On the Independent Axiomatizability of Modal and Intermediate Logics

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Abstract

This paper gives a solution to the old independent axiomatizability problem by presenting normal modal logics above **K4** and **Grz** and an intermediate logic without independent axiomatizations. Incidentally Blok's problem is solved: the lattices of varieties of topological Boolean and pseudo-Boolean algebras are not strongly atomic. We also study the relationship between independent axiomatizability of intermediate logics and their modal companions above **S4**.

Keywords: Modal logic, intermediate logic, independent axiomatization, differentiated frame.

1 Introduction

This paper gives a solution to an old problem connected with the efforts to describe the lattices of all normal modal and intermediate logics. The problem is as follows:

Does every normal modal or intermediate logic have an independent set of axioms?

For intermediate logics it was formulated by A. Tsytkin in [6], Problem 148.

A way to the negative solution to this problem is opened by the following observation in [5], which is presented here in a form suitable for our purpose:

LEMMA 1.1
Suppose a logic L_1 has an independent axiomatization. Then, for every finitely axiomatizable logic $L_2 \subset L_1$, the interval of logics $[L_2, L_1] = \{L : L_2 \subseteq L \subsetneq L_1\}$ contains an immediate predecessor of L_1, i.e. a logic $L \subset L_1$ which has no extension lying properly between L and L_1.

PROOF. If L_1 is finitely axiomatizable then the existence of an immediate predecessor of L_1 in $[L_2, L_1]$ follows from Zorn's Lemma.

Suppose now that L_1 has an infinite independent set of axioms $\{\varphi_i : i \in \omega\}$. Since L_2 is a finitely axiomatizable sublogic of L_1, there is $n < \omega$ such that L_2 is contained in the logic with the axioms $\varphi_0, \dots, \varphi_n$. Let L_3 be the logic with the axioms $\varphi_0, \dots, \varphi_n, \varphi_{n+2}, \varphi_{n+3}, \dots$. Since the set of L_1's axioms is independent, $L_2 \subset L_3 \subset L_1$ and $\varphi_{n+1} \notin L_3$. And now again Zorn's Lemma provides us with an immediate predecessor of L_1 in the interval $[L_3, L_1]$. ∎

Thus, to prove that there is a logic without an independent axiomatization it suffices to produce a finitely axiomatizable logic L_2 and its proper extension L_1 having no immediate predecessor in the interval $[L_2, L_1]$.

A lattice (e.g. the lattice of extensions of a given logic) is called *strongly coatomic* if its interval $[L_2, L_1]$ with $L_2 \subset L_1$ contains an immediate predecessor of L_1. Blok in [1] proved that the lattice of normal modal logics is not strongly coatomic (more exactly, he showed that the dual lattice of varieties of modal algebras is not *strongly atomic*). However, it seems unlikely that in the interval $[L_2, L_1]$, constructed by Blok and containing no immediate predecessor of L_1, the logic L_2 is finitely axiomatizable; in any case its semantic definition involves the set of squares of natural numbers which can hardly be described by a finite set of axioms.

We will strengthen appropriately Blok's result to construct logics without independent axiomatizations lying above **K4**, **S4**, **Grz** and intuitionistic logic, answering incidentally his question concerning the strong coatomicity of the lattices of intermediate logics and modal logics containing **S4**.

2 Preliminaries

We use standard notions and notation in the realm of non-classical logic. Here we mention only those of them that have variants.

We denote by $\Box^+\varphi$, $\Diamond^+\varphi$, $\Box^n\varphi$ and $\Diamond^n\varphi$ the formulas $\varphi \wedge \Box\varphi$, $\varphi \vee \Diamond\varphi$, $\underbrace{\Box\ldots\Box}_{n}\varphi$ and $\underbrace{\Diamond\ldots\Diamond}_{n}\varphi$, respectively; $\varphi(\psi/p)$ means the result of replacement of all occurrences of the variable p in φ with ψ.

All modal logics in this paper, except those in Section 6, are assumed to be normal, i.e. containing **K** and closed under modus ponens, substitution and necessitation $\varphi/\Box\varphi$. The smallest normal modal logic to contain a logic L and a set of formulas Γ is denoted by $L \oplus \Gamma$. Intermediate logics are consistent extensions of intuitionistic logic **Int** closed under modus ponens and substitution. $L + \Gamma$ means the closure of the set $L \cup \Gamma$ under the latter two rules.

Let L be a logic and Γ, Δ sets of formulas in the language of L. Γ is said to be an *independent set of axioms for L over Δ* if, for every $\Sigma \subseteq \Gamma$, L is the closure of $\Sigma \cup \Delta$ under the postulated inference rules of L iff $\Sigma = \Gamma$. For instance, we can talk about independent axiomatization of an intermediate logic over **Int** or that of a modal logic over **K**. If Γ is an independent set of axioms for L over $\Delta = \emptyset$ then Γ is called an (*absolutely*) *independent set of axioms for L*. A logic L is *independently axiomatizable* (*over Δ*) if there is an independent set of axioms for L (over Δ).

It is clear that the following lemma holds.

LEMMA 2.1
If a logic L is independently axiomatizable over a finitely axiomatizable logic then L is absolutely independently axiomatizable.

As to our semantic apparatus, we use here *differentiated general frames*. Recall that a general frame $\langle \mathfrak{F}, P\rangle$, where $\mathfrak{F} = \langle W, R\rangle$ is a Kripke frame and P a set of possible values in $\mathfrak{F}$, is *differentiated* if, for every two distinct points in W, there is a set in P containing only one of them. For more information on general frames consult [4], from which it follows in particular that every normal modal and intermediate logic is characterized by a class of rooted differentiated general frames.

All our frames are assumed to be transitive. We will define them by drawing diagrams (directed graphs) in which reflexive and irreflexive points are denoted by ∘ and •, respectively, and, for distinct points x and y, xRy means that there is a directed path from x to y. We write $x\overline{R}y$ if xRy or $x = y$. So $\mathfrak{F} = \langle W, R\rangle$ is *rooted* if there is $x \in W$ such that $x\overline{R}y$ for every $y \in W$; in this case x is called a *root* of $\mathfrak{F}$.

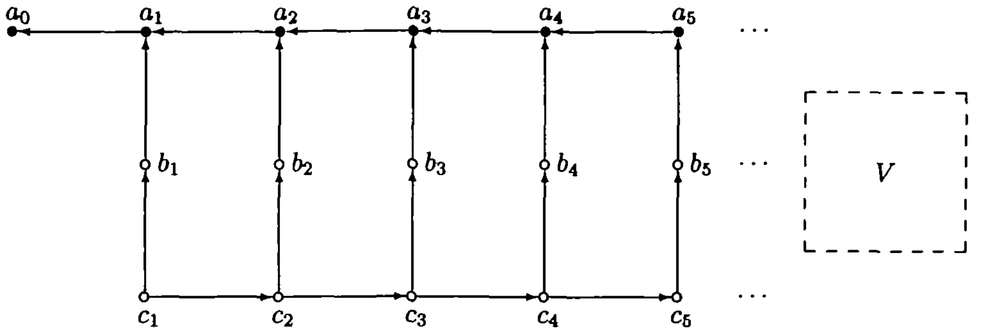

FIG. 1.

3 Modal logics above K4

First we give a solution to the independent axiomatizability problem for modal logics containing **K4**. Though afterwards stronger results will be obtained, we prefer to begin with logics above **K4** because in this case our construction is more transparent.

We require a number of modal formulas:

$$\alpha = p \wedge \neg \Diamond p,\ \alpha' = \alpha(\Diamond p/p),\ \alpha'' = \alpha'(\Diamond p/p) = \alpha(\Diamond^2 p/p),$$

$$\alpha_i = \alpha(\Diamond^i \top/p),\ \alpha_{i+1} = \alpha'(\Diamond^i \top/p),\ \alpha_{i+2} = \alpha''(\Diamond^i \top/p),$$

$$\beta = \Diamond\alpha \wedge \neg \Diamond^+ \alpha',\ \beta' = \beta(\Diamond p/p),$$

$$\beta_i = \beta(\Diamond^i \top/p) = \Diamond\alpha_i \wedge \neg \Diamond^+ \alpha_{i+1},$$

$$\beta_{i+1} = \beta'(\Diamond^i \top/p) = \Diamond\alpha_{i+1} \wedge \neg \Diamond^+ \alpha_{i+2},$$

$$\gamma = \Diamond\beta' \wedge \Diamond\alpha'' \wedge \neg \Diamond \beta,\ \gamma' = \gamma(\Diamond p/p),$$

$$\gamma_{i+1} = \gamma(\Diamond^i \top/p) = \Diamond\beta_{i+1} \wedge \Diamond\alpha_{i+2} \wedge \neg \Diamond \beta_i,$$

$$\gamma_{i+2} = \gamma'(\Diamond^i \top/p) = \Diamond\beta_{i+2} \wedge \Diamond\alpha_{i+3} \wedge \neg \Diamond \beta_{i+1} \quad (i \geq 0).$$

Define L_2 as

$$L_2 = \mathbf{K4} \oplus \{ax1, ax2, ax3, ax4, ax5.\psi :\ \psi \in \{\alpha, \beta, \gamma\}\},$$

where

$$ax1 = \alpha_0 \vee \Diamond^+\alpha_1,\ ax2 = \gamma \to \Diamond\gamma,\ ax3 = \gamma \to \Diamond\gamma',$$

$$ax4 = \Diamond\beta' \wedge \Diamond\alpha'' \to \Diamond\gamma,\ ax5.\psi = \Box^+(q \to \neg\psi) \vee \Box^+(\neg q \to \neg\psi).$$

It is not hard to verify that L_2 is consistent. Indeed, all its axioms are valid in the frame shown in Figure 1 with empty V.

Our first goal is to characterize the constitution of rooted differentiated frames for L_2. To this end we require the following substitution instances of its axioms:

$$ax2.i = \gamma_i \to \Diamond\gamma_i = ax2(\Diamond^i \top/p),$$

$$ax3.i = \gamma_i \to \Diamond\gamma_{i+1} = ax3(\Diamond^i\top/p),$$
$$ax4.i = \Diamond\beta_i \wedge \Diamond\alpha_{i+1} \to \Diamond\gamma_i = ax4(\Diamond^i\top/p) \ \ (i \geq 1),$$
$$ax5.\alpha_i = \Box^+(q \to \neg\alpha_i) \vee \Box^+(\neg q \to \neg\alpha_i) = ax5.\alpha(\Diamond^i\top/p),$$
$$ax5.\beta_i = \Box^+(q \to \neg\beta_i) \vee \Box^+(\neg q \to \neg\beta_i) = ax5.\beta(\Diamond^i\top/p),$$
$$ax5.\gamma_{i+1} = \Box^+(q \to \neg\gamma_{i+1}) \vee \Box^+(\neg q \to \neg\gamma_{i+1}) = ax5.\gamma(\Diamond^i\top/p), \ \ (i \geq 0).$$

For each $n \geq 1$, by $\mathfrak{F}(n, V)$ we denote the rooted subframe of the frame in Figure 1 generated by c_n; $\mathfrak{F}(1, V)$ is that frame itself. Here V is a (possibly empty) set of points which see all a_is and are seen from all c_is (as it follows from the diagram, b_is do not see points in V and are not seen from them); the accessibility relation between points in V is of no concern to us.

Observe that the points a_i, b_{i+1}, c_{i+1}, for $i \geq 0$, are characterized in $\mathfrak{F}(1, V)$ by the formulas α_i, β_{i+1}, γ_{i+1}, respectively, in the sense that under any valuation in $\mathfrak{F}(1, V)$ we have:

$$\{x : x \models \alpha_i\} = \{a_i\}, \ \{x : x \models \beta_{i+1}\} = \{b_{i+1}\}, \ \{x : x \models \gamma_{i+1}\} = \{c_{i+1}\},$$

and the points in V are exactly those points in $\mathfrak{F}(1, V)$ at which all $\Diamond\alpha_i$s are true and all $\Diamond\beta_{i+1}$s are false, for $i \geq 0$.

LEMMA 3.1
Suppose $\langle \mathfrak{F}, P \rangle$ is a rooted differentiated frame for L_2. Then $\mathfrak{F}$ is (isomorphic to) a rooted generated subframe of $\mathfrak{F}(1, V)$, for some V, and $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ are in P, for all points a_i, b_j, c_k in $\mathfrak{F}$.

PROOF. Let z be the root of $\mathfrak{F}$. As was done above, we classify the points in $\mathfrak{F}$ according to which of the formulas α_i, β_i and γ_i are true at them.

Say that a point x in $\mathfrak{F}$ *is of type* a_i (respectively, b_{i+1}, c_{i+1}) if α_i (respectively, β_{i+1}, γ_{i+1}) is true at x; x *is of type* a_ω if $x \models \Diamond\alpha_i$ and $x \not\models \Diamond\beta_j$, for all $i \geq 0$, $j \geq 1$.

Since $\langle \mathfrak{F}, P \rangle \models ax5.\alpha_i$, $\mathfrak{F}$ contains at most one point of type a_i, for each $i \geq 0$. Indeed, suppose there are two distinct points x, y of type a_i. Since $\langle \mathfrak{F}, P \rangle$ is differentiated, there is $X \in P$ such that $x \in X$ and $y \notin X$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking $\mathfrak{V}(q) = X$. Then $z \not\models ax5.\alpha_i$, which is a contradiction. Likewise, for each $i \geq 1$, there are at most one point of type b_i and one point of type c_i.

By the definition of α_i, each point x of type a_i, if any, is irreflexive and must see a point of type a_j, for every $j < i$, and every point accessible from x is of type a_j, for some $j < i$. Therefore, in view of their uniqueness, the points of type a_i, $i \geq 0$, form a descending chain in $\mathfrak{F}$.

By $ax3.i$, each point of type c_i for $i \geq 1$, if any, sees a point of type c_j, for every $j > i$, and, by the definition of γ_i, a point of type a_j, for every $j \geq 0$; besides, by $ax2.i$ and the uniqueness of points of type c_i, every such point is reflexive.

If some point x in $\mathfrak{F}$ sees a point of type a_i and neither sees a point of type a_{i+1} nor is of type a_{i+1} itself then, by the definition of β_i, x is of type b_i. Besides, by $ax1$, $ax4.i$ and the properties of points of types c_j and a_j established above, every point accessible from x is of one of the types $a_0, \ldots, a_i, b_i$. It follows in particular that x is reflexive. For if x is irreflexive then either it sees only points of types $a_0, \ldots, a_i$ and so is of type a_{i+1} itself, contrary to our assumption, or sees a point of type b_i, contrary to the uniqueness of such a point.

It should be clear from the arguments above that each point in $\mathfrak{F}$ is of at most one type. We show now that each point in $\mathfrak{F}$ *is* of some type indeed.

Let x be an arbitrary point in $\mathfrak{F}$. By $ax1$, among the points y such that $x\overline{R}y$ there is at least one point of type a_i, for some $i \geq 0$. If x sees only finitely many points of type a_i, $i \geq 0$, then,

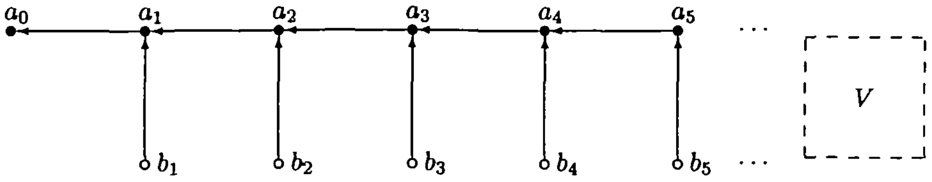

FIG. 2.

as was established above, x is either of type a_i or of type b_i, for some i. If x sees points of type a_i for all $i \geq 0$ then we have the following alternatives. First, x sees no point of type b_j, for $j \geq 1$, which means that x is of type a_ω. Second, x sees a point of type b_j, for some $j \geq 1$, and no point of type b_k, for $0 < k < j$, which means that x of type c_j. We have exhausted all the possibilities, and so each point in $\mathfrak{F}$, in particular z, is of some unique type.

The isomorphism we are after is quite clear now: we map every point of type a_i (respectively, b_{i+1}, c_{i+1}) to a_i (respectively, b_{i+1}, c_{i+1}). The uniqueness of points of types a_i, b_{i+1} and c_{i+1} guarantees that P satisfies the desirable condition. ∎

Now we are in a position to define L_1. Let $\mathcal{C}_1$ be the class of all differentiated frames for L_2 whose underlying Kripke frames have the form shown in Figure 2. Since $\mathfrak{F}(1,\emptyset) \models L_2$ and the frame in Figure 2 with empty V is a generated subframe of $\mathfrak{F}(1,\emptyset)$, $\mathcal{C}_1 \neq \emptyset$. We define L_1 as the logic characterized by the class $\mathcal{C}_1$, i.e. put

$$L_1 = \{\varphi : \forall \mathfrak{F} \in \mathcal{C}_1\ \mathfrak{F} \models \varphi\}.$$

LEMMA 3.2
(i)$L_2 \subset L_1$.
(ii) L_1 has no immediate predecessor in the interval $[L_2, L_1]$.

PROOF. (i) By the definition of L_1, we have $L_2 \subseteq L_1$. This inclusion is proper, since $\neg\gamma_1 \in L_1 - L_2$.

(ii) Suppose otherwise. Let L be an immediate predecessor of L_1 containing L_2. Since $L \subset L_1$, there exists a rooted differentiated frame $\langle \mathfrak{F}, Q\rangle$ such that $\langle \mathfrak{F}, Q\rangle \models L$ and $\langle \mathfrak{F}, Q\rangle \not\models L_1$. On the other hand, since $L_2 \subseteq L$, we have $\langle \mathfrak{F}, Q\rangle \models L_2$ and so, by Lemma 3, $\langle \mathfrak{F}, Q\rangle$ is of the form $\langle \mathfrak{F}(n, V), P\rangle$, for some $n \geq 1$, V and P. Then $\neg\gamma_n \notin L$; for, as we know, $c_n \models \gamma_n$.

Let $\mathcal{C}'$ be the class of frames containing all the frames in $\mathcal{C}_1$ and also the subframe of $\langle \mathfrak{F}(n, V), P\rangle$ generated by c_{n+1}, and let L' be the logic characterized by $\mathcal{C}'$. By the definition, $L \subseteq L' \subseteq L_1$. Moreover, $\langle \mathfrak{F}(n+1, U), Q\rangle \models \neg\gamma_n$, for every U and Q, from which $\neg\gamma_n \in L'$, and $c_{n+1} \models \gamma_{n+1}$, from which $\neg\gamma_{n+1} \notin L'$, while $\neg\gamma_{n+1} \in L_1$. Therefore, $L \subset L' \subset L_1$, contrary to L being an immediate predecessor of L_1. ∎

As a consequence of Lemmas 1.1 and 3.2 and the fact that L_2 is finitely axiomatizable we obtain our main result:

THEOREM 3.3
L_1 has no independent axiomatization.

REMARK 3.4
It is worth noting that L_1 is recursively axiomatizable. Indeed, using Lemma 3.1 one can readily

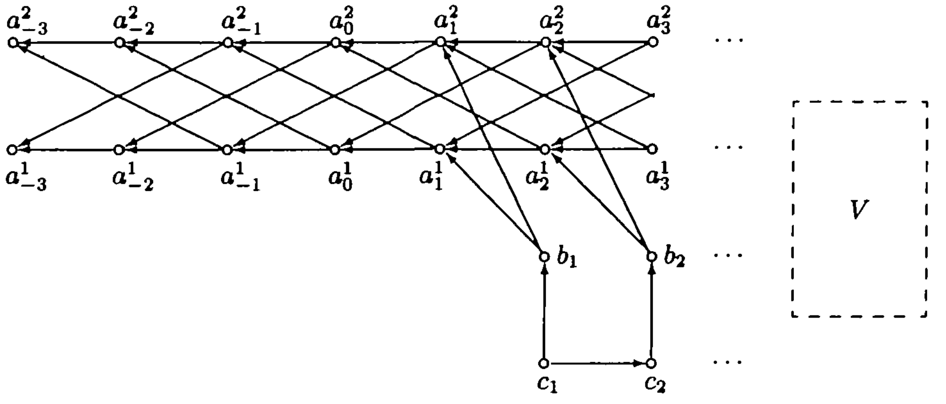

FIG. 3.

prove that

$$L_1 = L_2 \oplus \{\neg\gamma_i : i \geq 1\}.$$

4 Intermediate logics

Now we show how to modify the construction above in order to obtain much stronger logics without independent axiomatizations. First we consider intermediate logics.

The construction in Section 3 was based upon the frame in Figure 1 containing the descending chain $a_0, a_1, \dots$ of irreflexive points. We replace it with 'Fine's ladder' (cf. [3], p.26) consisting of the pairs of reflexive points $a_0^1, a_0^2, a_1^1, a_1^2, \dots$; see Figure 3 where the points a^1_{-i} and a^2_{-i}, for $i = 1, 2, 3$, play an auxiliary role.

Since in the case under consideration variable free formulas are not expressive enough—there are only two of them (up to equivalence, of course), namely, $\bot$ and $\top$—we shall use as a 'starting formula' the following one:

$$\delta = (p \to q \vee \neg q) \vee (\neg p \to q \vee \neg q).$$

It is not hard to see that a rooted Kripke frame $\mathfrak{F}$ refutes δ iff it contains a (not necessarily generated) subframe of the form shown in Figure 4, with a and b having no common successors in $\mathfrak{F}$. Since the frame in Figure 3 contains only one (modulo interchanging superscripts) subframe of that sort, without loss of generality we may assume that under any valuation refuting δ in the frame we have:

$$a^1_{-2} \models p,\ a^1_{-2} \not\models q \vee \neg q,\ a^1_{-3} \models q,$$
$$a^2_{-2} \models \neg p,\ a^2_{-2} \not\models q \vee \neg q,\ a^2_{-3} \models q.$$

Now, taking the formulas

$$\alpha^1_{-3} = p \wedge q \to \bot,\ \alpha^2_{-3} = \neg p \wedge q \to \bot,$$

$$\alpha^1_{-2} = p \to q \vee \neg q,\ \alpha^2_{-2} = \neg p \to q \vee \neg q,$$
$$\alpha^1_{i+1} = \alpha^2_i \to \alpha^1_i \vee \alpha^2_{i-1},\ \alpha^2_{i+1} = \alpha^1_i \to \alpha^2_i \vee \alpha^1_{i-1}\ \ (i \geq -2),$$

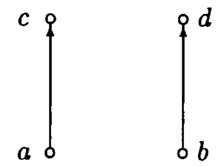

FIG. 4.

$$\beta_i = \alpha^1_{i+1} \wedge \alpha^2_{i+1} \to \alpha^1_i \vee \alpha^2_i \ (i \geq -3),$$
$$\gamma_{i+1} = \beta_i \to \beta_{i+1} \vee \alpha^1_{i+2} \vee \alpha^2_{i+2} \ (i \geq 0),$$

we obtain, under a valuation refuting δ, a classification of points in the frame in Figure 3 similar to that in Section 3:

$$\{x : x \Vdash\!\!\not\models \alpha^1_i\} = \{a^1_i\},\ \{x : x \Vdash\!\!\not\models \alpha^2_i\} = \{a^2_i\}\ (i \geq -3),$$
$$\{x : x \Vdash\!\!\not\models \beta_i\} = \begin{cases} \{b_i\} & \text{if } i \geq 1 \\ \emptyset & \text{if } -3 \leq i \leq 0, \end{cases}$$
$$\{x : x \Vdash\!\!\not\models \gamma_i\} = \{c_i\}\ (i \geq 1).$$

Here $x \Vdash\!\!\not\models \varphi \to \psi$ means $x \models \varphi$ and $x \not\models \psi$. It follows that

$$\{x : a^1_{-3}Rx\} = \{x : x \models p \wedge q\},\ \{x : a^2_{-3}Rx\} = \{x : x \models \neg p \wedge q\},$$
$$\{x : a^1_iRx\} = \{x : x \models \alpha^2_{i-1}\},\ \{x : a^2_iRx\} = \{x : x \models \alpha^1_{i-1}\}\ (i \geq -2),$$
$$\{x : b_1Rx\} = \{x : x \models \alpha^1_2 \wedge \alpha^2_2\},\ \{x : b_iRx\} = \{x : x \models \alpha^1_{i+1} \wedge \alpha^2_{i+1} \wedge \beta_1 \wedge \ldots \wedge \beta_{i-1}\},$$
$$\{x : c_1Rx\} = \{x : x \models \top\},\ \{x : c_iRx\} = \{x : x \models \beta_1 \wedge \ldots \wedge \beta_{i-1}\}\ (i \geq 2).$$

Therefore, every general frame based on this Kripke frame and refuting δ contains in its set of possible values the sets generated by each of the points a^i_j, b_k, c_l.

L_2 can be defined now by adding to **Int** the following axioms, for $i = 1, 2$ and $-3 \leq j \leq 1$:

$$Ax1.1 = \neg(\alpha^1_{-3} \wedge \alpha^2_{-3}) \vee \delta,$$
$$Ax1.2 = (\alpha^1_{-2} \wedge \alpha^2_{-2} \to \alpha^1_{-3} \vee (p \wedge q)) \vee \delta,$$
$$Ax1.3 = (\alpha^1_{-2} \wedge \alpha^2_{-2} \to \alpha^2_{-3} \vee (\neg p \wedge q)) \vee \delta,$$
$$Ax1.4 = (\beta_{-2} \wedge \beta_{-1} \wedge \beta_0) \vee \delta,$$
$$Ax3 = \zeta_2 \to \zeta_1 \vee \delta,\ \ Ax4 = \zeta_1 \to \eta_1 \vee \xi_2 \vee \xi'_2 \vee \delta,$$
$$Ax5.\alpha^i_j = \phi(\alpha^i_j),\ \ Ax5.\xi_2 = \phi(\xi_2),\ \ Ax5.\eta_1 = \phi(\eta_1),\ \ Ax5.\zeta_1 = \phi(\zeta_1),$$

where

$$\xi_{-3} = r_1,\ \xi'_{-3} = r_2,\ \xi_{-2} = s_1,\ \xi'_{-2} = s_2,$$
$$\xi_n = \xi'_{n-1} \to \xi_{n-1} \vee \xi'_{n-2},\ \xi'_n = \xi_{n-1} \to \xi'_{n-1} \vee \xi_{n-2}\ (n \geq -1),$$
$$\eta_n = \xi_{n+1} \wedge \xi'_{n+1} \to \xi_n \vee \xi'_n\ (n \geq 0),$$
$$\zeta_n = \eta_{n-1} \to \eta_n \vee \xi_{n+1} \vee \xi'_{n+1}\ (n \geq 1)$$

and $\phi(\varphi \to \psi)$ is an abbreviation for $(r \wedge \varphi \to \psi) \vee (\varphi \to r \vee \psi) \vee \delta$. The meaning and purpose of the axioms above are analogous to those of the corresponding axioms in Section 3.

LEMMA 4.1
Every frame validating δ is a frame for L_2.

PROOF. For each axiom φ of L_2, we have $\delta \to \varphi \in \mathbf{Int}$. ∎

It follows in particular that L_2 is consistent.

Given a formula φ, we denote by φ_i^* the result of substituting in φ the formulas α_i^1, α_i^2, α_{i+1}^1, α_{i+1}^2 $(i \geq -3)$ for the variables r_1, r_2, s_1, s_2, respectively. Observe that the formulas ξ_n, ξ_n', η_n, ζ_n are related to α_i^1, α_i^2, β_i, γ_i in the following way.

LEMMA 4.2
(i) $(\xi_n)_i^* \leftrightarrow \alpha_{n+i+3}^1 \in \mathbf{Int}$, $(\xi_n')_i^* \leftrightarrow \alpha_{n+i+3}^2 \in \mathbf{Int}$, for $n \geq -3$.
(ii) $(\eta_n)_i^* \leftrightarrow \beta_{n+i+3} \in \mathbf{Int}$, for $n \geq 0$.
(iii) $(\zeta_n)_i^* \leftrightarrow \gamma_{n+i+3} \in \mathbf{Int}$, for $n \geq 1$.

PROOF. (i) is proved by trivial induction on n; (ii) and (iii) follow from (i). ∎

REMARK 4.3
If we mean by φ_i^* the result of substituting in φ the formulas α_i^2, α_i^1, α_{i+1}^2, α_{i+1}^1 $(i \geq -3)$ for r_1, r_2, s_1, s_2, respectively, then Lemma 4.2 will hold with (i) replaced by
(iv) $(\xi_n)_i^* \leftrightarrow \alpha_{n+i+3}^2 \in \mathbf{Int}$, $(\xi_n')_i^* \leftrightarrow \alpha_{n+i+3}^1 \in \mathbf{Int}$, for $n \geq -3$.

As a consequence of Lemma 4.2 and Remark 4.3 we obtain that L_2 contains the following formulas:

$$Ax3.i = \gamma_{i+1} \to \gamma_i \vee \delta \quad (i \geq 1),$$

$$Ax4.i = \gamma_i \to \beta_i \vee \alpha_{i+1}^1 \vee \alpha_{i+1}^2 \vee \delta \quad (i \geq 1),$$

$$Ax5.\alpha_j^i = \phi(\alpha_j^i) \quad (i = 1, 2,\ j \geq -3),$$

$$Ax5.\beta_i = \phi(\beta_i) \quad (i \geq 1),$$

$$Ax5.\gamma_i = \phi(\gamma_i) \quad (i \geq 1).$$

Note by the way that we do not need any counterparts of the formulas $ax2$ and $ax2.i$ from Section 3, because intuitionistic frames are *a priori* reflexive.

LEMMA 4.4
If a rooted differentiated frame $\langle \mathfrak{F}, P \rangle$ for L_2 refutes δ then $\mathfrak{F}$ is (isomorphic to) a generated subframe of a frame of the form shown in Figure 3, with the sets generated by each of its points a_j^i, b_k, c_l belonging to P.

PROOF. The proof is similar to that of Lemma 3.1, so we leave almost all the details to the reader. The main difference concerns the use of δ.

Suppose $\mathfrak{F} = \langle W, R \rangle$ and z is the root of $\mathfrak{F}$ refuting δ under a valuation $\mathfrak{V}$ of the variables p and q in $\langle \mathfrak{F}, P \rangle$. As was observed above, $\mathfrak{F}$ must contain a subframe shown in Figure 4, with a and b having no common successors in $\mathfrak{F}$, and

$$c \not\Vdash \alpha_{-3}^1,\ d \not\Vdash \alpha_{-3}^2,\ a \not\Vdash \alpha_{-2}^1,\ b \not\Vdash \alpha_{-2}^2.$$

(As in the proof of Lemma 3.1, one might call c, d, a, b points of types a_{-3}^1, a_{-3}^2, a_{-2}^1, a_{-2}^2, respectively.)

Let us show first that each point x in $\mathfrak{F}$ sees c or d. Indeed, suppose otherwise. Since $z \not\models \delta$ and $\langle \mathfrak{F}, P \rangle \models Ax1.1$, we have $z \models \neg(\alpha_{-3}^1 \wedge \alpha_{-3}^2)$ and so $x \not\models \alpha_{-3}^1$ or $x \not\models \alpha_{-3}^2$. Assume for definiteness that $x \not\models \alpha^1_{-3}$, i.e. there is a point y such that xRy and $y \not\Vdash \alpha^1_{-3}$. Since $\neg xRc$, it follows that $y \neq c$. Then there is a set $X \in P$ such that either $y \in X, c \notin X$ or $y \notin X, c \in X$. In the former case, by taking $\mathfrak{V}(r) = X$ we obtain that $y \not\Vdash r \wedge p \wedge q \to \bot$ and $c \not\Vdash p \wedge q \to r \vee \bot$, whence $z \not\models Ax5.\alpha^1_{-3}$, which is a contradiction. The latter case is considered analogously.

Thus, $\mathfrak{F}$ has exactly two final points, namely c and d, with $x = c$ whenever $x \not\Vdash \alpha^1_{-3}$ and $x = d$ whenever $x \not\Vdash \alpha^2_{-3}$.

Now we show that each point in $\mathfrak{F}$ different from c and d sees a or b. Again we prove this by *reductio ad absurdum*. Suppose $x \neq c$, $x \neq d$, $\neg xRa$ and $\neg xRb$. As was established above, xRc or xRd. Assume for definiteness that xRc. Then $x \not\models \alpha^1_{-3}$. Besides we have $x \not\models p \wedge q$, for otherwise $x \not\Vdash \alpha^1_{-3}$, from which $x = c$. Since $z \models Ax1.2$, we obtain then either $x \not\models \alpha^1_{-2}$ or $x \not\models \alpha^2_{-2}$. In the former case there exists a point y accessible from x such that $y \neq a$ and $y \not\Vdash \alpha^1_{-2}$. Choosing a set in P separating y and a, we can prove in the same way as above that $z \not\models Ax5.\alpha^1_{-2}$. The latter case and the case when xRd are considered analogously. Taking into account the differentiatedness of our frame, we obtain also that $\{c\}, \{d\} \in P$, because $\{c\} = \{x : x \models p \wedge q\}$ and $\{d\} = \{x : x \models \neg p \wedge q\}$.

Observe that if xRa and $\neg xRd$ then $x = a$. Indeed, since c and d are the only final points in $\mathfrak{F}$, we have $x \models \neg\neg p$. Let us change, if necessary, the valuation $\mathfrak{V}$ in such a way that $\mathfrak{V}(p) = \mathfrak{V}(\neg\neg p)$. It should be clear that we still have $z \not\models \delta$ and besides $x \not\Vdash \alpha^1_{-2}$ and $a \not\Vdash \alpha^1_{-2}$. Therefore, in view of $\langle \mathfrak{F}, P\rangle \models Ax5.\alpha^1_{-2}$, we have $x = a$. Likewise we can show that if xRb and $\neg xRc$ then $x = b$.

The axioms $Ax5.\alpha^1_{-2}$ and $Ax5.\alpha^2_{-2}$ guarantee that a and b are the only points satisfying $a \not\Vdash \alpha^1_{-2}$ and $b \not\Vdash \alpha^2_{-2}$, respectively.

Let x be a point in $\mathfrak{F}$ different from a, b, c, d. Then xRa or xRb. Suppose xRa and $\neg xRb$. As was shown above, xRd and so $x \not\Vdash \alpha^1_{-1}$. And using the axiom $Ax5.\alpha^1_{-1}$ we see that x is the only point in $\mathfrak{F}$ with this property. Similarly, if $\neg xRa$ and xRb then x is the unique point such that $x \not\Vdash \alpha^2_{-1}$. Finally, if x sees both a and b then $x \not\models \alpha^1_{-2} \vee \alpha^2_{-2}$ and hence, by $Ax1.4$ (its conjunct β_2, to be more exact), $x \not\models \alpha^1_{-1} \wedge \alpha^2_{-1}$, i.e. x sees a point y such that either $y \not\Vdash \alpha^1_{-1}$ or $y \not\Vdash \alpha^2_{-1}$ and such y is unique.

Since $a \models \alpha^2_{-3}$ and $x \not\models \alpha^2_{-3}$ whenever $\neg aRx$, the subset of W generated by a is $\{x : x \models \alpha^2_{-3}\} \in P$. In the same way one can prove that the subset of W generated by b is also in P.

The rest of the proof resembles that of Lemma 3.1. ∎

Now we define L_1 as the intermediate logic characterized by the class of all differentiated frames validating δ and all differentiated frames for L_2 whose underlying Kripke frames have the form shown in Figure 3, but with the points c_i removed.

LEMMA 4.5
(i) $L_2 \subset L_1$.
(ii) L_1 has no immediate predecessor in the interval $[L_2, L_1]$.

PROOF. (i) $L_2 \subseteq L_1$ is a consequence of Lemma 4.1. That this inclusion is proper will follow from the fact that $\gamma_1 \in L_1 - L_2$. To establish the latter it suffices to show that the Kripke frame $\mathfrak{F} = \langle W, R\rangle$ in Figure 3 with $V = \emptyset$ validates all the axioms of L_2. We verify this claim only for $Ax3$ and $Ax5.\xi_2$.

Suppose that under some valuation in $\mathfrak{F}$ we have $x \not\models \zeta_1 \vee \delta$ and show that $x \not\models \zeta_2$. Since

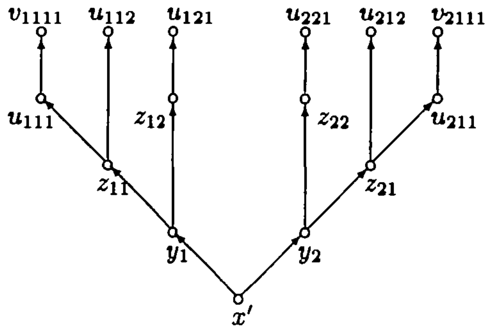

FIG. 5.

$x \not\models \zeta_1$, there is a point x' such that xRx' and

$$x' \models \eta_0, \tag{4.1}$$

$$x' \not\models \eta_1, \tag{4.2}$$

$$x' \not\models \xi_2 \vee \xi_2'. \tag{4.3}$$

The latter provides us with points y_1, y_2 such that $x'Ry_1$, $x'Ry_2$ and

$$y_1 \not\Vdash \xi_2, \quad y_2 \not\Vdash \xi_2'.$$

Clearly, y_1 and y_2 do not see each other and there are points $z_{11}, z_{12}, z_{21}, z_{22}$ such that $y_1 R z_{1i}$, $y_2 R z_{2i}$, for $i = 1, 2$, and

$$z_{11} \not\Vdash \xi_1, \quad z_{12} \not\Vdash \xi_0', \quad z_{21} \not\Vdash \xi_1', \quad z_{22} \not\Vdash \xi_0.$$

So there are points $u_{111}, u_{112}, u_{121}, u_{211}, u_{212}, u_{221}$ such that $z_{11} R u_{11i}$, $z_{12} R u_{121}$, $z_{21} R u_{21i}$, $z_{22} R u_{221}$, for $i = 1, 2$, and

$$u_{111} \not\Vdash \xi_0, \quad u_{112} \not\Vdash \xi_{-1}', \quad u_{121} \not\Vdash \xi_{-1}', \quad u_{211} \not\Vdash \xi_0', \quad u_{212} \not\Vdash \xi_{-1}, \quad u_{221} \not\Vdash \xi_{-1}.$$

The first and the fourth of these conditions give us points v_{1111} and v_{2111} such that $u_{111} R v_{1111}$, $u_{211} R v_{2111}$ and

$$v_{1111} \not\Vdash \xi_{-1}, \quad v_{2111} \not\Vdash \xi_{-1}'.$$

It follows that $\mathfrak{F}$ contains a subframe of the form shown in Figure 5 where the points in each of the following sets are not necessarily distinct:

$$\{u_{111}, z_{22}\}, \quad \{u_{211}, z_{12}\}, \quad \{u_{112}, u_{121}, v_{2111}\}, \quad \{u_{212}, u_{221}, v_{1111}\}.$$

Comparing this diagram with $\mathfrak{F}$ we can conclude that modulo interchanging superscripts one may assume that, for some $i \geq -3$,

$$\{u_{212}, u_{221}, v_{1111}\} = \{a_i^1\}, \quad \{u_{112}, u_{121}, v_{2111}\} = \{a_i^2\},$$

$$\{u_{111}, z_{22}\} = \{a^1_{i+1}\}, \ \{u_{211}, z_{12}\} = \{a^2_{i+1}\},$$
$$z_{11} = a^1_{i+2}, \ z_{21} = a^2_{i+2}, \ y_1 = a^1_{i+3}, \ y_2 = a^2_{i+3}.$$

Thus, for some $i \geq -3$ and $j = 0, 1, 2, 3$, we have

$$a^1_{i+j} \not\Vdash \xi_{j-1}, \ a^2_{i+j} \not\Vdash \xi'_{j-1},$$

from which by induction on j one can show that, for $j \geq -1$,

$$\{x : x \not\Vdash \xi_j\} = \{a^1_{i+j+1}\}, \ \{x : x \not\Vdash \xi'_j\} = \{a^2_{i+j+1}\}. \tag{4.4}$$

Immediately from (4.4) we obtain that

$$\{x : x \not\Vdash \eta_j\} = \{b_{i+j+1}\}, \ \{x : x \not\Vdash \zeta_j\} = \{c_{i+j+1}\}, \tag{4.5}$$

where $j \geq -1$, $i + j + 1 \geq 1$, and if $i + j + 1 \leq 0$ then

$$\{x : x \not\Vdash \eta_j\} = \{x : x \not\Vdash \zeta_j\} = \emptyset.$$

The conditions (4.1)–(4.5) show that

$$\neg x' R b_{i+1}, \ x' R b_{i+2}, \ x' R a^1_{i+3}, \ x' R a^2_{i+3},$$

which is possible only if $x' = c_{i+2}$. Therefore, we have $x R c_{i+2} R c_{i+3}$, $c_{i+3} \not\Vdash \zeta_2$ (by (4.5)) and so $x \not\models \zeta_2$.

Let us consider now $Ax5.\xi_2$ and suppose that under some valuation it is refuted in $\mathfrak{F}$. Using almost the same argument as above we can establish (4.4). So there is only one point x in $\mathfrak{F}$ such that $x \not\Vdash \xi_2$. However, by the refutability of $\phi(\xi_2)$, we must have two points x_1 and x_2 such that

$$x_1 \models r \wedge \xi'_1, \ x_1 \not\models \xi_1 \vee \xi'_0, \ x_2 \models \xi'_1, \ x_2 \not\models r \vee \xi_1 \vee \xi'_0.$$

It follows that $x_1 \neq x_2$, $x_1 \not\Vdash \xi_2$ and $x_2 \not\Vdash \xi_2$, which is a contradiction.
(ii) is proved in the same way as (ii) in Lemma 3.2. ∎

As a consequence of Lemmas 4.5 and 1.1 and the fact that L_2 is finitely axiomatizable, we obtain

THEOREM 4.6
There is an intermediate logic without an independent axiomatization.

Lemma 4.5 provides us with an interval $[L_2, L_1]$ of intermediate logics in which L_1 has no immediate predecessors. This result and the Blok–Esakia Theorem, according to which the lattices of varieties of pseudo-Boolean (alias Heyting) algebras and Grzegorczyk algebras are isomorphic, give a solution to Blok's problem in [1]:

THEOREM 4.7
(i) The lattice of varieties of pseudo-Boolean algebras is not strongly atomic.
(ii) The lattice of varieties of topological Boolean (and even Grzegorczyk) algebras is not strongly atomic.

Slightly modifying the construction above one can prove

THEOREM 4.8

(i) There exists a normal extension of the Gödel–Löb provability logic $\mathbf{GL} = \mathbf{K4} \oplus \Box(\Box p \to p) \to \Box p$ without an independent axiomatization.

(ii) The lattice of varieties of diagonalizable algebras is not strongly atomic.

PROOF. The only essential difference from the proof of Theorem 4.6 is that all reflexive circles $\circ$ in Figure 3 are replaced with irreflexive bullets $\bullet$ and instead of intuitionistic formulas we use their modal counterparts in **GL** (for instance, obtained from them by prefixing $\Box^+$ to their every subformula). The Kripke frame $\mathfrak{F}$ in Figure 3 (with $V = \emptyset$) does not validate the Löb axiom. However, by defining the set of possible values P in $\mathfrak{F}$ as the family of all finite and cofinite subsets in $\mathfrak{F}$, we obtain $\langle \mathfrak{F}, P\rangle \models \Box(\Box p \to p) \to \Box p$. ∎

5 Modal logics above S4

Now we consider the correlation between the independent axiomatizability of intermediate logics and normal modal logics above **S4**. We remind the reader that there is a lattice homomorphism ρ from the lattice of normal extensions of **S4** onto the lattice of extensions of **Int**, which is defined as follows: for every normal logic $M \supseteq \mathbf{S4}$,

$$\rho M = \{\varphi : T\varphi \in M\},$$

where T is the *Gödel translation* prefixing $\Box$ to every subformula of an intuitionistic formula. The logic M is called a *modal companion* of ρM. The set of all modal companions of an intermediate logic $L = \mathbf{Int} + \{\varphi_i : i \in I\}$ forms the infinite interval of logics $[\tau L, \sigma L]$, where

$$\tau L = \mathbf{S4} \oplus \{T\varphi_i : i \in I\} \subseteq \mathbf{S5},$$

$$\sigma L = \tau L \oplus \mathbf{Grz} = \tau L \oplus \Box(\Box(p \to \Box p) \to p) \to p,$$

with σ being the Blok–Esakia isomorphism between the lattices of extensions of **Int** and normal extensions of the Grzegorczyk system **Grz** mentioned at the end of Section 4. It should be clear that if M is an immediate predecessor of τL_1 in the interval $[\tau L_2, \tau L_1]$ then ρM is an immediate predecessor of L_1 in the interval $[L_2, L_1]$. (Indeed, by the definition of τL_1, we have $\rho M \subset L_1$. And if $\rho M \subset \rho M + \varphi \subset L_1$, for some φ, then $T\varphi \notin M$, $M \oplus T\varphi = \tau L_1$ and so, since ρ is a homomorphism, $\rho(M \oplus T\varphi) = \rho M + \varphi = L_1$, which is a contradiction.) For more information on modal companions of intermediate logics and references consult [2].

It follows directly from these facts and Lemma 4.5 that in the intervals $[\tau L_2, \tau L_1]$ and $[\sigma L_2, \sigma L_1]$, where L_1 and L_2 are the intermediate logics constructed in Section 4, the modal logics τL_1 and σL_1 have no immediate predecessors, respectively. Thus we obtain

THEOREM 5.1

There is a normal modal logic in the interval $[\mathbf{S4}, \mathbf{S5}]$ and a normal logic containing **Grz** without independent axiomatizations.

Another consequence of the properties of τ and σ mentioned above is

THEOREM 5.2

For every intermediate logic L, the following conditions are equivalent:

- L is independently axiomatizable over **Int**;
- τL is independently axiomatizable over **S4**;

- σL is independently axiomatizable over **Grz**.

The maps ρ, τ and σ can be characterized with the help of the modal and intuitionistic canonical formulas, which are denoted here by $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ and $\beta(\mathfrak{F},\mathfrak{D},\bot)$, respectively; for a brief exposition and further references consult [7] or [2]. Namely, a normal logic $M \supseteq \mathbf{S4}$ is a modal companion of an intermediate logic

$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i,\mathfrak{D}_i,\bot) : i \in I\}$$

iff M can be represented in the form

$$M = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i,\mathfrak{D}_i,\bot) : i \in I\} \oplus \{\alpha(\mathfrak{G}_j,\mathfrak{E}_j,\bot) : j \in J\},$$

where each $\mathfrak{G}_j$, for $j \in J$, contains at least one proper cluster; in particular,

$$\tau L = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i,\mathfrak{D}_i,\bot) : i \in I\},$$

$$\sigma L = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i,\mathfrak{D}_i,\bot) : i \in I\} \oplus \alpha(\circ\circ, \emptyset).$$

Here $\circ\circ$ is the two-point cluster.

THEOREM 5.3
If an intermediate logic L has an infinite independent axiomatization over **Int** then every logic in the interval $[\tau L, \sigma L]$ is independently axiomatizable (over **S4**).

PROOF. Suppose $L = \mathbf{Int} + \{\varphi_i : i \in \omega\}$ with independent axioms φ_i. According to the characterization above, every logic $M \in [\tau L, \sigma L]$ can be represented as

$$M = \mathbf{S4} \oplus \{T\varphi_i : i \in \omega\} \oplus \{\alpha(\mathfrak{F}_i,\mathfrak{D}_i,\bot) : i \in \omega\},$$

where each $\mathfrak{F}_i$, for $i \in \omega$, contains a proper cluster. Therefore,

$$M = \mathbf{S4} \oplus \{T\varphi_i \wedge \alpha(\mathfrak{F}_i,\mathfrak{D}_i,\bot) : i \in \omega\}.$$

The latter axiomatization is independent over **S4**, for otherwise we would have, for some $i \in \omega$,

$$T\varphi_i \in M' = \mathbf{S4} \oplus \{T\varphi_j \wedge \alpha(\mathfrak{F}_j,\mathfrak{D}_j,\bot) : j \in \omega,\ j \neq i\}$$

and hence

$$\varphi_i \in \rho M' = \mathbf{Int} + \{\varphi_j : j \in \omega,\ j \neq i\},$$

which is a contradiction. By Lemma 2.1, M is absolutely independently axiomatizable. ∎

That L in Theorem 5.3 is *infinitely* independently axiomatizable over **Int** is essential. For, as is shown by the following theorem, **Int** itself has a modal companion without an independent axiomatization.

THEOREM 5.4
The interval $[\tau\mathbf{Int}, \sigma\mathbf{Int}] = [\mathbf{S4}, \mathbf{Grz}]$ contains a logic without an independent axiomatization.

PROOF. (A sketch.) We point out how to change the proof of Theorem 4.6 in order to obtain the logic we need.

As a 'starting formula' δ, we take a modal formula which is refuted in a rooted Kripke frame $\mathfrak{F}$ iff $\mathfrak{F}$ contains a subframe shown in Figure 4, a and b have no common successors in $\mathfrak{F}$ and d

(or c) is contained either in a proper cluster or in an infinite strictly ascending chain. Besides, in the frame in Figure 3 we replace a^2_{-3} with the two-point cluster.

Then we construct a finite number of axioms for L_2 in such a way that the modal counterpart of Lemma 4.4 holds for every rooted differentiated frame for L_2 refuting δ. And L_1 is defined as the logic characterized by the class of all differentiated (reflexive) frames validating δ and all differentiated frames for L_2 of the form shown in Figure 3 with a^2_{-3} replaced by the two-point cluster and the points c_i, $i \geq 1$, removed. This class contains all the finite partially ordered frames (since all of them validate δ) which means that $\rho L_1 = \mathbf{Int}$. The fact that L_1 has no independent axiomatization is proved in the same way as in Sections 3 and 4. ∎

That the property of independent axiomatizability is not in general preserved while passing from an intermediate logic to its arbitrary modal companion can hardly be regarded as a great surprise. Many other properties (such as the decidability, finite model property, Kripke completeness, etc.) behave in this respect in the same way. What is rather unexpected is that unlike the other 'good' properties of logics (at least those known to us) the independent axiomatizability is not in general preserved under the map ρ.

THEOREM 5.5
There is an independently axiomatizable normal modal logic $M \supset \mathbf{S4}$ such that ρM does not have an independent axiomatization.

PROOF. We are going to construct an independently axiomatizable modal logic M such that $\rho M = L_1$, where L_1 is the intermediate logic without an independent axiomatization constructed in the proof of Theorem 4.6. By the definition of L_1 and the proof of Lemma 4.5 (i), each subframe $\mathfrak{G}_i$ of the frame in Figure 3 generated by b_i, for $i \in \omega$, validates L_1 and so each frame $\mathfrak{F}_i$, which is obtained from $\mathfrak{G}_i$ by replacing b_i with the two-point cluster is a frame for τL_1. For $i \in \omega$, we denote by β^*_i the formula

$$T(\alpha^1_{i+1} \wedge \alpha^2_{i+1}) \to T(\alpha^1_i \vee \alpha^2_i) \vee (\Box(\Box(r \to \Box r) \to r) \to r),$$

where α^k_l's are taken from the proof of Theorem 4.6. It is not hard to verify that $\mathfrak{F}_i \not\models \beta^*_i$ and $\mathfrak{F}_j \models \beta^*_i$, for every $j \neq i$. Therefore, the set $\{\beta^*_i : i \in \omega\}$ is independent over τL_1.

Let $\{\varphi_i : i \in \omega\}$ be a set of axioms for L_1 over $\mathbf{Int}$. Then, by defining M as

$$\mathbf{S4} \oplus \{T\varphi_i : i \in \omega\} \oplus \{\beta^*_i : i \in \omega\}, \tag{5.1}$$

we clearly have $\tau L \subset M \subset \sigma L$, with

$$\mathbf{S4} \oplus \{T\varphi_i \wedge \beta^*_i : i \in \omega\}$$

being an independent axiomatization of M. ∎

REMARK 5.6
It may be of interest that it is impossible to extract an independent set of axioms for M from the axiomatization (5.1). By using the logic L_1 constructed in the proof of Theorem 4.6, it is not difficult to construct an intermediate logic with the same property.

6 Open problems

We conclude the paper with some questions to which we could not find answers.

The first three questions concern the difference between absolutely independent axiomatizability and independent axiomatizability over a finitely axiomatizable logic.

- Is an absolutely independently axiomatizable logic L_1 containing a finitely axiomatizable logic L_2 independently axiomatizable over L_2?
- Does the conversion of Lemma 1.1 hold?
- Do Theorems 5.2 and 5.3 hold in the case of absolutely independent axiomatizability?

Our fourth question is connected with the fact that there are two ways of axiomatizing modal logics, namely, with the rule of necessitation and without it. The results above establish the existence of modal logics having no independent axiomatizations only of the former kind. In the proof of Theorem 3.3 the rule of necessitation was used together with the formulas $ax3.i$, which can be rewritten as $\Box\neg\gamma_{i+1} \to \neg\gamma_i$, to ensure that $\neg\gamma_i$ is in an extension of L_2 whenever $\neg\gamma_j$ belongs to it, for some $j > i$. Without this rule the set $\{\neg\gamma_i : i \geq 1\}$ is independent over L_2, and it is not hard to show that $L_1 = L_2 + \{\Box^+\neg\gamma_i : i \geq 1\}$. In the proof of Theorem 5.1 we used the Blok–Esakia isomorphism between the lattices of intermediate logics and *normal* extensions of **Grz**, with the condition of normality being essential here (for details see [2]).

- Do there exist modal logics having no independent axiomatizations without the postulated rule of necessitation (for instance, extensions of Solovay's system S)?

One can show, using the mystical part V of the frames in Figure 1 and 3 that all the logics without independent axiomatizations above have rooted frames of infinite width and depth. Besides, the frames in Figure 1 and 3 are closely related to the frame which was used in [3] for constructing an incomplete modal logic. So our three final questions are:

- Do there exist Kripke complete (modal or intermediate) logics without independent axiomatizations?
- Do there exist (modal or intermediate) logics without independent axiomatizations but with the finite model property?
- Do there exist (modal or intermediate) logics of finite width or finite depth without independent axiomatizations?

(As to the last question, our conjecture is that such logics do not exist.)

Acknowledgements

The work was partially supported by grant number 93-01-16006 from the Russian Fundamental Research Foundation and by the Soros Foundation. We should like to thank anonymous referees for pointing out some mistakes in the first version of this paper.

References

[1] W. J. Blok. The lattice of varieties of modal algebras is not strongly atomic. *Algebra Universalis*, **11**, 285–294, 1980.

[2] A. V. Chagrov and M. V. Zakharyaschev. Modal companions of intermediate propositional logics. *Studia Logica*, **51**, 49–82, 1992.

[3] K. Fine. An incomplete logic containing $S4$. *Theoria*, **40**, 23–29, 1974.

[4] R. I. Goldblatt. Metamathematics of modal logic. *Reports on Mathematical Logic*, **6**, 41–77, 1976.

[5] Yu. G. Kleyman. On certain questions in theory of varieties of groups. *Izvestia of the USSR Academy of Sciences (Mathematical series)*, **47**, 37–74, 1983. (In Russian.)

[6] Logic Notebook. Institute of Mathematics, Novosibirsk, 1986. (In Russian.)

[7] M. V. Zakharyaschev. Canonical formulas for modal and superintuitionistic logics: a short outline. In *Modal Logic and its Neighbours '92*. M. de Rijke, ed., Kluwer Academic Publishers, in press.

Received 28 July 1993