backslash \{x\})$.*
6. *If R is symmetric and Euclidean, then $R^x = W^x \times W^x$ or $R^x = \varnothing$.*
7. *If R is transitive and Euclidean, then both*:
   (a) *either $R^x = W^x \times W^x$ or $R^x = W^x \times (W^x \backslash \{x\})$,*
   (b) *$R^x = W^x \times A^x$.*

Proof. *Ad* 4. Let R be symmetric and $A^x \neq \varnothing$. Then for some $y \in A^x$ we have $x\,R\,y$ and $y\,R\,x$. So $x\,R^2\,x$ and $x \in A^x$.

Ad 5. Let R be Euclidean. Suppose that $x\,R\,x$ and $y, z \in W^x$, i.e., there are $n, m > 0$, $y_1, \ldots, y_{n-1}, z_1, \ldots, z_{m-1} \in W^x$ such that $x\,R\,x\,R\,y_1, \ldots, y_{n-1}\,R\,y_n = y$ and $x\,R\,x\,R\,z_1, \ldots, z_{n-1}\,R\,z_n = z$. Then, by assumption, we obtain: $y_1\,R\,y_1$, $y_1\,R\,x$, $z_1\,R\,z_1$ and $z_1\,R\,x$. So, by induction, we obtain that $x\,R\,y_i$ and $x\,R\,z_i$. Hence $y\,R\,z$. Therefore $W^x \times W^x \subseteq R^x$.

Now suppose that $y, z \in W^x \backslash \{x\}$. Then, firstly, there are $n, m > 0$, $y_1, \ldots, y_{n-1}, z_1, \ldots, z_{m-1} \in W^x$ such that $x\,R\,y_1, \ldots, y_{n-1}\,R\,y_n = y$ and $xRz_1, \ldots, z_{n-1}Rz_n = z$. Then y_1Rz_1 and, by induction, y_iRz_j. So yRz. Thus $(W^x \backslash \{x\}) \times (W^x \backslash \{x\}) \subseteq R^x$. Secondly, if $\langle x, x\rangle \notin R$, then $R^x \neq W^x \times W^x$. Hence, if $y\,R^x\,z$, then $z \neq x$. Therefore $R^x \subseteq W^x \times W^x \backslash \{x\}$.

Ad 6. Let R be symmetric and Euclidean. If $A^x = \varnothing$, then $W^x = \{x\}$ and so either $R^x = \{x\} \times \{x\}$ or $R^x \subseteq \{x\} \times \varnothing = \varnothing$, by (5). If $A^x \neq \varnothing$, then—as in the proof of (4)—we have $x\,R^2\,x$. So also $x\,R\,x$, since R is also transitive. Hence $R^x = W^x \times W^x$, by (5).

Ad 7. Let R be transitive and Euclidean. For (a): If $R^x \neq W^x \times W^x$ then $\langle x, x\rangle \notin R$, by (5). Now suppose that $y, z \in W^x$ and $z \neq x$. Then, by (3), $x\,R\,z$ and either $x = y$ or $x\,R\,y$. Hence $y\,R\,z$, since R is Euclidean. Thus, $W^x \times (W^x \backslash \{x\}) \subseteq R^x$. So $R^x = W^x \times (W^x \backslash \{x\})$, by (5).

For (b): If $x \notin A^x$ then $A^x = W^x \backslash \{x\}$. If $x \in A^x$ then $A^x = W^x$. So in both cases $R^x = W^x \times A^x$, by (a). ■

By Lemmas 2.1 and 2.2 we get the following [cf. 5, Corollary 2.3]:

Corollary 2.3. *For an arbitrary frame $\langle W, R\rangle$ and $x \in W$:*

1. *If R is reflexive, transitive and Euclidean, then $\langle W^x, R^x\rangle \in \mathsf{U}$.*
2. *If R is transitive and Euclidean, then $\langle W^x, R^x\rangle \in \mathsf{sU}$.*

3. *If R is symmetric and transitive (so also Euclidean), then $R^x = \varnothing$ or $\langle W^x, R^x \rangle \in \mathsf{U}$.*
4. *If R is serial and transitive (Euclidean), then $\langle W^x, R^x \rangle \in \mathsf{sU}^+$.*

Given the above facts we notice that classes of semi-universal models for K45, KB4, KD45 and these logics extended by $(\mathtt{Q})$, $(\mathtt{T_q})$, $(\mathtt{Alt}_n)$ and/or $(\mathtt{Talt}_m)$ are connected with some classes of generated models. We make use of models generated from relational models [cf. 1, p. 97]. Let $\mathscr{M} = \langle W, R, V \rangle$ and $x \in W$. Then the model generated by $x \in W$ is the model $\mathscr{M}^x = \langle W^x, R^x, V^x \rangle$ in which W^x and R^x are as in Lemma 2.2 and for all $\alpha \in \mathtt{At}$ and $y \in W^x$ we have $V^x(\alpha, y) = V(\alpha, y)$. Of course, V^x preserves classical conditions for truth-value operators and satisfies condition $(\mathrm{V}^R_\Box)$ for R^x. So also for all $\varphi \in \mathtt{For}$ and $y \in W^x$ we have $V^x(\varphi, y) = V(\varphi, y)$. Moreover, φ is true in $\mathscr{M}$ iff for any $x \in W$, φ is true in $\mathscr{M}^x$ [see 1, Theorems 3.10 and 3.11].

For any class $\boldsymbol{M}$ of models we put the following class of generated models $\mathcal{G}(\boldsymbol{M}) := \{\mathscr{M}^x : \mathscr{M} \in \boldsymbol{M}$ and x is in $\mathscr{M}\}$. We have:

FACT 2.4 **(cf. 1, Theorem 3.12)**. *For any $\varphi \in \mathtt{For}$: φ is valid in $\boldsymbol{M}$ iff φ is valid in $\mathcal{G}(\boldsymbol{M})$.*

The following two lemmas will be used later.

LEMMA 2.5. *For arbitrary non-empty sets W and S such that $W \cap S = \varnothing$: if φ is not true in a universal frame on W, then φ is not true in the properly semi-universal frame $\langle W \cup S, (W \cup S) \times W \rangle$.*[8]

As a consequence we get: if φ is true in a non-empty semi-universal frame $\langle W, W \times A \rangle$ then φ is also true in the universal frame on A, i.e., in the frame $\langle A, A \times A \rangle$.

PROOF. Assume that for a model $\mathscr{M} = \langle W, W \times W, V \rangle$ and for some $x \in W$ we have $V(\varphi, x) = 0$. Then there is a model $\mathscr{M}_* = \langle W \cup S, (W \cup S) \times W, V_* \rangle$ such that $V_*(\varphi, x) = 0$. In fact, we construct $v \colon \mathtt{At} \times (W \cup S) \to \{0, 1\}$ such that for any $\alpha \in \mathtt{At}$,

$$v(\alpha, y) := \begin{cases} V(\alpha, y) & \text{if } y \in W, \\ \text{an arbitrary value from } \{0,1\} & \text{if } y \in S. \end{cases}$$

Let $\mathscr{M}_*$ be the model $\langle W \cup S, (W \cup S) \times (W \cup S), V_* \rangle$, where V_* is the extension of v. Obviously, $\mathscr{M}_*^x = \mathscr{M}$. Hence $V_*(\varphi, x) = V_*^x(\varphi, x) = V(\varphi, x) = 0$. ∎

[8]It has already been noticed without proof in [5, p. 170]

Lemma 2.6. *For arbitrary non-empty sets W and S such that $W \cap S = \varnothing$: if φ is not true in a universal frame on W then φ is not true in the universal frame on $W \cup S$.*

As a consequence we get: if φ is true in a universal frame on W and $X \subsetneq W$, then φ is true in the universal frame on $W \backslash X$.

Proof. As in the proof of Lemma 2.5, we construct a model $\mathscr{M}_*$, but now we put for any $\alpha \in \mathtt{At}$

$$v(\alpha, y) := \begin{cases} V(\alpha, y) & \text{if } y \in W, \\ V(\alpha, x) & \text{if } y \in S. \end{cases}$$

Let $\mathscr{M}_*$ be the model $\langle W \cup S, (W \cup S) \times W, V_* \rangle$, where V_* is the extension of v. It is easy to see that for any subformula ψ of φ we have: $V_*(\psi, x) = V_*(\psi, y)$, for any $y \in S$. Hence $V_*(\varphi, x) = V(\varphi, x) = 0$. ■

We have a counterpart of the above lemma for semi-universal frames.

Lemma 2.7. *For all non-empty sets W, A and S such that $A \subsetneq W$ and $W \cap S = \varnothing$: if φ is not true in a properly semi-universal frame $\langle W, W \times A \rangle$, then φ is not true in the properly semi-universal frame $\langle W \cup S, (W \cup S) \times (A \cup S) \rangle$.*

As a consequence of we get: if φ is true in a properly semi-universal frame $\langle W, W \times A \rangle$ and $X \subsetneq A$, then φ is true in the non-empty properly semi-universal frame $\langle W \backslash X, (W \backslash X) \times (A \backslash X)$.

Proof. Assume that for a model $\mathscr{M} = \langle W, W \times A, V \rangle$ and for some $x \in W$ we have $V(\varphi, x) = 0$. Then there is a model $\mathscr{M}_* = \langle W \cup S, (W \cup S) \times (A \cup S), V_* \rangle$ such that $V_*(\varphi, x) = 0$. We consider two cases.

Firstly, if $x \in A$, we construct $v \colon \mathtt{At} \times (W \cup S) \to \{0, 1\}$ as in the proof of Lemma 2.6. Let $\mathscr{M}_*$ be the model $\langle W \cup S, (W \cup S) \times (A \cup S), V_* \rangle$, where V_* is the extension of v. It is easy to see that for any subformula ψ of φ we have: $V_*(\psi, x) = V_*(\psi, y)$, for any $y \in S$. Hence $V_*(\varphi, x) = V(\varphi, x) = 0$.

Secondly, if $x \in W \backslash A$, for a certain $x_0 \in A$ we construct v as above; the only change is that we take x_0 instead of x. It is easy to see that for any subformula ψ of φ we have: $V_*(\psi, x_0) = V_*(\psi, y)$, for any $y \in S$. Hence $V_*(\varphi, x) = V(\varphi, x) = 0$. ■

2.2. Semi-Universal Frames for Normal Extension of K45

For a shorter formulation of theorems we accept the following convention. Let $\mathsf{sU^w}$ be the class of semi-universal frames with $R = W \times (W \backslash \{w\})$, for some $w \in W$. Instead of $\mathsf{sU^w}$ we can take $\{\mathfrak{F}_\varnothing\} \cup \mathsf{sU^{w+}}$. Obviously,

$\langle W,R\rangle \in \mathsf{sU}^{\mathsf{w}+}$ iff $\langle W,R\rangle \in \mathsf{sU}^{\mathsf{w}}$ and $\mathrm{Card}W > 1$. Obviously, all frames from sU^{w} are properly semi-universal.

Note that for any $k \geqslant 2$ the frame $\langle\{1,\dots,k\},\{1,\dots,k\}\times\{2,\dots,k\}\rangle$ belongs to $\mathsf{sU}^{\mathsf{w}+}_{\mathrm{fin}}$. Let $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\mathrm{fin}}$ be the set of all such frames extended by the single frame $\mathfrak{F}_\varnothing$. Furthermore, let $\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}$ be the set of universal frames based on $\{1,\dots,k\}$, for any $k \geqslant 1$. Obviously, by Theorem 1.3(5), the logic S5 is determined by the set $\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}$.

In the light of the facts form above point and Theorem 1.3(5) we get [cf. 5, Theorem 2.5 and a remark at the end of Section 2]:

THEOREM 2.8. 1. K45 *is determined by the following classes*: sU, psU, $\{\mathfrak{F}_\varnothing\}\cup\mathsf{psU}^+$, $\{\mathfrak{F}_\varnothing\}\cup\mathsf{sU}^{\mathsf{w}+}$, $\{\mathfrak{F}_\varnothing\}\cup\mathsf{sU}^{\mathsf{w}+}_{\mathrm{fin}}$, $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\mathrm{fin}}$.

2. KB4 *is determined by the classes*: $\{\mathfrak{F}_\varnothing\}\cup\mathsf{U}$, $\{\mathfrak{F}_\varnothing\}\cup\mathsf{U}_{\mathrm{fin}}$, $\{\mathfrak{F}_\varnothing\}\cup\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}$.

3. KD45 *is determined by the classes*: sU^+, psU^+, $\mathsf{sU}^{\mathsf{w}+}$, $\mathsf{sU}^{\mathsf{w}+}_{\mathrm{fin}}$, $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\mathrm{fin}}\setminus\{\mathfrak{F}_\varnothing\}$.

PROOF. *Ad* 1. In virtue of Theorem 1.3(5), K45 is determined by the class of all transitive Euclidean frames. By Lemma 2.1, this class includes the following sequence of classes: $\mathsf{sU} \supsetneq \mathsf{psU} \supsetneq \{\mathfrak{F}_\varnothing\}\cup\mathsf{psU}^+ \supsetneq \{\mathfrak{F}_\varnothing\}\cup\mathsf{sU}^{\mathsf{w}+}$. Therefore in virtue of Fact 2.4, Corollary 2.3(2), and Lemmas 2.2(7) and 2.5, K45 is determined by $\{\mathfrak{F}_\varnothing\}\cup\mathsf{sU}^{\mathsf{w}+}$. By filtrations, K45 is determined both by $\{\mathfrak{F}_\varnothing\}\cup\mathsf{sU}^{\mathsf{w}+}_{\mathrm{fin}}$ and $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\mathrm{fin}}$.

Ad 2. In virtue of Theorem 1.3(5), KB4 is determined by the class of all symmetric transitive (Euclidian) frames. By Lemma 2.1, this class includes the class $\{\mathfrak{F}_\varnothing\}\cup\mathsf{U}$. Therefore in virtue of Fact 2.4, Corollary 2.3(3) and Lemma 2.2(6), KB4 is determined by $\{\mathfrak{F}_\varnothing\}\cup\mathsf{U}$. By filtrations, KB4 is determined both by $\{\mathfrak{F}_\varnothing\}\cup\mathsf{U}_{\mathrm{fin}}$ and $\{\mathfrak{F}_\varnothing\}\cup\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}$.

Ad 3. In virtue of Theorem 1.3(5), KD45 is determined by the class of all serial transitive Euclidean frames. By Lemma 2.1, this class includes the following sequence of classes: $\mathsf{sU}^+ \supsetneq \mathsf{psU}^+ \supsetneq \mathsf{sU}^{\mathsf{w}+}$. Therefore in virtue of Fact 2.4, Corollary 2.3(4), and Lemmas 2.2(7) and 2.5, KD45 is determined by $\mathsf{sU}^{\mathsf{w}+}$. By filtrations, KD45 is determined both by $\mathsf{sU}^{\mathsf{w}+}_{\mathrm{fin}}$ and $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\mathrm{fin}}\setminus\{\mathfrak{F}_\varnothing\}$. ■

Now we get a new determination theorem for $\mathrm{S5}\oplus(\mathtt{Alt}_n)$ and normal logics of each of the following forms: $\mathrm{K}X\oplus(\mathtt{Alt}_n)$, $\mathrm{K}X\oplus(\mathtt{Talt}_n)$ and $\mathrm{K}X\oplus\{\mathtt{Alt}_n,\mathtt{Talt}_m\}$, where $X = 45, \mathrm{B4}, \mathrm{D45}$ and $n > m$. For $\mathrm{S5}\oplus(\mathtt{Alt}_n)$ see also Theorem 1.2(8).

For a shorter formulation of theorems, for any $n > 0$, let U_n and $\mathsf{U}_{\leqslant n}$ be the sets of universal frames with cardinality equal to n and less than or equal to n (i.e., $\mathrm{Card}W = n$ and $\mathrm{Card}W \leqslant n$), respectively. Now let $\mathsf{U}^{\mathbb{N}}_{\leqslant n}$

be the set of universal frames based on $\{1,\dots,k\}$, for any $k \in \{1,\dots,n\}$. We have $\mathsf{U}^{\mathbb{N}}_{\leqslant n} \subsetneq \mathsf{U}_{\leqslant n}$ and $\mathsf{U}^{\mathbb{N}}_{\leqslant n} \subsetneq \mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}$.

Furthermore, for any $n > 0$, let sU_n and $\mathsf{sU}_{\leqslant n}$ be the set of semi-universal frames having cardinality equal to n and less than or equal to n, respectively. By analogy, we define the appropriate classes of properly semi-universal frames psU_n and $\mathsf{psU}_{\leqslant n}$. We put $\mathsf{sU}^{\mathsf{w}}_n := \mathsf{sU}^{\mathsf{w}} \cap \mathsf{sU}_n$, $\mathsf{sU}^{\mathsf{w}}_{\leqslant n} := \mathsf{sU}^{\mathsf{w}} \cap \mathsf{sU}_{\leqslant n}$ and $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant n} := \mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\mathrm{fin}} \cap \mathsf{sU}^{\mathsf{w}}_{\leqslant n}$, i.e., $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant 1} = \{\mathfrak{F}_{\varnothing}\}$ and for $n \geqslant 2$, $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant n}$ is the set of frames of the form $\langle\{1,\dots,k\},\{1,\dots,k\}\times\{2,\dots,k\}\rangle$, for any $k \in \{2,\dots,n\}$.

Theorem 2.9. *For arbitrary $n, m > 0$:*[9]

1. $\mathrm{S5} \oplus (\mathtt{Alt}_n)$ *is determined by the following classes:* $\mathsf{U}_{\leqslant n}$, $\mathsf{U}^{\mathbb{N}}_{\leqslant n}$ *and* U_n, *as well by the single universal frame based on* $\{1,\dots,n\}$.
2. $\mathrm{K45} \oplus (\mathtt{Alt}_n)$ *is determined by the class of semi-universal frames with* $\mathrm{Card}A \leqslant n$. *Furthermore, this logic is determined by the following classes: the class of properly semi-universal frames with* $\mathrm{Card}A = n$ *extended by the single frame* $\mathfrak{F}_{\varnothing}$, $\mathsf{U}_{\leqslant n} \cup \mathsf{psU}_{\leqslant n+1}$, $\mathsf{psU}_{\leqslant n+1}$, psU_{n+1}, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{psU}^{+}_{\leqslant n+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{psU}^{+}_{n+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{sU}^{\mathsf{w}+}_{\leqslant n+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{sU}^{\mathsf{w}+}_{n+1}$, $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant n+1}$, *as well by the pair of frames,* $\mathfrak{F}_{\varnothing}$ *and* $\langle\{1,\dots,n+1\},\{1,\dots,n+1\}\times\{2,\dots,n+1\}\rangle$.
3. $\mathrm{KB4} \oplus (\mathtt{Alt}_n)$ *is determined by the class of frames which are empty or universal with* $\mathrm{Card}W \leqslant n$. *Furthermore, this logic is determined by the following classes:* $\{\mathfrak{F}_{\varnothing}\} \cup \mathsf{U}_{\leqslant n}$, $\{\mathfrak{F}_{\varnothing}\} \cup \mathsf{U}_n$, $\{\mathfrak{F}_{\varnothing}\} \cup \mathsf{U}^{\mathbb{N}}_{\leqslant n}$, *as well by the pair of frames,* $\mathfrak{F}_{\varnothing}$ *and the universal frame based on* $\{1,\dots,n\}$.
4. $\mathrm{KD45} \oplus (\mathtt{Alt}_n)$ *is determined by the class of semi-universal frames such that* $0 < \mathrm{Card}A \leqslant n$. *Furthermore, this logic is determined by the following classes: the class of properly semi-universal frames with* $\mathrm{Card}A = n$, $\mathsf{U}_{\leqslant n} \cup \mathsf{psU}^{+}_{\leqslant n+1}$, $\mathsf{psU}^{+}_{\leqslant n+1}$, psU^{+}_{n+1}, $\mathsf{sU}^{\mathsf{w}+}_{\leqslant n+1}$, $\mathsf{sU}^{\mathsf{w}+}_{n+1}$, $\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant n+1}\setminus\{\mathfrak{F}_{\varnothing}\}$, *as well by the single frame* $\langle\{1,\dots,n+1\},\{1,\dots,n+1\}\times\{2,\dots,n+1\}\rangle$.
5. $\mathrm{K45}\oplus(\mathtt{Talt}_m)$ *is determined by the class of semi-universal frames which are universal or have* $\mathrm{Card}A \leqslant m$. *Furthermore, this logic is determined by the classes:* $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{U}\cup\mathsf{psU}^{+}_{\leqslant m+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{U}\cup\mathsf{sU}^{\mathsf{w}+}_{\leqslant m+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup \mathsf{U}\cup\mathsf{sU}^{\mathsf{w}+}_{m+1}$, $\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}\cup\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant m+1}$, $(\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}\setminus\mathsf{U}^{\mathbb{N}}_{\leqslant m})\cup\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant m+1}$, *as well by the set*

[9] Note that $\mathrm{S5} = \mathrm{KD45} \oplus (\mathtt{T_q}) = \mathrm{S5} \oplus (\mathtt{T_q}) = \mathrm{S5} \oplus (\mathtt{Talt}_m)$, $\mathrm{KB4} = \mathrm{K45} \oplus (\mathtt{T_q}) = \mathrm{KB4}\oplus(\mathtt{T_q}) = \mathrm{KB4}\oplus(\mathtt{Talt}_m)$ and $\mathrm{KB4}\oplus\{\mathtt{Alt}_n,\mathtt{T_q}\} = \mathrm{KB4}\oplus\{\mathtt{Alt}_n,\mathtt{Talt}_m\} = \mathrm{KB4}\oplus(\mathtt{Alt}_n)$. Furthermore, if $m \geqslant n$ then $\mathrm{K} \oplus \{\mathtt{Alt}_n,\mathtt{Talt}_m\} = \mathrm{K} \oplus (\mathtt{Alt}_n)$.

$\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}\backslash\mathsf{U}^{\mathbb{N}}_{\leqslant m}$ *extended by the pair of frames,* $\mathfrak{F}_{\varnothing}$ *and* $\langle\{1,\ldots,m+1\}, \{1,\ldots,m+1\}\times\{2,\ldots,m+1\}\rangle$.

6. KD45 $\oplus$ $(\mathtt{Talt}_m)$ *is determined by the class of semi-universal frames which are universal or have* $0 < \mathrm{Card}A \leqslant m$. *Furthermore, this logic is determined by the classes*: $\mathsf{U}\cup\mathsf{psU}^{+}_{\leqslant m+1}$, $\mathsf{U}\cup\mathsf{sU}^{\mathsf{w}+}_{\leqslant m+1}$, $\mathsf{U}\cup\mathsf{sU}^{\mathsf{w}+}_{m+1}\backslash\{\mathfrak{F}_{\varnothing}\}$, $\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}\cup\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant m+1}$, $(\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}\backslash\mathsf{U}^{\mathbb{N}}_{\leqslant m})\cup\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant m+1}\backslash\{\mathfrak{F}_{\varnothing}\}$, *as well by the set* $\mathsf{U}^{\mathbb{N}}_{\mathrm{fin}}\backslash\mathsf{U}^{\mathbb{N}}_{\leqslant m}$ *extended by the single frame* $\langle\{1,\ldots,m+1\},\{1,\ldots,m+1\}\times\{2,\ldots,m+1\}\rangle$.

Moreover, if $n > m$:

7. K45$\oplus\{\mathtt{Alt}_n,\mathtt{Talt}_m\}$ *is determined by the class of semi-universal frames which are universal with* $\mathrm{Card}W \leqslant n$ *or have* $\mathrm{Card}A \leqslant m$. *Furthermore, this logic is determined by the following classes*: $\mathsf{U}_{\leqslant n}\cup\mathsf{psU}_{\leqslant m+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{U}_{\leqslant n}\cup\mathsf{psU}^{+}_{\leqslant m+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{U}_{n}\cup\mathsf{psU}^{+}_{m+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{U}_{\leqslant n}\cup\mathsf{sU}^{\mathsf{w}+}_{\leqslant m+1}$, $\{\mathfrak{F}_{\varnothing}\}\cup\mathsf{U}_{n}\cup\mathsf{sU}^{\mathsf{w}+}_{m+1}$, $\mathsf{U}^{\mathbb{N}}_{\leqslant n}\cup\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant m+1}$, *as well by the triple of frames,* $\mathfrak{F}_{\varnothing}$, $\langle\{1,\ldots,n\},\{1,\ldots,n\}\times\{1,\ldots,n\}\rangle$ *and* $\langle\{1,\ldots,m+1\},\{1,\ldots,m+1\}\times\{2,\ldots,m+1\}\rangle$.

8. KD45 $\oplus$ $\{\mathtt{Alt}_n,\mathtt{Talt}_m\}$ *is determined by the class of semi-universal frames which are universal with* $\mathrm{Card}W \leqslant n$ *or have* $0 < \mathrm{Card}A \leqslant m$. *Furthermore, this logic is determined by the classes*: $\mathsf{U}_{\leqslant n}\cup\mathsf{psU}^{+}_{\leqslant m+1}$, $\mathsf{U}_{n}\cup\mathsf{psU}^{+}_{m+1}$, $\mathsf{U}_{\leqslant n}\cup\mathsf{sU}^{\mathsf{w}+}_{\leqslant m+1}$, $\mathsf{U}_{n}\cup\mathsf{sU}^{\mathsf{w}+}_{m+1}$, $\mathsf{U}^{\mathbb{N}}_{\leqslant n}\cup\mathsf{sU}^{\mathsf{w}\mathbb{N}}_{\leqslant m+1}\backslash\{\mathfrak{F}_{\varnothing}\}$, *as well by the pair of frames,* $\langle\{1,\ldots,n\},\{1,\ldots,n\}\times\{1,\ldots,n\}\rangle$ *and* $\langle\{1,\ldots,m+1\},\{1,\ldots,m+1\}\times\{2,\ldots,m+1\}\rangle$.

Remark 2.1. 1. S5$\oplus(\mathtt{C}_0)$ = S5$\oplus\{\mathtt{Alt}_1,\mathtt{T}_{\mathtt{q}}\}$ = S5$\oplus(\mathtt{Alt}_1)$ and for any $n > 0$: S5 $\oplus$ $(\mathtt{C}_n)$ = S5 $\oplus$ $\{\mathtt{Alt}_{n+1},\mathtt{Talt}_n\}$ = S5 $\oplus$ $(\mathtt{Alt}_{n+1})$ (see Remark 1.3).

2. KD45 $\oplus$ $(\mathtt{C}_0)$ = KD45 $\oplus$ $\{\mathtt{Alt}_1,\mathtt{T}_{\mathtt{q}}\}$ = S5 $\oplus$ $(\mathtt{Alt}_1)$ and for any $n \geqslant 1$: KD45 $\oplus$ $(\mathtt{C}_n)$ = KD45 $\oplus$ $\{\mathtt{Alt}_{n+1},\mathtt{Talt}_n\}$. So for any $n \geqslant 1$, the logic KD45$\oplus(\mathtt{C}_n)$ is determined by the pair of frames: $\langle\{1,\ldots,n+1\},\{1,\ldots,n+1\}\times\{1,\ldots,n+1\}\rangle$ and $\langle\{1,\ldots,n+1\},\{1,\ldots,n+1\}\times\{2,\ldots,n+1\}\rangle$.

3. KB4 $\oplus$ $(\mathtt{C}_0)$ = KB4 $\oplus$ $\{\mathtt{Alt}_1,\mathtt{T}_{\mathtt{q}}\}$ = KB4 $\oplus$ $(\mathtt{Alt}_1)$ and for any $n \geqslant 1$: KB4 $\oplus$ $(\mathtt{C}_n)$ = KB4 $\oplus$ $\{\mathtt{Alt}_{n+1},\mathtt{Talt}_n\}$ = KB4 $\oplus$ $(\mathtt{Alt}_{n+1})$.

4. K45 $\oplus$ $(\mathtt{C}_0)$ = K45 $\oplus$ $\{\mathtt{Alt}_1,\mathtt{T}_{\mathtt{q}}\}$ = KB4 $\oplus$ $(\mathtt{Alt}_1)$ and for any $n \geqslant 1$: K45$\oplus(\mathtt{C}_n)$ = K45$\oplus\{\mathtt{Alt}_{n+1},\mathtt{Talt}_n\}$. So for any $n \geqslant 1$, the logic K45$\oplus(\mathtt{C}_n)$ is determined by the triple of frames: $\mathfrak{F}_{\varnothing}$, the universal frame based on $\{1,\ldots,n+1\}$ and the frame $\langle\{1,\ldots,n+1\},\{1,\ldots,n+1\}\times\{2,\ldots,n+1\}\rangle$.

Proof of Theorem 2.9 *Ad* 1. Theorem 1.2(8) says that S5 $\oplus$ $(\mathtt{Alt}_n)$ is determined by the class $\mathsf{U}_{\leqslant n}$. This class includes the classes $\mathsf{U}^{\mathbb{N}}_{\leqslant n}$ and U_n.

Therefore, by Lemma 2.6, this logic is determined by $\mathbf{U}_n$, as well by the single universal frame based on the set $\{1, \ldots, n\}$.

Ad 2. In virtue of Theorem 1.3(5), K45 $\oplus$ ($\mathtt{Alt}_n$) is determined by the class of transitive Euclidean frames such that for any $x \in W$, $\mathrm{Card}R[x] \leqslant n$. Hence, by virtue of Lemmas 2.1(5), 2.2(7), 2.5 and 2.7, Corollary 2.3(2) and Fact 2.4, this logic is determined by the listed classes.

Ad 3. In virtue of Theorem 1.3(5), KB4 $\oplus$ ($\mathtt{Alt}_n$) is determined, respectively, by the class of symmetric Euclidean frames such that for any $x \in W$, $\mathrm{Card}R[x] \leqslant n$. Hence, by Lemmas 2.1(5), 2.2(6) and 2.6, Corollary 2.3(3) and Fact 2.4, this logic is determined by the listed classes.

Ad 4. In virtue of Theorem 1.3(5), KD45 $\oplus$ ($\mathtt{Alt}_n$) is determined by the class of serial transitive Euclidean frames such that for any $x \in W$, $\mathrm{Card}R[x] \leqslant n$. Hence, by Lemmas 2.1(5), 2.2(7), 2.5 and 2.7, Corollary 2.3(4) and Fact 2.4, this logic is determined by the listed classes.

Ad 5. In virtue of Theorem 1.3(5), K45 $\oplus$ ($\mathtt{Talt}_m$) is determined by the class of transitive Euclidean frames, where for any $x \in W$ either $x\,R\,x$ or $\mathrm{Card}R[x] \leqslant m$. Hence, by Lemmas 2.1(6), 2.2(7), 2.5 and 2.7, Corollary 2.3(2) and Fact 2.4, this logic is determined by the listed classes.

Ad 6. In virtue of Theorem 1.3(5), KD45 $\oplus$ ($\mathtt{Talt}_m$) is determined by the class of serial transitive Euclidean frames, where for any $x \in W$ either $x\,R\,x$ or $\mathrm{Card}R[x] \leqslant m$. Hence, by virtue of Lemmas 2.1(6), 2.2(7), 2.5 and 2.7, Corollary 2.3(4) and Fact 2.4, this logic is determined by the listed classes.

Ad 7. In virtue of Theorem 1.3(5), K45 $\oplus$ $\{\mathtt{Alt}_n, \mathtt{Talt}_m\}$ is determined by the class of transitive Euclidean frames, where $\forall_{x\in W}\, \mathrm{Card}R[x] \leqslant n$ and $\forall_{x\in W}(x\,R\,x$ or $\mathrm{Card}R[x] \leqslant m)$. So, by Lemma 2.1(5,6), Corollary 2.3(2) and Fact 2.4, this logic is determined by the class of semi-universal frames such that either $\mathrm{Card}A \leqslant m$ or both $W = A$ and $\mathrm{Card}W \leqslant n$. Hence, by virtue of Lemmas 2.2(7), 2.6 and 2.7, this logic is determined by the listed classes.

Ad 8. In virtue of Theorem 1.3(5), KD45 $\oplus$ $\{\mathtt{Alt}_n, \mathtt{Talt}_m\}$ is determined by the class of serial transitive Euclidean frames, where $\forall_{x\in W}\, \mathrm{Card}R[x] \leqslant n$ and $\forall_{x\in W}(x\,R\,x$ or $\mathrm{Card}R[x] \leqslant m)$. So, by Lemma 2.1(5,6), Corollary 2.3(4) and Fact 2.4, this logic is determined by the class of non-empty semi-universal frames where either $\mathrm{Card}A \leqslant m$ or both $W = A$ and $\mathrm{Card}W \leqslant n$. Hence, by virtue of Lemmas 2.2(7), 2.6 and 2.7, this logic is determined by the listed classes. ∎

2.3. Simplified Frames for Normal Extensions of K45

In the light of the following lemma, any semi-universal frame $\langle W, R\rangle$ may be identified with a *simplified frame* of the form $\langle W, A\rangle$, where W is a non-empty set and A is a subset of W. If $A = W$ then we call $\langle W, A\rangle$ *universal.* If $A \neq \varnothing$ then we call $\langle W, A\rangle$ *non-empty.* If $A = \varnothing$ we call $\langle W, \varnothing\rangle$ *empty.* As already mentioned in footnote , empty semi-universal frames and empty frames are identical. Instead of empty frames we can use the empty frame $\mathfrak{F}_\varnothing$.

On any simplified frame $\langle W, A\rangle$ we construct a *simplified model* $\langle W, A, V\rangle$, where V is a function which to any pair built out of a formula and a world from W assigns a truth-value with respect to A. More precisely, $V\colon \mathtt{For} \times W \to \{0, 1\}$ preserves classical conditions for truth-value operators and for any $\varphi \in \mathtt{For}$ and $x \in W$:

$$(\mathrm{V}^A_\Box) \qquad V(\Box\varphi, x) = 1 \quad \text{iff} \quad \forall_{y\in A}\, V(\varphi, y) = 1.$$

Lemma 2.10 **(5, Lemma 2.6)**. *Let W be a non-empty set, $A \subseteq W$ and $v\colon \mathtt{At} \times W \to \{0, 1\}$. Moreover,*

- *let $\langle W, W \times A, V\rangle$ be a semi-universal model in which V is the extension of v by conditions for truth-value operators and $(\mathrm{V}^R_\Box)$ for $R = W \times A$;*
- *let $\langle W, A, V'\rangle$ be a simplified model in which V' is the extension of v by classical conditions for truth-value operators and $(\mathrm{V}^A_\Box)$ for A.*

Then $V = V'$. Thus, the semi-universal model $\langle W, W \times A, V\rangle$ may be identified with the simplified model $\langle W, A, V'\rangle$.

In the light of Theorem 2.8 and Lemma 2.10 we obtain that the logics K45, KB4 (= KB5) and KD45 are determined by suitable special classes of simplified frames [see 5, Theorem 2.5]. Simply, in Theorem 2.8 we replace a given class $\boldsymbol{C}$ of semi-universal frames with the following class $\mathsf{S}_{\boldsymbol{C}} := \{\langle W, A\rangle : \langle W, W \times A\rangle \in \boldsymbol{C}\}$ of simplified frames.

Moreover, in virtue of Theorem 2.9 and Lemma 2.10 we obtain that also logics from the theorem are determined by special classes of simplified frames. Again it is enough to replace the term ‘semi-universal’ with the term ‘simplified’ and the class $\boldsymbol{C}$ of semi-universal frames with the class $\mathsf{S}_{\boldsymbol{C}}$ of suitable simplified frames. If $\boldsymbol{C}$ is a class composed of universal frames, then as a name of $\mathsf{S}_{\boldsymbol{C}}$ we can take the same name as for $\boldsymbol{C}$. In other cases, if $\boldsymbol{C}$ has one of the names used in Theorems 2.8 and 2.9, then the name of $\mathsf{S}_{\boldsymbol{C}}$ can be obtained by replacing ‘**sU**’ (resp. ‘**psU**’) with ‘**S**’ (resp. ‘**pS**’).

Obviously, the logics S5, Triv and Ver are also determined by special classes of simplified frames (cf. Theorem 1.3): S5 is determined by the class of finite universal simplified frames; Triv and Ver are determined by the single universal simplified frame $\mathfrak{F}_1$ and the single empty frame $\mathfrak{F}_\varnothing$, respectively.

3. Versions of Nagle's Fact for the Remaining Logics

In [5] to each of logics K45, KB4 (= KB5) and KD45 is assigned a suitable class consisting of finite semi-universal frames which satisfy conditions for normal extensions of K5 presented by Nagle in [2].

NAGLE'S FACT **(2, p. 325)**. *Every normal logic containing* (5) *is determined by a set consisting of finite Euclidean frames* $\langle W, R\rangle$ *which satisfy one and only one of the following conditions*:

(a) *W is a singleton and $R = \varnothing$,*

(b) *$R = W \times W$,*

(c) *there is a unique "initial" world $w \in W$ such that $(W\backslash\{w\}) \times (W\backslash\{w\})$ is included in R and $w\,R\,x$, for some $x \in W\backslash\{w\}$.*

For all normal extensions of the logic K45 condition (c) can be replaced by the following:

(c′) W is not a singleton and there is a $w \in W$ such that $R = W \times (W\backslash\{w\})$.

LEMMA 3.1. 1. *Every frame satisfying* (c′) *also satisfies* (c).

2. *Every properly semi-universal frame satisfying* (c) *also satisfies* (c′).

PROOF. *Ad* 1. Suppose that $\langle W, R\rangle$ satisfies (c′). Then $W\backslash\{w\} \neq \varnothing$, the product $(W\backslash\{w\}) \times (W\backslash\{w\})$ is included in the product $W \times (W\backslash\{w\})$ and $w\,R\,x$, for any $x \in W\backslash\{w\}$. So $\langle W, R\rangle$ satisfies (c).

Ad 2. Suppose that $\langle W, R\rangle$ satisfies (c) and $R = W \times A$, for some $A \subsetneq W$. Then $W\backslash\{w\} \neq \varnothing$; and so W is not a singleton. Moreover, $W\backslash\{w\} \subseteq A \subsetneq W$. So also $A \subseteq W\backslash\{w\}$. ■

Notice that only one of conditions (a), (b), (c′) can be met. Therefore, instead of 'satisfy one and only one of conditions (a), (b) and (c′)' we may just write 'satisfy one of conditions (a), (b) and (c′)'. Obviously, instead of empty frames satisfying condition (a) we can use the single frame $\mathfrak{F}_\varnothing$, the frames satisfying (b) are universal and the properly semi-universal frames satisfying (c′) are the frames from $\mathbf{sU}^{\mathbf{w}+}$.

In the light of Nagle's Fact we obtain that a normal logic is a normal extension of K45 iff it is determined by a subclass of the class of all semi-universal frames. Moreover, also for KB4, KD45 and S5 we obtain similar results with respect to suitable classes of semi-universal or universal frames.

THEOREM 3.2. 1. *A normal extension of* K45 *is determined by a subclass of $\mathbf{sU}_{\mathrm{fin}}$. Furthermore, a normal extension of* K45 *is determined by suitable subclasses of the classes*: $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\mathrm{fin}} \cup \mathbf{sU}^{\mathbf{w}+}_{\mathrm{fin}}$ *and* $\mathbf{U}^{\mathbb{N}}_{\mathrm{fin}} \cup \mathbf{sU}^{\mathbf{w}\mathbb{N}}_{\mathrm{fin}}$.

2. *A normal extension of* KB4 *is determined by a subclass of the class* $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\text{fin}}$*. Furthermore, a normal extension of* KB4 *is determined by suitable subclass of* $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\text{fin}}^{\mathbb{N}}$.
3. *A normal extension of* KD45 *is determined by a subclass of* $\mathbf{sU}_{\text{fin}}^{+}$*. Furthermore, a normal extension of* KD45 *is determined by suitable subclasses of the following classes*: $\mathbf{U}_{\text{fin}} \cup \mathbf{sU}_{\text{fin}}^{\mathbf{w}+}$ *and* $\mathbf{U}_{\text{fin}}^{\mathbb{N}} \cup \mathbf{sU}_{\text{fin}}^{\mathbf{w}\mathbb{N}} \backslash \{\mathfrak{F}_\varnothing\}$.
4. *A normal extension of* S5 *is determined by a subclass of* $\mathbf{U}_{\text{fin}}$*. Furthermore, a normal extension of* S5 *is determined by a subclass of* $\mathbf{U}_{\text{fin}}^{\mathbb{N}}$.

Remark 3.1. For a normal extension of K45 (resp. KD45) we can not apply such simplifications of the classes of frames as for the logic K45 (resp. KD45) in Theorem 2.8. Indeed, for example, KB4 and S5 are normal extensions of K45; and S5 is a normal extension of KD45.

Proof of Theorem 3.2 *Ad* 1. Let Λ be a normal extension of K45. Then, by virtue of Nagle's Fact, Λ is determined by a subset $\boldsymbol{C}$ of the set of finite Euclidean frames which satisfy one and only one of conditions (a)–(c). We prove that any frame of $\boldsymbol{C}$ is either empty or belongs to $\mathbf{U}_{\text{fin}} \cup \mathbf{sU}_{\text{fin}}^{\mathbf{w}+}$. In fact, if $\langle W, R\rangle \in \boldsymbol{C}$, then R is transitive, because (4) is valid in $\langle W, R\rangle$. We must consider only the case when $\langle W, R\rangle$ satisfies condition (c), i.e., there is a $w \in W$ such that $(W\backslash\{w\}) \times (W\backslash\{w\}) \subseteq R$ and for some $x \in W\backslash\{w\}$ we have $w\,R\,x$. Then W is not a singleton and for any $y \in W\backslash\{w\}$ we have $w\,R\,y$, since $w\,R\,x$ and $x\,R\,y$. So $W \times (W\backslash\{w\}) \subseteq R$.

Now assume for a contradiction that there is a $y \in W\backslash\{w\}$ such that yRw. Then, by the transitivity of R, for any $z \in W\backslash\{w\}$ we have $z\,R\,w$, since $z\,R\,y$ and $y\,R\,w$. Moreover, $w\,R\,w$, since $w\,R\,y$ and $y\,R\,w$. Hence $R = W \times W$. So we obtain a contradiction: R satisfies (b). Thus, $R = W \times (W\backslash\{w\})$.

Ad 2. Let Λ be a normal extension of KB4. Then, by point 1, Λ is determined by a subset $\boldsymbol{C}$ of frames which are empty or belong to $\mathbf{U}_{\text{fin}} \cup \mathbf{sU}_{\text{fin}}^{\mathbf{w}+}$. We prove that any frame of $\boldsymbol{C}$ is either empty or universal. In fact, if $\langle W, R\rangle \in \boldsymbol{C}_\Lambda$, then R is symmetric and transitive, because (B) and (4) are valid in $\langle W, R\rangle$. Assume for a contradiction that $\langle W, R\rangle$ satisfies condition (c′), i.e., W is not a singleton and there is a unique $w \in W$ such that $R = W \times (W\backslash\{w\})$. But $w\,R\,w$, since for any $x \in W\backslash\{w\}$ we have $w\,R\,x$ and $x\,R\,w$. So we obtain a contradiction.

Ad 3. Let Λ be a normal extension of KD45. Then, by point 1, Λ is determined by a subset $\boldsymbol{C}$ of frames which are empty or belong to $\mathbf{U}_{\text{fin}} \cup \mathbf{sU}_{\text{fin}}^{\mathbf{w}+}$. We show that $\boldsymbol{C} \subseteq \mathbf{U}_{\text{fin}} \cup \mathbf{sU}_{\text{fin}}^{\mathbf{w}+}$. In fact, if $\langle W, R\rangle \in \boldsymbol{C}$, then R is serial, because (D) is valid in $\langle W, R\rangle$. Hence $R \neq \varnothing$.

Ad 4. Let Λ be a normal extension of S5. Then, by point 3, Λ is determined by a subset $\boldsymbol{C}$ of $\mathbf{U}_{\mathrm{fin}} \cup \mathbf{sU}^{\mathbf{w}+}_{\mathrm{fin}}$. We show that $\boldsymbol{C} \subseteq \mathbf{U}_{\mathrm{fin}}$. In fact, if $\langle W, R\rangle \in \boldsymbol{C}$, then R is symmetric, because (B) is valid in $\langle W, R\rangle$. Hence $\langle W, R\rangle$ does not satisfy condition (c'). ■

In the light of Theorems 1.2(8), 3.2(2) and 3.2, and Lemma 2.1, we can prove the following:

Theorem 3.3. *For arbitrary $n, m > 0$:*

1. *A normal extension of* S5 $\oplus$ ($\mathtt{Alt}_n$) *is determined by a subset of* $\mathbf{U}_{\leqslant n}$.
2. *A normal extension of* K45 $\oplus$ ($\mathtt{Alt}_n$) *is determined by a subclass of the class of finite semi-universal frames with* Card$A \leqslant n$. *Furthermore, a normal extension of* K45 $\oplus$ ($\mathtt{Alt}_n$) *is determined by suitable subsets of the following sets*: $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\leqslant n} \cup \mathbf{sU}^{\mathbf{w}+}_{\leqslant n+1}$ *and* $\mathbf{U}^{\mathbb{N}}_{\leqslant n} \cup \mathbf{sU}^{\mathbf{w}\mathbb{N}}_{\leqslant n+1}$.
3. *A normal extension of* KB4 $\oplus$ ($\mathtt{Alt}_n$) *is determined by a subclass of the class of finite frames which are empty or universal with* Card$W \leqslant n$. *Furthermore, a normal extension of* K45 $\oplus$ ($\mathtt{Alt}_n$) *is determined by suitable subsets of the following sets*: $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\leqslant n}$ *and* $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}^{\mathbb{N}}_{\leqslant n}$.
4. *A normal extension of* KD45 $\oplus$ ($\mathtt{Alt}_n$) *is determined by a subclass of the class of finite semi-universal frames with* $0 <$ Card$A \leqslant n$. *Furthermore, a normal extension of* KD45 $\oplus$ ($\mathtt{Alt}_n$) *is determined by suitable subsets of the sets*: $\mathbf{U}_{\leqslant n} \cup \mathbf{sU}^{\mathbf{w}+}_{\leqslant n+1}$ *and* $\mathbf{U}^{\mathbb{N}}_{\leqslant n} \cup \mathbf{sU}^{\mathbf{w}\mathbb{N}}_{\leqslant n+1} \setminus \{\mathfrak{F}_\varnothing\}$.
5. *A normal extension of* K45 $\oplus$ ($\mathtt{Talt}_m$) *is determined by a subclass of the class of finite semi-universal frames which are either universal or have* Card$A \leqslant m$. *Furthermore, a normal extension of* K45 $\oplus$ ($\mathtt{Talt}_m$) *is determined by suitable subsets of the following sets*: $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\mathrm{fin}} \cup \mathbf{sU}^{\mathbf{w}+}_{\leqslant m+1}$ *and* $\mathbf{U}^{\mathbb{N}}_{\mathrm{fin}} \cup \mathbf{sU}^{\mathbf{w}\mathbb{N}}_{\leqslant m+1}$.
6. *A normal extension of* KD45 $\oplus$ ($\mathtt{Talt}_m$) *is determined by a subclass of the class of finite semi-universal frames which are either universal or have* $0 <$ Card$A \leqslant m$. *Furthermore, a normal extension of* K45 $\oplus$ ($\mathtt{Talt}_m$) *is determined by suitable subsets of the following sets*: $\mathbf{U}_{\mathrm{fin}} \cup \mathbf{sU}^{\mathbf{w}+}_{\leqslant m+1}$ *and* $\mathbf{U}^{\mathbb{N}}_{\mathrm{fin}} \cup \mathbf{sU}^{\mathbf{w}\mathbb{N}}_{\leqslant m+1} \setminus \{\mathfrak{F}_\varnothing\}$.

Moreover, if $n > m$:

7. *A normal extension of* K45 $\oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$ *is determined by a subclass of the class of finite semi-universal frames which are either universal with* Card$W \leqslant n$ *or have* Card$A \leqslant m$. *Furthermore, a normal extension of* K45 $\oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$ *is determined by suitable subsets of the following sets*: $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\leqslant n} \cup \mathbf{sU}^{\mathbf{w}+}_{\leqslant m+1}$ *and* $\mathbf{U}^{\mathbb{N}}_{\leqslant n} \cup \mathbf{sU}^{\mathbf{w}\mathbb{N}}_{\leqslant m+1}$.

8. *A normal extension of* KD45 $\oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$ *is determined by a subclass of the class of finite semi-universal frames which are either universal with* Card$W \leqslant n$ *or have* $0 <$ Card$A \leqslant m$. *Furthermore, a normal extension of* KD45 $\oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$ *is determined by suitable subsets of the following sets*: $\mathbf{U}_{\leqslant n} \cup \mathbf{sU}^{\mathbf{w}+}_{\leqslant m+1}$ *and* $\mathbf{U}^{\mathbb{N}}_{\leqslant n} \cup \mathbf{sU}^{\mathbf{w}\mathbb{N}}_{\leqslant m+1} \setminus \{\mathfrak{F}_\varnothing\}$.

Proof. *Ad* 1. Let Λ be a normal extension of S5 $\oplus$ ($\mathtt{Alt}_n$). Then Λ is also a normal extension of KB4. So, by Theorem 3.2(2), Λ is determined by a subset of $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\rm fin}$. Let $\langle W, R\rangle$ be a member of this subset. However, $R \neq \varnothing$, because ($\mathtt{D}$) is valid in $\langle W, R\rangle$. Moreover, Card$W \leqslant n$, because ($\mathtt{Alt}_n$) is valid in $\langle W, R\rangle$.

Ad 2. Let Λ be a normal extension of K45 $\oplus$ ($\mathtt{Alt}_n$). Then Λ is also a normal extension of K45. So, by Theorem 3.2(1), Λ is determined by a subset of $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\rm fin} \cup \mathbf{sU}^{\mathbf{w}+}_{\rm fin}$. Let $\langle W, R\rangle$ be a member of this subset. Then $R = W \times A$, where either $A = \varnothing$, or $A = W$, or $A = W\setminus\{w\}$, for some $w \in W$. However, Card$A \leqslant n$, because ($\mathtt{Alt}_n$) is valid in $\langle W, R\rangle$.

Ad 3. Let Λ be a normal extension of KB4 $\oplus$ ($\mathtt{Alt}_n$). Then Λ is also a normal extension of KB4. So, by Theorem 3.2(2), Λ is determined by a subset of $\{\mathfrak{F}_\varnothing\}$ and $\mathbf{U}_{\rm fin}$. Let $\langle W, R\rangle$ be a member of this subset. However, Card$W \leqslant n$, because ($\mathtt{Alt}_n$) is valid in $\langle W, R\rangle$.

Ad 4. Let Λ be a normal extension of KD45 $\oplus$ ($\mathtt{Alt}_n$). Then Λ is also a normal extension of KD45. So, by Theorem 3.2(3), Λ is determined by a subset of $\mathbf{U}_{\rm fin} \cup \mathbf{sU}^{\mathbf{w}+}_{\rm fin}$. The rest as in the proof of point 2.

Ad 5. Let Λ be a normal extension of K45 $\oplus$ ($\mathtt{Talt}_n$). Then Λ is also a normal extension of K45. So, by Theorem 3.2(1), Λ is determined by a subset of $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\rm fin} \cup \mathbf{sU}^{\mathbf{w}+}_{\rm fin}$. Let $\langle W, R\rangle$ be a member of this subset, where $R = W \times A$, for some $A \subseteq W$. Because ($\mathtt{Talt}_n$) is valid in $\langle W, R\rangle$, we have $\forall_{x\in W}(x\,R\,x$ or Card$R[x] \leqslant n)$. Hence, $A = W$ or Card$A \leqslant n$, by Lemma 2.1(6). Moreover, either $A = \varnothing$, or $A = W$, or $A = W\setminus\{w\}$, for some $w \in W$. So either $A = \varnothing$, or $A = W$, or Card$W \leqslant n+1$.

Ad 6. Let Λ be a normal extension of KD45 $\oplus$ ($\mathtt{Talt}_n$). Then Λ is also a normal extension of KD45. So, by Theorem 3.2(3), Λ is determined by a subset of $\mathbf{U}_{\rm fin} \cup \mathbf{sU}^{\mathbf{w}+}_{\rm fin}$. The rest as in the proof of point 5.

Ad 7. Let Λ be a normal extension of K45 $\oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$. Then Λ is also a normal extension of K45. So, by Theorem 3.2(1), Λ is determined by a subset of $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\rm fin} \cup \mathbf{sU}^{\mathbf{w}+}_{\rm fin}$. Let $\langle W, R\rangle$ be a member of this subset, where $R = W \times A$, for some $A \subseteq W$. Because ($\mathtt{Alt}_n$) is valid in $\langle W, R\rangle$, we have Card$A \leqslant n$. Because ($\mathtt{Talt}_m$) is valid in $\langle W, R\rangle$, we have $\forall_{x\in W}(x\,R\,x$ or Card$R[x] \leqslant m)$. Hence, $A = W$ or Card$A \leqslant m$, by Lemma 2.1(6). Moreover, either $A = \varnothing$, or $A = W$, or $A = W\setminus\{w\}$, for some $w \in W$. So either $A = \varnothing$, or both $A = W$ and $\mathrm{Card}W \leqslant n$, or both $A = W$ and $\mathrm{Card}W \leqslant m$, or both $A = W \backslash \{w\}$ and $\mathrm{Card}A \leqslant m$. Hence either $\langle U, R\rangle = \mathfrak{F}_\varnothing$, or $\langle U, R\rangle \in \mathbf{U}_{\leqslant n}$, or $\langle U, R\rangle \in \mathbf{sU}^{\mathbf{w}+}_{\leqslant m+1}$.

Ad 8. Let Λ be a normal extension of $\mathrm{KD45} \oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$. Then Λ is also a normal extension of KD45. So, by virtue of Theorem 3.2(3), Λ is determined by a subset of the set $\mathbf{U}_{\mathrm{fin}} \cup \mathbf{sU}^{\mathbf{w}+}_{\mathrm{fin}}$. The rest as in the proof of point 7. ■

Furthermore, the following fact also occurs.

FACT 3.4. *For arbitrary $n, m > 0$:*

1. *If a logic is determined by a subclass of $\mathbf{sU}_{\mathrm{fin}}$ then it is a normal extension of* K45*.*
2. *If a logic is determined by a subclass of $\{\mathfrak{F}_\varnothing\} \cup \mathbf{U}_{\mathrm{fin}}$ then it is a normal extension of* KB4*.*
3. *If a logic is determined by a subclass of $\mathbf{sU}^{+}_{\mathrm{fin}}$ then it is a normal extension of* KD45*.*
4. *If a logic is determined by a subclass of $\mathbf{U}_{\mathrm{fin}}$ then it is a normal extension of* S5*.*
5. *If a logic is determined by a subclass of $\mathbf{U}_{\leqslant n}$ then it is a normal extension of* $\mathrm{S5} \oplus (\mathtt{Alt}_n)$*.*
6. *If a logic is determined by a subclass of the class of finite semi-universal frames with* $\mathrm{Card}A \leqslant n$*, then it is a normal extension of* $\mathrm{K45} \oplus (\mathtt{Alt}_n)$*.*
7. *If a logic is determined by a subclass of the class of finite frames which are empty or universal with* $\mathrm{Card}W \leqslant n$*, then it is a normal extension of* $\mathrm{KB4} \oplus (\mathtt{Alt}_n)$*.*
8. *If a logic is determined by a subclass of the class of finite semi-universal frames with* $0 < \mathrm{Card}A \leqslant n$*, then it is a normal extension of* $\mathrm{KD45} \oplus (\mathtt{Alt}_n)$*.*
9. *If a logic is determined by a subclass of the class of finite semi-universal frames which are universal or have* $\mathrm{Card}A \leqslant m$*, then it is a normal extension of* $\mathrm{K45} \oplus (\mathtt{Talt}_m)$*.*
10. *If a logic is determined by a subclass of the class of finite semi-universal frames which are universal or have* $0 < \mathrm{Card}A \leqslant m$*, then it is a normal extension of* $\mathrm{KD45} \oplus (\mathtt{Talt}_m)$*.*

Moreover, if $n > m$:

11. *If a logic is determined by a subclass of the class of finite semi-universal frames which are either universal with* $\mathrm{Card} W \leqslant n$ *or have* $\mathrm{Card} A \leqslant m$*, then it is a normal extension of* $\mathrm{K45} \oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$.
12. *If a logic is determined by a subclass of the class of finite semi-universal frames which are universal with* $\mathrm{Card} W \leqslant n$ *or have* $0 < \mathrm{Card} A \leqslant m$*, then it is a normal extension of* $\mathrm{KD45} \oplus \{\mathtt{Alt}_n, \mathtt{Talt}_m\}$.

Proof. *Ad* 1. If a logic is determined by a subclass of $\mathbf{sU}_{\mathrm{fin}}$, then, it is normal and contains (4) and (5), since these formulas are valid in this subclass, by Lemma 2.1(1).

Ad 2. If a logic is determined by a subclass of $\mathfrak{F}_{\varnothing} \cup \mathbf{U}_{\mathrm{fin}}$, then it is normal and contains (4), (5) and (B), since these formulas are valid in this subclass, by Lemma 2.1(1,3).

Ad 3. If a logic is determined by a subclass of $\mathbf{sU}^{+}_{\mathrm{fin}}$, then it is normal and contains (4), (5) and (D), since these formulas are valid in this subclass, by Lemma 2.1(1,4).

Ad 4. If a logic is determined by a subclass of $\mathbf{U}_{\mathrm{fin}}$, then it is normal and contains (T) and (5), since these formulas are valid in this subclass, by Lemma 2.1(1,2).

Ad 5. If a logic is determined by a subset of $\mathbf{U}_{\leqslant n}$, then it is normal and contains (T), (5) and ($\mathtt{Alt}_n$), since these formulas are valid in this subset, by Lemma 2.1(1,2).

Ad 6. If a logic is determined by a subclass of the class of finite semi-universal frames with $\mathrm{Card} A \leqslant n$, then it is normal and contains (4), (5) and ($\mathtt{Alt}_n$), since these formulas are valid in this subclass, by Lemma 2.1(1).

Ad 7. If a logic is determined by a subclass of the class of finite frames which are empty or universal with $\mathrm{Card} W \leqslant n$, then it is normal and contains (B), (4) and ($\mathtt{Alt}_n$), since these formulas are valid in this subclass, by Lemma 2.1(1,3) and the assumption.

Ad 8. If a logic is determined by a finite subclass of the class of semi-universal frames with $0 < \mathrm{Card} A \leqslant n$, then it is normal and contains (D), (4), (5) and ($\mathtt{Alt}_n$), since these formulas are valid in this subclass, by Lemma 2.1(1,4) and the assumption.

Ad 9. If a logic is determined by a subclass of the class of finite semi-universal frames which are universal or have $\mathrm{Card} A \leqslant n$, then it is normal and contains (4), (5) and ($\mathtt{Talt}_n$), since these formulas are valid in this subclass, by Lemma 2.1(1) and the assumption.

Ad 10. If a logic is determined by a subclass of the class of finite semi-universal frames which are universal or have $0 < \mathrm{Card}A \leqslant n$, then it is normal and contains ($\mathtt{D}$), (4), (5) and ($\mathtt{Talt}_n$), since these formulas are valid in this subclass, by Lemma 2.1(1,4) and the assumption.

Ad 11. If a logic is determined by a subclass of the class of finite semi-universal frames which are either universal with $\mathrm{Card}W \leqslant n$ or have $\mathrm{Card}A \leqslant m$, then it is normal and contains (4) and (5), since these formulas are valid in this subclass, by Lemma 2.1(1). Moreover, by the assumption, in any frame of this subclass either ($\mathtt{Alt}_m$) is valid or both ($\mathtt{T}$) and ($\mathtt{Alt}_n$) are valid. So both $\ulcorner(\mathtt{T}) \vee (\mathtt{Alt}_m)\urcorner$ and $\ulcorner(\mathtt{Alt}_n) \vee (\mathtt{Alt}_m)\urcorner$ are valid in all frames of this subclass. Hence also both ($\mathtt{Talt}_m$) and ($\mathtt{Alt}_n$) are valid in this subclass. So ($\mathtt{Talt}_m$) and ($\mathtt{Alt}_n$) belong to Λ.

Ad 12. If a logic is determined by a subclass of the class of finite semi-universal frames which are either universal with $\mathrm{Card}W \leqslant n$ or have $0 < \mathrm{Card}A \leqslant m$, then it is normal and contains ($\mathtt{D}$), (4) and (5), since these formulas are valid in this subclass, by Lemma 2.1(1,4). ■

As already mentioned in the introduction and point 2.3, instead of a semi-universal frame $\langle W, R \times A\rangle$ we can use the simplified frame $\langle W, A\rangle$. So instead of finite frames satisfying condition (c) we can use simplified frames which satisfy the following condition corresponding to (c):

(c$''$) W is not a singleton and there is a $w \in W$ such that $A = W\setminus\{w\}$.

It is easy to show that, as in point 2.3, also in Theorems 3.2 and 3.3 it is enough to replace the term 'semi-universal' with the term 'simplified' and the class $\boldsymbol{C}$ of semi-universal frames with the class $\mathsf{S}_{\boldsymbol{C}}$ of suitable simplified frames. Furthermore, for these simplified frameworks we can use the names proposed on Sect. 2.3.

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