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Algorithmic Problems Concerning First–Order Definability of Modal Formulas on the Class of All Finite Frames

Abstract. The main result is that is no effective algorithmic answer to the question: *how to recognize whether arbitrary modal formula has a first-order equivalent on the class of finite frames.* Besides, two known problems are solved: it is proved algorithmic undecidability of finite frame consequence between modal formulas; the difference between global and local variants of first-order definability of modal formulas on the class of transitive frames is shown.

Introduction

The problem of describing the "behaviour" of modal, or other propositional intensional formulas on the class of finite frames seems rather natural. Almost all modal logics resulting from formalization of substantial ideas have turned out to have the finite model property. Moreover, different semantic constructions in the finite case are effectively intertranslatable (there is no incompleteness effect). In this paper we consider the question, for any modal formula, *how to recognize whether it has a first-order equivalent on the class of finite frames.* Our main result is that no effective algorithmic answer to this question is possible.

Earlier, in [6], algorithmic undecidability was proved for first-order definability without restrictions on frame cadrinality. The proof there uses essentially undecidable calculi and, so, it cannot be transferred directly to finite frames. But another construction, used in [4] to show the undecidability of finite frame consequence between modal formulas (a research problem first stated in [2]) turned out to be a suitable replacement. Thereby the proof increases considerably in size, since we have to manipulate first-order definability of rather large formulas (cf. the forthcoming paper [7] concerning superintuitionistic logics). One indirect purpose of the present work is to give a simplest possible variant for modal formulas. For this purpose it turned out convenient to use finite GL-frames, having a transitive and

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irreflexive alternative relation. Note that, in the infinite case, it is almost pointless to use this frame class, as it is not first-order definable.

This paper is organized as follows. In Section 1, the revelant result from [4] about the undecidability of semantic consequence for finite GL-frames is proved, as welll as some generalizations thereof. The main technical lemma of this Section, which shows how to reduce the Halting Problem for Minsky Machines to the finite frame consequence problem, will be used in Section 4 to obtain the main results of the present paper. In Section 2, some technical facts are proved about first-order definability of modal formulas. This contains the messy complications we mentioned. Readers interested only in the general idea of our main results may omit the details of constructing the relevant first-order equivalents. In Section 3, formulas closely connected with the constructions of Section 1 and 2 are presented, which have no first-order equivalents on the class of finite frames. The results of Sections 1-3 are combined in Section 4 to obtain short proofs of the undecidability of various problems connected with first-order definability on the class of finite frames. In the course of this exposition, two notions of first-order definability will be considered, a "local" one and a "global" one. In particular, in Section 3 the problem raised in [1] concerning the difference between global and local definability on the class of all transitive frames is decided.

Some definitions

The usual definitions and results from Modal Logic are employed, including its connections with standard logic. In particular, any Kripke frame can be considered both as a semantic structure for modal formulas and as a model for the first-order language with equality and a single binary predicate. We say that a modal formula φ and a first-order sequence ψ are *globally equivalent* on a frame class $\mathbf{K}$ if, for any frame F in $\mathbf{K}$, $F \models \varphi$ iff $F \models \psi$. A modal formula φ is *globally first-order definable* on the class $\mathbf{K}$ if there exists a first-order sentence globally equivalent to it on $\mathbf{K}$. A modal formula φ and a first-order formula ψ with one free variable are *locally equivalent* on $\mathbf{K}$ if, for any frame F in $\mathbf{K}$ and any world d, $\langle F, d\rangle \models \varphi$ iff $F \models \psi[d]$. A modal formula φ is *locally first-order definable* on the frame class $\mathbf{K}$ iff there is a first-order formula with one free variable which is locally equivalent to it on $\mathbf{K}$. It is clear that local first-order definability implies global first-order definability on any frame class. In this paper, we are mainly interested in the class $\mathbf{K}$ of finite frames (but our results remain true for finite transitive irreflexive ones). Henceforth, all frames considered will be finite. If we want to allow that a class contains infinite frames, then we will mention this specifically. Here is some useful notation:

$\varphi \models_{fin} \psi$: for any frame $\mathcal{F}$, $\mathcal{F} \models \varphi$ implies $\mathcal{F} \models \psi$, and
$\varphi \models^{loc}_{fin} \psi$: for any $\mathcal{F}$ and world d, $\langle \mathcal{F}, d\rangle \models \varphi$ implies $\langle \mathcal{F}, d\rangle \models \psi$.

The relations $\models_{fin}$ and $\models^{loc}_{fin}$ are different. An example is provided by:

$$LF \wedge \Diamond\top \models_{fin} \bot \quad \text{but} \quad LF \wedge \Diamond\top \not\models^{loc}_{fin} \bot,$$

where $LF = \Box(\Box p \to p) \to \Box p$ is the formula that axiomatizes Löb's Logic GL. This principle is non-first-order in general, but "Transitivity and Irreflexivity" is a first-order equivalent (both global and local) for it over the finite frames. We will often "translate" first-order formulas into English: e.g., the preceding term "Irreflexivity" is our translation of the formula $\forall x \neg xRx$. Finally, we will freely use various modal and first-order abbreviation: such as $\boxdot\varphi$ for $\Box\varphi \& \varphi$, $x = y = z$ for $x = y \wedge y = z$, $xRyRz$ for $xRy \wedge yRz$, xR^3y for $\exists u \exists v(xRuRvRy)$. Other useful notations will be explained in due course.

1. Reduction of Minsky Machine Computability to Modal Consequence on Finite Frames

A 'Minsky Machine' is a two-tape effective machine operating on two integers s_1 and s_2. A *Minsky Machine Program* is a finite set of instructions I of the forms:

(1) $q_\alpha \to q_\beta T_1 T_0$: in state q_α, add 1 to s_1, and go to q_β;
(2) $q_\alpha \to q_\beta T_0 T_1$: in state q_α, add 1 to s_2, and go to q_β;
(3) $q_\alpha \to q_\beta T_{-1} T_0 (q_\gamma T_0 T_0)$: in state q_α, subtract 1 from s_1, if $s_1 \neq 0$; and go to q_β, otherwise go to q_γ;
(4) $q_\alpha \to q_\beta T_0 T_{-1} (q_\gamma T_0 T_0)$: in state q_α, subtract 1 from s_2, if $s_2 \neq 0$; and go to q_β, otherwise go to q_γ;

A *Minsky Machine Configuration* is an ordered triple (i, j, k) of natural numbers, where i is a state number, $j = s_1$, and $k = s_2$. We write $P : (\alpha, m, n) \to (\beta, k, l)$ to express that the program P starting at configuration (α, m, n) can reach configuration (β, k, l). Henceforth, we fix the symbols q_α and q_β for the initial and final states of the machine. Finally, we introduce two conventions which do not influence standard facts about Minsky machines, but are useful for our constructions: (i) all machines considered are deterministic, i.e., they do not contain different instructions with the same left parts, (ii) "blocking" states do not occur in our machines, i.e., if some nonfinal state is in the right part of some machine instruction, then there is an instruction with that state in the left part.

In this section we use two variants of undecidable halting problems of Minsky machines:

- There is a Minsky Machine P such that no algorithm recognizes for any configuration whether P eventually halts (in a final state), starting from that configuration.

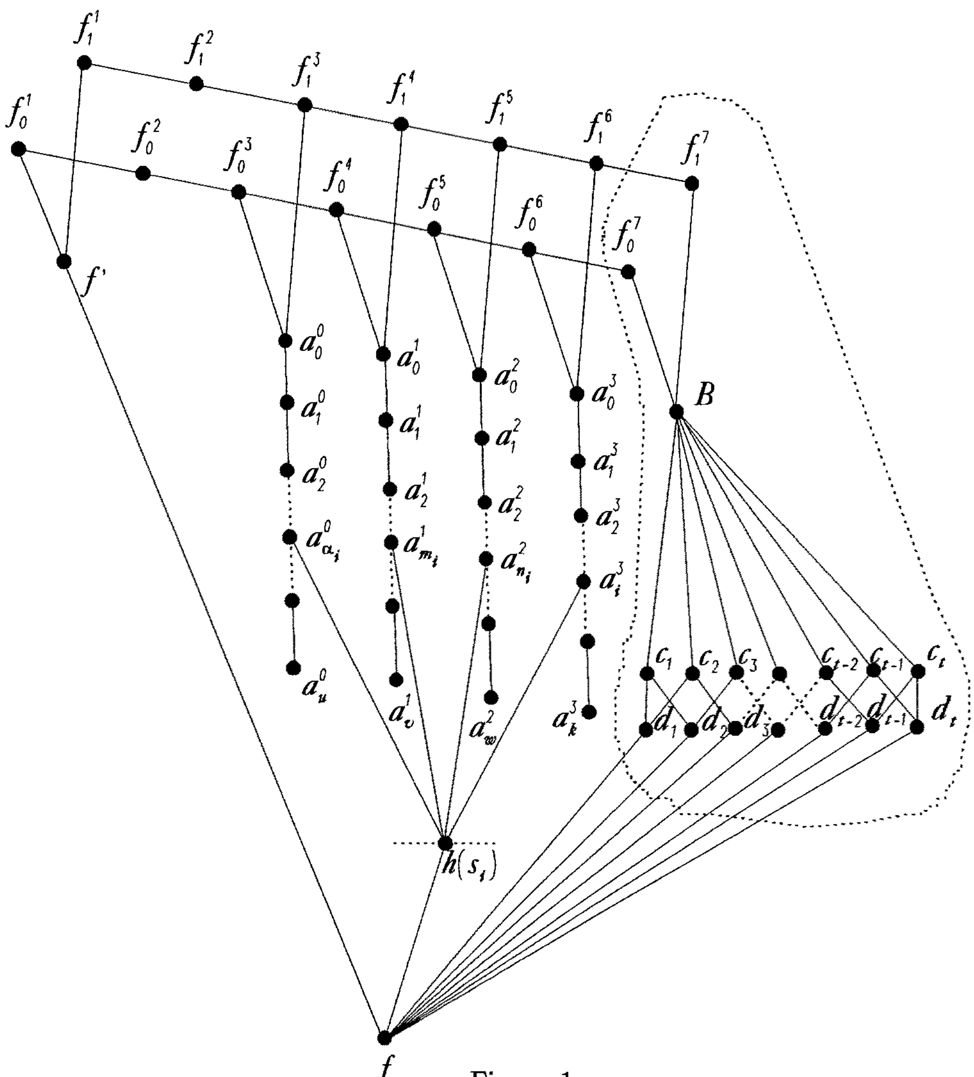

Figure 1

- There is a configuration (α, m, n) such that no algorithm recognizes for any Minsky machine program whether it eventually halts, starting from (α, m, n).

Simulation of Minsky machine work by modal formulas was applied in some precending papers of different authors, originally Isard [12], see also [3], [5], where the similar technique is applied.
Next, we introduce an important set of formulas. Although we do not need their semantic "sense" right now, it is useful to have a picture in mind, in accordance with which they are defined. The irreflexive transitive frame of Figure 1 is such a picture; where the part of the frame surrounded by the dotted line is not needed in this Section. Here is some explanation of its key features. The frame is constructed from a machine program P and an initial configuration (α, m, n). The presence of the worlds $h(s_i)$, where $s_i = (\alpha_i, m_i, n_i)$, expresses the fact that s_i is the configuration after the first $i-1$ steps of a computation starting from (α, m, n). Now set:

$$F' = \Box^2 \bot \to \Box p \vee \Box \neg p, \quad F = \Box F'.$$

These formulas are falsifiable in this frame by the same valuation, F at world f, and F' at f' (and only there). For as soon as F is falsifiable, say by a valuation V, then

$$V(p, f_0^1) = 1, \; V(p, f_1^1) = 0, \quad \text{or} \quad V(p, f_0^1) = 0, \; V(p, f_1^1) = 1.$$

We continue to introduce modal formulas, emphasizing their connection with the worlds in which they are true:

$$F_0^1 = \Box \bot \wedge p, \quad F_1^1 = \Box \bot \wedge \neg p,$$

$$F_i^j = \Box^j \bot \wedge \Diamond^{j-1} F_i^1 \wedge \Diamond^{j-1} F_{i-1}^1 \; (i \in \{0, 1\}, 2 \leqslant j \leqslant 7),$$

$$A_0^i = \Diamond F_0^{i+3} \wedge \Diamond F_1^{i+3} \wedge \Box^{i+4} \bot \; (i \leqslant 3),$$

$$A_j^i = \Diamond^j A_0^i \wedge \neg \Diamond^{j+1} A_0^i \wedge \textstyle\bigwedge_{i \neq k=0}^{3} \neg \Diamond A_0^k \; (0 \leqslant i \leqslant 3, j > 0),$$

$$S(\gamma, A_k^1, A_l^2) = \textstyle\bigwedge_{i=0}^{\gamma} \Diamond A_i^0 \wedge \neg \Diamond A_{\gamma+1}^0 \wedge \Diamond A_k^1 \wedge \neg \Diamond \Diamond A_k^1 \wedge \Diamond A_l^2 \wedge \neg \Diamond \Diamond A_l^2$$
(where $\gamma, k, l \geqslant 0$).

The formula $S(\gamma, A_k^1, A_l^2)$ corresponds to the configuration (γ, k, l). But for describing instructions we need formulas speaking about arbitrary configurations. Therefore, we set:

$$
\begin{array}{rcl}
Q_1 & = & (\Diamond A_0^1 \vee A_0^1) \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^2 \wedge \neg\Diamond A_0^3 \wedge p_1 \wedge \neg\Diamond p_1, \\
Q_2 & = & \Diamond A_0^1 \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^2 \wedge \neg\Diamond A_0^3 \wedge \Diamond p_1 \wedge \neg\Diamond\Diamond p_1, \\
R_1 & = & (\Diamond A_0^2 \vee A_0^2) \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^1 \wedge \neg\Diamond A_0^3 \wedge p_2 \wedge \neg\Diamond p_2, \\
R_2 & = & \Diamond A_0^2 \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^1 \wedge \neg\Diamond A_0^3 \wedge \Diamond p_2 \wedge \neg\Diamond\Diamond p_2, \\
T_1 & = & \Diamond A_0^3 \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^1 \wedge \neg\Diamond A_0^2 \wedge p_3 \wedge \neg\Diamond\Diamond p_3, \\
T_2 & = & \Diamond A_0^3 \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^1 \wedge \neg\Diamond A_0^2 \wedge \Diamond p_3 \wedge \neg\Diamond p_3, \\
S(\gamma, Q_i, R_j) & = & \bigwedge_{k=0}^{\gamma} \Diamond A_k^0 \wedge \neg\Diamond A_{\gamma+1}^0 \wedge \Diamond Q_i \wedge \neg\Diamond\Diamond Q_i \wedge \Diamond R_j \wedge \neg\Diamond\Diamond R_j, \\
S(\gamma, Q_1, A_0^2) & = & \bigwedge_{k=0}^{\gamma} \Diamond A_k^0 \wedge \neg\Diamond A_{\gamma+1}^0 \wedge \Diamond Q_1 \wedge \neg\Diamond\Diamond Q_1 \wedge \Diamond A_0^2 \wedge \neg\Diamond\Diamond A_0^2, \\
S(\gamma, A_0^1, R_1) & = & \bigwedge_{k=0}^{\gamma} \Diamond A_k^0 \wedge \neg\Diamond A_{\gamma+1}^0 \wedge \Diamond A_0^1 \wedge \neg\Diamond\Diamond A_0^1 \wedge \Diamond R_1 \wedge \neg\Diamond\Diamond R_1, \\
 & & (\text{where } \gamma \geqslant 0, i, j \in \{1, 2\}).
\end{array}
$$

We shall write $\varphi \equiv \psi$ if $\varphi \leftrightarrow \psi$ is true in all finite transitive and irreflexive frames, i.e. $GL \vdash \varphi \leftrightarrow \psi$. $\varphi \equiv \psi$ implies that φ and ψ are always semantically interchangeable in models of GL.

LEMMA 1.1. *Let φ^* be the result of the substitution in φ of $\Diamond^k A_0^1$ for p_1, $\Diamond^\ell A_0^2$ for p_2 and $\Diamond A_0^3$ for p_3. Then*

i) $Q_1^* \equiv A_k^1$, $Q_2^* \equiv A_{k+1}^1$;

ii) $R_1^* \equiv A_\ell^2$, $R_2^* \equiv A_{\ell+1}^2$;

iii) $T_1^* \equiv A_m^3$, $T_2^* \equiv A_{m+1}^3$;

iv) $(S(\gamma, Q_i, R_j))^* \equiv S(\gamma, A_{k+(i-1)}^1, A_{l+(j-1)}^2)$ $(i, j \in \{1, 2\})$;

v) $(S(\gamma, Q_1, A_0^2))^* \equiv S(\gamma, A_k^1, A_0^2)$;

vi) $(S(\gamma, A_0^1, R_1))^* \equiv S(\gamma, A_0^1, A_\ell^2)$;

PROOF. Immediate ■

The formulas that simulate instructions use subformulas of the form T_i for the "calculation of some number of steps". For instruction I set:

if I is of the form $q_\gamma \to q_\delta T_1 T_0$, then
$AxI = \neg F \wedge \Diamond(S(\gamma, Q_1, R_1) \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1) \wedge \Diamond T_2 \to \Diamond(S(\delta, Q_2, R_1) \wedge \Diamond T_2 \wedge \neg\Diamond\Diamond T_2)$;

if I is of the form $q_\gamma \to q_\delta T_0 T_1$, then
$AxI = \neg F \wedge \Diamond(S(\gamma, Q_1, R_1) \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1) \wedge \Diamond T_2 \to \Diamond(S(\delta, Q_1, R_2) \wedge \Diamond T_2 \wedge \neg\Diamond\Diamond T_2)$;

if I is of the form $q_\gamma \to q_\delta T_{-1} T_0 (q_\varepsilon\ T_0 T_0)$, then
$(AxI = (\neg F \wedge \Diamond(S(\gamma, Q_2, R_1) \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1) \wedge \Diamond T_2 \to \Diamond(S(\delta, Q_1, R_1) \wedge \Diamond T_2 \wedge \neg\Diamond\Diamond T_2)) \wedge (\neg F \wedge \Diamond(S(\gamma, A_0^1, R_1) \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1) \wedge \Diamond T_2 \to \Diamond(S(\varepsilon, A_0^1, R_1) \wedge \Diamond T_2 \wedge \neg\Diamond\Diamond T_2));$

if I is of the form $q_\gamma \to q_\delta T_0 T_{-1} (q_\varepsilon\ T_0 T_0)$, then
$(AxI = (\neg F \wedge \Diamond(S(\gamma, Q_1, R_2) \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1) \wedge \Diamond T_2 \to \Diamond(S(\delta, Q_1, R_1) \wedge \Diamond T_2 \wedge \neg\Diamond\Diamond T_2)) \wedge (\neg F \wedge \Diamond(S(\gamma, Q_1, A_0^2) \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1) \wedge \Diamond T_2 \to \Diamond(S(\varepsilon, Q_1, A_0^2) \wedge \Diamond T_2 \wedge \neg\Diamond\Diamond T_2));$

Next, for a Minsky program P set

$$AxP = \bigwedge_{I \in P} AxI$$

and then define the formula $A(P)$ as follows:

$$A(P) \;=\; LF \wedge AxP \wedge LinT \wedge UT,$$

where

$$\begin{aligned} LinT \;&=\; \neg(\neg F \wedge \Diamond(\Diamond A_0^3 \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^1 \wedge \neg\Diamond A_0^2 \wedge \boxdot r \wedge \neg q) \\ &\quad \wedge \Diamond(\Diamond A_0^3 \wedge \neg\Diamond A_0^0 \wedge \neg\Diamond A_0^1 \wedge \neg\Diamond A_0^2 \wedge \boxdot q \wedge \neg r)); \\ UT \;&=\; \neg(\neg F \wedge \Diamond(q \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1 \wedge \Diamond A_0^0 \wedge \Diamond A_0^1 \wedge \Diamond A_0^2) \\ &\quad \wedge \Diamond(q \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1 \wedge \Diamond A_0^0 \wedge \Diamond A_0^1 \wedge \Diamond A_0^2)). \end{aligned}$$

Finally, define the formula $B(\alpha, m, n)$ as follows:

$$\begin{aligned} B(\alpha, m, n) \;&=\; \neg F \wedge \Diamond(S(\alpha, A_m^1, A_n^2) \wedge \Diamond A_1^3 \wedge \neg\Diamond\Diamond A_1^3) \\ &\quad \to \neg\Diamond(S(\beta, Q_1, R_1) \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1). \end{aligned}$$

Now, the main technical result of this Section is the following:

LEMMA 1.2. $A(P) \models_{fin} B(\alpha, m, n)$ *iff the program P, starting at configuration (α, m, n) cannot reach a final state (with number β).*

PROOF. "If". This is the direction for which our specific finite frames were developed. Let the Minsky program P continue indefinitely, starting from the configuration (α, m, n). We show that then, $A(P) \models_{fin} B(\alpha, m, n)$, by a reductio ad absurdum. Suppose that, for some finite frame $\mathcal{F}$

(1) $\mathcal{F} \models A(P)$,

(2) $\mathcal{F} \not\models B(\alpha, m, n)$.

Condition (2) means that, for some valuation V, in some world a from $\mathcal{F}$, we have $V(B(\alpha, m, n), a) = 0$ (we abbreviate this as: $a \not\models B(\alpha, m, n)$). I.e.,

(3) $a \not\models F$,

(4) $a \models \Diamond(S(\alpha, A^1_m, A^2_n) \wedge \Diamond A^3_1 \wedge \neg\Diamond\Diamond A^3_1)$,

(5) $a \models \Diamond(S(\beta, Q_1, R_1) \wedge \Diamond T_1 \wedge \neg\Diamond\Diamond T_1)$.

Condition (4) implies, that there is a world b in $\mathcal{F}$ such that aRb and

(6) $b \models S(\alpha, A^1_m, A^2_n)$,

(7) $b \models \Diamond A^3_1$,

(8) $b \not\models \Diamond\Diamond A^3_1$.

Condition (5) implies, that there is a world c in $\mathcal{F}$ such that aRc and

(9) $c \models \Diamond S(\beta, Q_1, R_1)$,

(10) $c \models \Diamond T_1$,

(11) $c \not\models \Diamond\Diamond T_1$.

Condition (7) implies, that there is a world (denote it by a^3_1) such that bRa^3_1 and $a^3_1 \models A^3_1$, i.e.,

(12) $a^3_1 \models \Diamond A^3_0 \wedge \neg\Diamond A^0_0 \wedge \neg\Diamond A^1_0 \wedge \neg\Diamond A^2_0$,

(13) $a^3_1 \not\models \Diamond\Diamond A^3_0$.

Condition (10) implies that there must be a world x such that cRx and $x \models T_1$, i.e.,

(14) $x \models \Diamond A^3_0 \wedge \neg\Diamond A^0_0 \wedge \neg\Diamond A^1_0 \wedge \neg\Diamond A^2_0$,

(15) $x \models p_3 \wedge \neg\Diamond p_3$.

Now we need a further auxiliary result.

LEMMA 1.3. *For any frame $\mathcal{F}$, if $\mathcal{F} \models LF \wedge LinT$ and the formula F is falsifiable at one of its worlds h by some valuation, then the set of worlds of this frame which are accessible from h, in which the formula $\Diamond A^3_0 \wedge \neg\Diamond A^0_0 \wedge$*

$\neg\Diamond A_0^1 \wedge \neg\Diamond A_0^2$ is true, is strictly linearly ordered by the relation of accessibility.

PROOF. This is a standard exercise. ■

We continue the proof of Lemma 1.2. By Lemma 1.3, conditions (1), (3) above imply that the set of worlds in $\mathcal{F}$ that are accessible from a at which the conditions (12), (14) hold, form a chain strictly linearly ordered by the alternative relation in which a_1^3 is the R-greatest element, because of (13). Denote this chain by

$$a_k^3 R a_{k-1}^3 R a_{k-2}^3 R \cdots R a_2^3 R a_1^3.$$

We can characterize its elements as follows by the formulas A_i^3 : $a_i^3 \models A_j^3$ iff $i = j$. Now, (14) implies, that $x = a_s^3$ for some $s, 1 \leqslant s \leqslant k$, so that we have

(16) $a_s^3 \models T_1$ (from (14), (15)),

(17) $c \models \Diamond A_s^3$,

(18) $c \not\models \Diamond\Diamond A_s^3$ (from (11)).

Now that we have succeeded in identyfying x with some a_s^3, and with the help of the conjunctive members of $A(P)$, which are true in $\mathcal{F}$ by (1), we shall get "step by step" from a_1^3 to x. This turns out possible exactly thanks to the finiteness of F.

Let P, starting from (α, m, n), one application of some instruction at a time, give the successive configurations (α_2, m_2, n_2), (α_3, m_3, n_3), ...; where we identify (α, m, n) with (α_1, m_1, n_1). Note, that by the given conditions, $\alpha_i \neq \beta$ for any $i \in \omega$. In this sequence of configurations, we shall only be interested in finite initial segment of length s:

$$(\alpha_1, m_1, n_1),\ (\alpha_2, m_2, n_2), \ldots, (\alpha_s, m_s, n_s).$$

Note that for each i, $1 \leqslant i \leqslant k-1$,

$$\begin{aligned} a \models\ & \Diamond(S(\alpha_i, A_{m_i}^1, A_{n_i}^2) \wedge \Diamond A_i^3 \wedge \neg\Diamond\Diamond A_i^3) \rightarrow \\ & \Diamond(S(\alpha_{i+1}, A_{m_{i+1}}^1, A_{n_{i+1}}^2) \wedge \Diamond A_{i+1}^3 \wedge \neg\Diamond\Diamond A_{i+1}^3). \end{aligned}$$

Indeed, from (1), we have by substitution in a suitable conjunct of AxP:

$$\begin{aligned} a \models\ & \neg F \wedge \Diamond(S(\alpha_i, A_{m_i}^1, A_{n_i}^2) \wedge \Diamond A_i^3 \wedge \neg\Diamond\Diamond A_i^3) \wedge A_{i+1}^3 \rightarrow \\ & \Diamond(S(\alpha_{i+1}, A_{m_{i+1}}^1, A_{n_{i+1}}^2) \wedge \Diamond A_{i+1}^3 \wedge \neg\Diamond\Diamond A_{i+1}^3), \end{aligned}$$

and (3) and aRa^3_{i+1} together with $a^3_{i+1} \models A^3_{i+1}$ yield that $a \models \neg F \wedge \Diamond A^3_{i+1}$, which gives the desired result. Applying (4) and "Modus Ponens" successively one obtains then, for any i, $1 \leqslant i \leqslant k-1$, that

$$a \models \Diamond(S(\alpha_{i+1}, A^1_{m_{i+1}}, A^2_{n_{i+1}}) \wedge \Diamond A^3_{i+1} \wedge \neg\Diamond\Diamond A^3_{i+1})$$

and hence in particular,

$$a \models \Diamond(S(\alpha_s, A^1_{m_s}, A^2_{n_s}) \wedge \Diamond A^3_s \wedge \neg\Diamond\Diamond A^3_s).$$

The last condition implies that there is a world d such that aRd and

(19) $d \models S(\alpha_s, A^1_{m_s}, A^2_{n_s})$,

(20) $d \models \Diamond A^3_s$,

(21) $d \not\models \Diamond\Diamond A^3_s$.

From (20), (21), using Lemma 1.3 and the fact that $a^3_i \models A^3_i, dRa^3_s$ while not dRa^3_{s+1}, because there is a unique world among $a^3_1, \ldots, a^3_k$ in which the formula T_1 holds, it follows with the help of (14), (15) and $x = a^3_s$ that

(22) $d \models \Diamond T_1 \wedge \neg\Diamond\Diamond T_1$.

Now, we collect all neccessary conditions. From (9), (10), (11) we obtain

(23) $c \models \Diamond T_1 \wedge \neg\Diamond\Diamond T_1 \wedge \Diamond A^0_0 \wedge \Diamond A^1_0 \wedge \Diamond A^2_0$,

and from (19), (22),

(24) $d \models \Diamond T_1 \wedge \neg\Diamond\Diamond T_1 \wedge \Diamond A^0_0 \wedge \Diamond A^1_0 \wedge \Diamond A^2_0$.

The worlds c and d are different, because (9) implies that $c \models \bigwedge_{i=0}^{\beta} \Diamond A^0_i \wedge \neg\Diamond A^0_{\beta+1}$, and from (19) we get $d \models \bigwedge_{i=0}^{\alpha_s} \Diamond A^0_i \wedge \neg\Diamond A^0_{\alpha_s+1}$, by the condition that $\alpha_s \neq \beta$. This difference allows us to define the valuation V for a new variable q in such a way that $c \models q$, $d \not\models q$. Together with (23), (24), and also (3), this gives $a \not\models UT$ which contradicts (1).

This contradiction shows that $A(P) \models_{fin} B(\alpha, m, n)$: and thus, we have proved the "If" direction of Lemma 1.2.

"Only if". Let us now assume that the program P can reach a final state starting from the configuration (α, m, n). We will show that $A(P) \not\models_{fin} B(\alpha, m, n)$, by constructing a suitable finite frame. Let

$$s_1 = (\alpha, m, n) = (\alpha_1, m_1, n_1),\ s_2 = (\alpha_2, m_2, n_2), \dots, s_k = (\alpha_k, m_k, n_k)$$

be the list of all configurations, successively obtained by program P, starting at (α, m, n). Set

$$u = max(\alpha_1, \dots, \alpha_k),\ v = max(m_1, \dots, m_k),\ w = max(n_1, \dots, n_k).$$

Consider the frame $\mathcal{F}$ sketched in Figure 1. We remind the reader that the part of $\mathcal{F}$ which is "enclosed by the dotted line" remains disregarded for the present.

LEMMA 1.4. *$\mathcal{F} \models A(P)$.*

PROOF. This is a routine verification. For analogous scrupulous verifications see [6], [3]. ■

Now introduce a valuation on $\mathcal{F}$ such that

$$x \models p \Leftrightarrow x = f_0^1;\ x \models p_1 \Leftrightarrow x = a_{m_k}^1;$$
$$x \models p_2 \Leftrightarrow x = a_{n_k}^2;\ x \models p_3 \Leftrightarrow x = a_k^3.$$

LEMMA 1.5. *This valuation has the effect that $f \not\models B(\alpha, m, n)$.*

PROOF. This is again a routine verification. ■

From Lemmas 1.4 and 1.5 we obtain the desired result, and Lemma 1.2 is proved. ■

Because $A(P)$ and the formula $B(\alpha, m, n)$ are in fact constructed effectively from the program P and the configuration (α, m, n), the undecidability of the earlier two halting problems gives the following two results.

THEOREM 1.6. *There is a formula φ such that the problem of recognizing, given any formula ψ, whether $\varphi \models_{fin} \psi$, is algorithmically undecidable.*

THEOREM 1.7. *There is a formula ψ such that the problem of recognizing, given any formula φ, whether $\varphi \models_{fin} \psi$, is algorithmically undecidable.*

From these theorems we obtain

COROLLARY 1.8. *There is no algorithm which recognizes, given any two modal formulas φ and ψ, whether $\varphi \models_{fin} \psi$.*

Taking into account the simple fact that $\{\langle\varphi,\psi\rangle | \varphi \not\models_{fin} \psi\}$ is recursively enumerable, we obtain from 1.6 – 1.8 this further

COROLLARY 1.9.

a) *The set $\{\langle\varphi,\psi\rangle | \varphi \models_{fin} \psi\}$ is not recursively enumerable.*

b) *There is a formula φ such that $\{\psi | \varphi \models_{fin} \psi\}$ is not recursively enumerable.*

c) *There is a formula ψ such that $\{\varphi | \varphi \models_{fin} \psi\}$ is not recursively enumerable.*

From the proofs of statements 1.6 – 1.9 we obtain the following analogues for $\models^{loc}_{fin}$.

THEOREM 1.10. *There is a formula φ such that the problem of recognizing, given any modal formula ψ, wheter $\varphi \models^{loc}_{fin} \psi$, is algorithmically undecidable*

THEOREM 1.11. *There is a formula ψ such that the problem of recognizing, given any modal formula φ, wheter $\varphi \models^{loc}_{fin} \psi$, is algorithmically undecidable*

COROLLARY 1.12. *There is no algorithm which recognizes, given any two modal formulas φ and ψ, whether $\varphi \models^{loc}_{fin} \psi$.*

COROLLARY 1.13.

a) *The set $\{\langle\varphi,\psi\rangle | \varphi \models^{loc}_{fin} \psi\}$ is not recursively enumerable.*

b) *There is a formula φ such that $\{\psi | \varphi \models^{loc}_{fin} \psi\}$ is not recursively enumerable.*

c) *There is a formula ψ such that $\{\varphi | \varphi \models^{loc}_{fin} \psi\}$ is not recursively enumerable.*

Now change the definition of $A(P)$ by adding a conjunct $\Diamond\top$. In this case the above proofs will work for $\models^{loc}_{fin}$, but no longer for $\models_{fin}$, because after such a change, we will have $A(P) \wedge \Diamond T \models_{fin} B(\alpha, m, n)$, indepedently of the choice of $P, (\alpha, m, n)$. Thus, one obtains

THEOREM 1.14.

a) $\{\langle\varphi,\psi\rangle | \varphi \models_{fin} \psi,\ \varphi \not\models^{loc}_{fin} \psi\}$ *is not recursive.*

b) There is a formula φ *such that* $\{\psi | \varphi \models_{fin} \psi,\ \varphi \not\models^{loc}_{fin} \psi\}$ *is not recursive.*

c) There is a formula ψ *such that* $\{\varphi | \varphi \models_{fin} \psi,\ \varphi \not\models^{loc}_{fin} \psi\}$ *is not recursive.*

We do not know if it is possible to replace "recursive" in the above statements by "recursively enumerable".

2. A First-Order Definable Variant On Finite Frames For Formulas Describing Minsky Machines

To obtain the main result of this paper we need an additional property for the formulas $A(P)$ from the previous section – namely their first-order definability on finite frames. Although the above construction may have produced formulas of this kind, we shall complicate the organization of $A(P)$ to simplify the proof of its first-order definability. It will have to be checked, of course, that the new version will still do the job of the original formula $A(P)$. Now we shall merely add conjuncts that are true in the frame of Figure 1, as may be verified immediately, and this is harmless. The new $A(P)$ is defined as follows:

$$\begin{aligned} A(P) = {} & LF \wedge UF \wedge Last_2 \wedge UF_0^2 \wedge \ldots \wedge UF_0^6 \wedge UA_0^0 \wedge UA_0^1 \wedge UA_0^2 \\ & \wedge UA_0^3 \wedge Lin\Diamond A_0^0 \wedge Lin\Diamond A_0^1 \wedge Lin\Diamond A_0^2 \wedge Lin\Diamond A_0^3 \wedge UT \wedge AxP. \end{aligned}$$

where

$$\begin{aligned} Last_2 &= \Box(q \wedge r \to \Diamond T) \vee \Box(q \wedge \neg r \to \Diamond T) \vee \Box(\neg q \wedge r \to \Diamond T), \\ UF &= \boxdot(q \to F) \vee \boxdot(\neg q \to F), \end{aligned}$$

and for $X \notin \{F, S\}$,

$$UX = (\Box(X \to \Box(Y_1 \vee \ldots \vee Y_m)) \wedge (\Box(q \to X) \vee \Box(\neg q \to X))) \vee F,$$

where $X, Y_1, \ldots, Y_m$ are formulas corresponding to different worlds $x, y_1, \ldots, y_m$ in Figure 1, such that $xRy_1, \ldots, xRy_m$ and whenever xRz, then $z \in \{y_1, \ldots, y_m\}$, and finally

$$\begin{aligned} Lin\Diamond A_0^i &= \Box(\Diamond A_0^i \wedge \textstyle\bigwedge_{i \neq j=0}^{3} \neg\Diamond A_0^j \wedge \boxdot r \to q) \\ & \quad \vee \Box(\Diamond A_0^i \wedge \textstyle\bigwedge_{i \neq j=0}^{3} \neg\Diamond A_0^j \wedge \boxdot q \to r) \vee F. \end{aligned}$$

Now we use the new variant of $A(P)$. For this formula, the old Lemma 1.2 still holds. Moreover, we have:

LEMMA 2.1. *The new formula $A(P)$ is first-order definable on finite frames (both globally and locally).*

PROOF. We shall prove the first-order definability step by step, for which we introduce the following notation. For a formula of the form $A_1 \wedge \ldots \wedge A_{i-1} \wedge \ldots \wedge A_m, A_i^*$ will denote $A_1 \wedge \ldots \wedge A_i$. For example, $(UF_0^2)^* = LF \wedge UF \wedge Last_2 \wedge UF_0^1 \wedge UF_0^2$ and $(AxP)^* = A(P)$ for the new formula $A(P)$. We shall construct a first-order local equivalent for the conjunction of the first i conjuncts of $A(P)$, increasing i till we get to the full $A(P)$. The first-order equivalent on finite frames ("first-order equivalent", for short) for formulas $(A)^*$ will be denoted by $foe(A)^*$. For instance, we have

$$foe((LF)^*) \;=\; foe(LF) = \text{"transivity and irreflexivity"}.$$

In all conjuncts of $A(P)$, exept the first, the formula F occurs. We describe the conditions of its refutation. The following auxiliary predicate will be useful:

$$l(x) = n \quad \Leftrightarrow \quad \text{"a chain of worlds of length } n \text{ is accessible from } x \text{ and chains of length } n+1 \text{ are not accessible"}.$$

LEMMA 2.2. *The formula F is refuted at world x of frame $\mathcal{F}$ iff there are worlds y, z_1, z_2 in $\mathcal{F}$ such that y is accessible from x and $l(y) = 1$, $l(z_1) = l(z_2) = 0$, while $yRz_1, yRz_2, z_1 \neq z_2$. This assertion presupposes that the refuting valuation has $z_1 \models p, z_2 \not\models p$.*

PROOF. This may be verified immediately. ■

For convenience, we denote the right-hand condition in Lemma 2.2 by $CR(x, y, z_1, z_2)$.

REMARK. Lemma 2.2 implies that the formulas F and $\forall y \forall z_1 \forall z_2 \neg CR(x, y, z_1, z_2)$ are locally equivalent.

Now we set

$$\begin{aligned} foe(UF)^* \;=\; & foe(LF) \wedge \forall x_1 \forall x_2((xRx_1 \vee x = x_1) \wedge (xRx_2 \vee x = x_2) \wedge \\ & \exists y \exists z_1 \exists z_2 CR(x_2, y, z_1, z_2) \rightarrow x = x_1 \wedge x = x_2). \end{aligned}$$

This formula expresses the uniqueness of the world in which F is refuted.

LEMMA 2.3. *$(UF)^*$ and $foe(UF)^*$ are locally equivalent.*

PROOF. Let $(F)^*$ be refuted at world a of frame $\mathcal{F}$. We show that $\mathcal{F} \not\models foe(UF)^*(a/x)$. $\langle \mathcal{F}, a\rangle \not\models (UF)^*$ means that there is a valuation V such that $a \not\models^V (UF)^*$ (later on, we shall omit V as usually). Then $a \not\models LF$ or $a \models LF$, but $a \not\models UF$. We may suppose that $\langle \mathcal{F}, x\rangle \models foe(LF)$, because otherwise, the desired result is obtained at once. Then we have $a \not\models UF$, i.e.,

(1) $a \not\models \boxdot(p \to F)$,

(2) $a \not\models \boxdot(\neg p \to F)$.

From (1) we obtain that in $\mathcal{F}$ there is a world a_1 such that aRa_1 or $a = a_1$ and

(3) $a_1 \models p$,

(4) $a_1 \not\models F$.

From (2) we obtain that in $\mathcal{F}$ there is also a world a_2 such that aRa_2 or $a = a_2$ with

(5) $a_2 \not\models p$,

(6) $a_2 \not\models F$.

From (4) and (6), using Lemma 2.2 we obtain that in $\mathcal{F}$ there are worlds $b', c_1', c_2', b'', c_1'', c_2''$ such that the formulas $CR(a_1, b', c_1', c_2')$ and $CR(a_2, b'', c_1'', c_2'')$, and hence the formulas $\exists y \exists z_1 \exists z_2 CR(a_1, y, z_1, z_2)$, $\exists y \exists z_1 \exists z_2 CR(a_2, y, z_1, z_2)$ are true. Besides, by (3) and (5) it follows, that $a \neq a_1$ or $a \neq a_2$. Collecting all facts obtained so far, one obtains that

$$\mathrm{F} \not\models (aRa_1 \vee a = a_1) \wedge (aRa_2 \vee a = a_2) \wedge \exists y \exists z_1 \exists z_2 CR(a_1, y, z_1, z_2)$$
$$\wedge \exists y \exists z_1 \exists z_2 CR(a_2, y, z_1, z_2) \to (a = a_1 \wedge a = a_2),$$

which refutes the second conjunct of $foe(UF)^*(a/x)$, i.e.

$$\mathcal{F} \not\models foe(UF)^*(a/x).$$

Next, let, for some world a of some frame $\mathcal{F}$, $\mathcal{F} \not\models foe(UF)^*(a/x)$. We show that $(UF)^*$ is refuted at a by some valuation on $\mathcal{F}$. First, we may suppose that $\mathcal{F} \models foe(LF)(a/x)$, because otherwise, we can refute LF at the world a. Now the cone of $\mathcal{F}$ generated by the world a is transitive and irreflexive, and there are a_1, a_2 in $\mathcal{F}$ such that aRa_1 or $a = a_1$ and aRa_2 or $a = a_2$, and $a \neq a_1$ or $a \neq a_2$, and moreover, there are $b', c_1', c_2', b'', c_1'', c_2''$ such that in $\mathcal{F}$ the formulas $CR(a_1, b', c_1', c_2')$, $CR(a_2, b'', c_1'', c_2'')$ hold. Because of $l(b') = l(b'') = 1$, we have

(7) $b' \models \Box^2 \bot$,

(8) $b'' \models \Box^2 \bot$.

Two cases arise here: $\{c_1', c_2'\} \cap \{c_1'', c_2''\} = \emptyset$ and $\{c_1', c_2'\} \cap \{c_1'', c_2''\} \neq \emptyset$. In the former case, we can choose a valuation such that:

$$c_1' \models p, \quad c_2' \not\models p, \quad c_1'' \models p, \quad c_2'' \not\models p.$$

In the latter case, say $c_1' = c_2''$, we can choose a valuation such that:

$$c_1' \models p, \quad c_2' \not\models p, \quad c_1'' \not\models p.$$

Thus, in both cases we have

(9) $b' \not\models \Box p \vee \Box \neg p$,

(10) $b'' \not\models \Box p \vee \Box \neg p$,

From (7), (9) and (8), (10) we obtain $b' \not\models F', b'' \not\models F'$, and hence $a \not\models F, a_2 \not\models F$. Recall that at least two worlds from a, a_1, a_2 are different, say $a \neq a_2$. Then we can suppose that

$$a \models q, \quad a_2 \not\models q,$$

whence $a \not\models \boxdot(q \to F) \vee \boxdot(\neg q \to F)$, and so $a \not\models (UF)^*$. Thus, Lemma 2.3 is proved. ■

Now, we define $foe(Last_2)$:

$$\begin{aligned} foe(Last_2) \;=\; & \forall y \forall z \forall u (xRy \wedge xRz \wedge xRu \wedge y \neq u \wedge y \neq z \wedge u \neq z \to \\ & l(y) \neq 0 \vee l(z) \neq 0 \vee l(u) \neq 0). \end{aligned}$$

LEMMA 2.4. *The formulas $Last_2$ and $foe(Last_2)$ are locally equivalent.*

PROOF. This is a standard exercise. ■

This statement allows us to define $foe(Last_2)^*$ as follows:

$$foe(Last_2)^* \;=\; foe(UF)^* \wedge foe(Last_2).$$

Thus, as a consequence of Lemmas 2.3 and 2.4 we have the following statement:

LEMMA 2.5. *The formulas* $(Last_2)^*$ *and* $foe(Last_2)^*$ *are locally equivalent.*

Now we define $foe(UF_0^2)^*$:

$$\begin{aligned} foe(UF_0^2)^* \quad = \quad & foe(Last_2)^* \wedge \forall y \forall z_1 \forall z_2 \forall u_1 \forall u_2 (CR(x,y,z_1,z_2) \wedge xRu_1 \\ & \wedge xRu_2 \wedge \neg u_1 R z_2 \wedge \neg u_2 R z_2) \vee (u_1 R z_2 \wedge u_2 R z_2 \wedge \neg u_1 R z_1 \\ & \wedge \neg u_2 R z_1)) \wedge l(u_1) = l(u_2) = 1 \rightarrow u_1 = u_2). \end{aligned}$$

Applied to the frame of Figure 1 the conjunct added to $foe(Last_2)^*$ means that the world from which the world f_0^1 (or f_1^1) is accessible exactly by one step, but f_1^1 (respectively, f_0^1) is not accessible, is unique. The conjuncts to be added later to $UF_0^3, \ldots, UF_0^6$ have an analogous sense.

LEMMA 2.6. *The formulas* $(UF_0^2)^*$ *and* $foe(UF_0^2)^*$ *are locally equivalent.*

Our next step is the following definition, for all $i, 3 \leqslant i \leqslant 6$:

$$\begin{aligned} foe(UF_0^i)^* \quad = \quad & foe(UF_0^{i-1})^* \wedge \forall y \forall z_1 \forall z_2 \forall u_1 \forall u_2 (CR(x,y,z_1,z_2) \\ & \wedge((u_1 R^{i-1} z_1 \wedge u_2 R^{i-1} z_1 \wedge \neg u_1 R z_2 \wedge \neg u_2 R z_2) \\ & \vee(u_1 R^{i-1} z_2 \wedge u_2 R^{i-1} z_2 \wedge \neg u_1 R z_1 \wedge \neg u_2 R z_1)) \\ & \wedge l(u_1) = l(u_2) = 1 \rightarrow u_1 = u_2). \end{aligned}$$

LEMMA 2.7. *The formulas* $(UF_0^i)^*$ *and* $foe(UF_0^i)^*$ *are locally equivalent for* $3 \leqslant i \leqslant 6$.

PROOF. By a stepwise argument for each successive i $(3 \leqslant i \leqslant 6)$. ∎

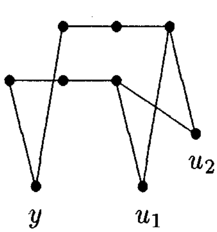

Figure 2.

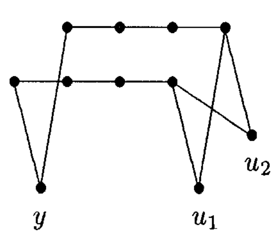

Figure 3.

Let now $foe(UA_0^0)$ be the negation of a formula expressing the following situation: worlds are accessible from world x which form a subframe generated by worlds y, u_1, u_2. This is sketched in Figure 2. Set

$$foe(UA_0^0)^* \ = \ foe(UF_0^6)^* \wedge foe(UA_0^0).$$

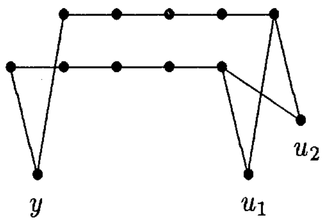

Figure 4.

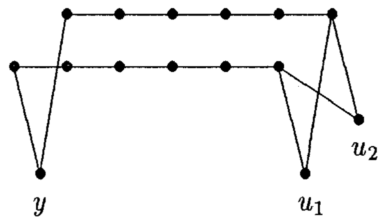

Figure 5.

The formulas $foe(UA_0^1), foe(UA_0^2), foe(UA_0^3)$ are defined analogously, using the frames in Figures 3, 4, 5, respectively, instead of Figure 2. Now define

$$foe(UA_0^i)^* \;=\; foe(UA_0^{i-1})^* \wedge foe(UA_0^i),\ 1 \leqslant i \leqslant 3.$$

LEMMA 2.8. *The formulas* $(UA_0^i)^*$ *and* $foe(UA_0^i)^*$ *are locally equivalent, for* $1 \leqslant i \leqslant 3$.

PROOF. Again, this is constructed successively for each i, $0 \leqslant i \leqslant 3$.

Now, if $foe(UA_0^3)$ is true in some world x of some frame, then the generated subframes sketched in Figures 2–5 cannot be accessible from this world, whereas the generated subframes sketched in figures 6–9 can be. We shall say that, if one of these generated subframes is accessible from world x in a frame, then world v_i (see Figures 6–9) is the 'i-th marked world'. Note that the formulas $(UA_0^i)^*$ and $foe(UA_0^i)^*$ state, in particular, that in a frame at most one i-th marked world is accessible from some world. For later convenience, we introduce the following predicates $ch_i(u)$:

$ch_i(u)$: the i-th marked world is accessible from u or is equal to u and the other marked worlds are not accessible from u,$0 \leqslant i \leqslant 2$;
$ch_3(u)$: the third marked world is accessible from u, while other marked worlds are not accessible.

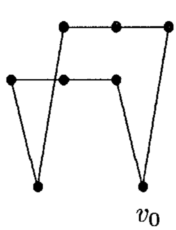

Figure 6.

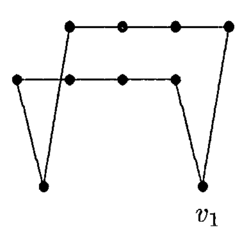

Figure 7.

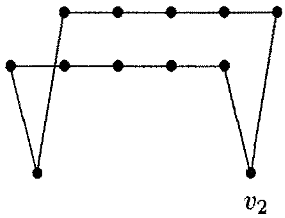

Figure 8.

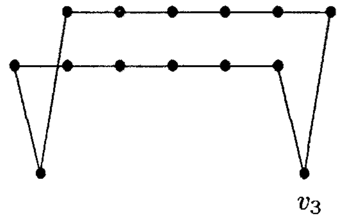

Figure 9.

Let $foeLin\Diamond A_0^i$ express the following: a set of worlds accessible from x and having the property ch_i is linearly ordered. Set

$$\begin{aligned} foe(Lin\Diamond A_0^0)^* &= foe(UA_0^3)^* \wedge foeLin\Diamond A_0^0, \\ foe(Lin\Diamond A_0^i)^* &= foe(Lin\Diamond A_0^{i-1})^* \wedge foeLin\Diamond A_0^i, \qquad 1 \leqslant i \leqslant 3. \end{aligned}$$

LEMMA 2.9. *The formulas* $(Lin\Diamond A_0^i)^*$ *and* $foe(Lin\Diamond A_0^i)^*$ *are locally equivalent for each* i, $0 \leqslant i \leqslant 3$.

PROOF. The proof is again by successive construction for each i, $0 \leqslant i \leqslant 3$. ∎

Next, for $(UT)^*$ we define the following first-order equivalent:

$$\begin{aligned} foe(UT)^* = \; & (foeLin\Diamond A_0^3)^* \wedge \forall y \forall z_1 \forall z_2 \forall u \forall v (CR(x,y,z_1,z_2) \wedge xRu \\ & \wedge xRv \wedge \exists w_0(uRw_0 \wedge vRw_0 \wedge ch_0(w_0)) \wedge \exists w_1(uRw_1 \wedge vRw_1 \\ & \wedge ch_1(w_1)) \wedge \exists w_2(uRw_2 \wedge vRw_2 \wedge ch_2(w_2)) \wedge \exists w_3(uRw_3 \\ & \wedge vRw_3 \wedge \neg uR^2w_3 \wedge \neg vR^2 w_3 \wedge ch_3(w_3)) \rightarrow u = v). \end{aligned}$$

LEMMA 2.10. *The formulas* $(UT)^*$ *and* $foe(UT)^*$ *are locally equivalent.*

PROOF. Via a calculation using all previous Lemmas of this Section. ∎

Next, we get to conjuncts simulating the instructions of our Minsky Machine. For future use, we make the following abbreviations (where u, w_1, w_2 are always free variables):

$$\begin{aligned} & fodS(\gamma, Q_1, R_1) = \\ & \exists t_1 \exists t_2 \cdots \exists t_{\gamma+1}(uRt_1Rt_2 \cdots Rt_{\gamma+1} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+1})) \\ & \wedge \neg \exists t_1 \exists t_2 \cdots \exists t_{\gamma+2}(uRt_1Rt_2 \cdots Rt_{\gamma+2} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+2})) \\ & \wedge uRw_1 \wedge \neg uR^2w_1 \wedge ch_1(w_1) \wedge uRw_2 \wedge \neg uR^2 w_2 \wedge ch_2(w_2), \end{aligned}$$

$fodS(\gamma, Q_2, R_1) =$
$\exists t_1 \exists t_2 \cdots \exists t_{\gamma+1}(uRt_1Rt_2 \cdots Rt_{\gamma+1} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+1}))$
$\wedge \neg\exists t_1 \exists t_2 \cdots \exists t_{\gamma+2}(uRt_1Rt_2 \cdots Rt_{\gamma+2} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+2}))$
$\wedge uR^2w_1 \wedge \neg uR^3w_1 \wedge ch_1(w_1) \wedge uRw_2 \wedge \neg uR^2w \wedge ch_2(w_2)$,

$fodS(\gamma, Q_1, R_2) =$
$\exists t_1 \exists t_2 \cdots \exists t_{\gamma+1}(uRt_1Rt_2 \cdots Rt_{\gamma+1} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+1}))$
$\wedge \neg\exists t_1 \exists t_2 \cdots \exists t_{\gamma+2}(uRt_1Rt_2 \cdots Rt_{\gamma+2} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+2}))$
$\wedge uRw_1 \wedge \neg uR^2w_1 \wedge ch_1(w_1) \wedge uR^2w_2 \wedge \neg uR^3w_2 \wedge ch_2(w_2)$,

$fodS(\gamma, A_0^1, R_1) =$
$\exists t_1 \exists t_2 \cdots \exists t_{\gamma+1}(uRt_1Rt_2 \cdots Rt_{\gamma+1} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+1}))$
$\wedge \neg\exists t_1 \exists t_2 \cdots \exists t_{\gamma+2}(uRt_1Rt_2 \cdots Rt_{\gamma+2} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+2}))$
$\wedge uRw_1 \wedge ch_1(w_1) \wedge \forall w'(uR^2w' \to \neg ch_1(w')) \wedge uRw_2 \wedge \neg uR^2w_2 \wedge ch_2(w_2)$,

$fodS(\gamma, Q_1, A_0^2) =$
$\exists t_1 \exists t_2 \cdots \exists t_{\gamma+1}(uRt_1Rt_2 \cdots Rt_{\gamma+1} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+1}))$
$\wedge \neg\exists t_1 \exists t_2 \cdots \exists t_{\gamma+2}(uRt_1Rt_2 \cdots Rt_{\gamma+2} \wedge ch_0(t_1) \wedge ch_0(t_2) \wedge \cdots \wedge ch_0(t_{\gamma+2}))$
$\wedge uRw_1 \wedge \neg uR^2w_1 \wedge ch_1(w_1) \wedge uRw_2 \wedge ch_2(w_2) \wedge \forall w'(uR^2w' \to \neg ch_2(w'))$.

Now we can continue our definitions. For any instruction I we write:

if I $= q_\gamma \to q_\delta T_1 T_0$, then $foe((UT)^* \wedge AxI) = foe(UT)^* \wedge$
$\forall y \forall z_1 \forall z_2 \forall w_1 \forall w_2 \forall w_3 (CR(x,y,z_1,z_2) \wedge \exists u(xRu \wedge fodS(\gamma, Q_1, R_1) \wedge uRw_3 \wedge$
$\neg uR^2w_3 \wedge ch_3(w_3)) \wedge \exists w_3'(xRw_3' \wedge w_3'Rw_3 \wedge ch_3(w_3')) \to$
$\exists u(xRu \wedge fodS(\delta, Q_2, R_1) \wedge \exists w_3'(uRw_3'Rw_3 \wedge \neg w_3'R^2w_3 \wedge ch_3(w_3')))))$.

if I $= q_\gamma \to q_\delta T_0 T_1$, then the definition of $foe((UT)^* \wedge AxI)$ can be obtained from the previous one by substituting $fodS(\delta, Q_1, R_2)$ for $fodS(\delta, Q_2, R_1)$,

if I $= q_\gamma \to q_\delta T_{-1} T_0 (q_\varepsilon T_0 T_0)$, then $foe((UT)^* \wedge AxI) = foe(UT)^*$
$\wedge \forall y \forall z_1 \forall z_2 \forall w_1 \forall w_2 \forall w_3 (CR(x,y,z_1,z_2) \wedge \exists u(xRu \wedge fodS(\gamma, Q_2, R_1) \wedge uRw_3$
$\wedge \neg uR^2w_3 \wedge ch_3(w_3)) \wedge \exists w_3'(xRw_3' \wedge w_3'Rw_3 \wedge ch_3(w_3')) \to \exists u(xRu$
$\wedge fodS(\delta, Q_1, R_1) \wedge \exists w_3'(uRw_3'Rw_3 \wedge \neg w_3'R^2w_3 \wedge ch_3(w_3'))))$
$\wedge \forall y \forall z_1 \forall z_2 \forall w_1 \forall w_2 \forall w_3 (CR(x,y,z_1,z_2) \wedge \exists u(xRu \wedge fodS(\gamma, A_0^1, R_1) \wedge uRw_3$
$\wedge \neg uR^2w_3 \wedge ch_3(w_3)) \wedge \exists w_3'(xRw_3' \wedge w_3'Rw_3 \wedge ch_3(w_3')) \to \exists u(xRu$
$\wedge fodS(\varepsilon, A_0^1, R_1) \wedge \exists w_3'(uRw_3'Rw_3 \wedge \neg w_3'R^2w_3 \wedge ch_3(w_3'))))$.

if I $= q_\gamma \to q_\delta T_0 T_{-1} (q_\varepsilon T_0 T_0)$, then the definition of $foe((UT)^* \wedge AxI)$ can be obtained from the previous one by substituting $fodS(\delta, Q_1, R_2)$ for $fodS(\gamma, Q_2, R_1)$, and $fodS(\delta, Q_1, A_0^2)$ for $fodS(\delta, A_0^1, R_1)$.

LEMMA 2.11. *The formulas $(UT)^* \wedge AxI$ and $foe((UT)^* \wedge AxI)$ are locally equivalent for any instruction I.*

Finally, we define

$$foe(AxP)^* = \bigwedge_{I \in P} foe((UT)^* \wedge AxI)$$

Evidently, local equivalence of φ_1 and ψ_1, φ_2 and ψ_2 implies local equivalence of $\varphi_1 \wedge \varphi_2$ and $\psi_1 \wedge \psi_2$, and hence Lemma 2.11 implies:

LEMMA 2.12. *The formulas $(AxP)^*$ and $foe(AxP)^*$ are locally equivalent.*

Because $(AxP)^*$ is $A(P)$, Lemma 2.1 is then obtained from Lemma 2.12. Now change $A(P)$ by adding the conjunct $\Diamond T$. From Lemma 2.12 we also get

LEMMA 2.13. *The formulas $A(P) \wedge \Diamond T$ and $foe(AxP)^* \wedge \exists y(xRy)$ are locally equivalent, and so $A(P) \wedge \Diamond T$ is locally first-order definable.*

Note, that a global first-order equivalent of $A(P) \wedge \Diamond T$ is $\bot$ or, for example, $\forall x(x \neq x)$.

It only remains to be verified that our new first-order version of $A(P)$ works as before. Consider the proof of the main technical Lemma 1.2 from Section 1. Its "if" part depended on the conjuncts LF, $LinT$, UT, AxP, but these still occur in the new variant of $A(P)$. For its "only if" part, it was important that $A(P)$ should be true in the frame of Figure 1, and that $A(P) \wedge \Diamond T$ be true in world f of this frame. For the latter purpose, because of Lemmas 2.12 and 2.13, it suffices to have the two formulas $\forall x foe(AxP)^*$ and $foe(AxP)^* \wedge \exists y\,(xRy(x/f))$ true in this frame. And this is obvious from their definitions (which were in fact inspired by Figure 1) plus the proofs of Lemmas 2.12 and 2.13. Thus, we obtain a following results:

PROPOSITION 2.14. *For the variant of the formula $A(P)$ defined in this section, Lemma 1.2 holds.*

PROPOSITION 2.15. *For the variant of the formula $A(P)$ defined in this section, Lemma 1.2 also holds with $\models^{loc}_{fin}$ instead of $\models_{fin}$.*

3. Modal Formulas Without First-Order Equivalents on Finite Frames

In this Section, we consider the part of Figure 1 surrounded by the dotted line. Essentially, the results formulated here are small modifications of results obtained by K. Doets in [9]. First we formulate some key facts which we shall need.

Recall the definition of n-*equivalence* of models, notation: $A \equiv_n B$. We use game-theoretical terminology as in [10], [9] ([8] has this notion in a different but equivalent form). Note that we have only equality and one binary relation in the signature (or language) of our models. Now the n-*game* on A and B, $G(A, B, n)$, has two players, I and II, which move alternatively. I is allowed the first move; each player is allowed n moves altogether. A move consist in the selection of an element in either W_1 or W_2 (where W_1 and W_2 are the universes of A and B, respectively). However, if player I chooses an element in W_1 (W_2), then player II has to counter in W_2 (W_1). Therefore, a move of player I and the following counter-move of player II form an ordered pair in $W_1 \times W_2$. When the game is over, the set of ordered pairs of moves is at most an n-element relation $h \subseteq W_1 \times W_2$. II has won the play by definition if h is a partial isomorphism between W_1 and W_2. Now, A and B are called 'n-equivalent' if II has a winning strategy for $G(A, B, n)$. Here is the basic logical property of $\equiv_n$. If $A \equiv_n B$ then, for any first-order formula φ of quantifier depth at most n,

$$A \models \varphi \text{ iff } B \models \varphi.$$

Following [9], let PZ_k be the nontransitive frame of Figure 10. The unpainted cicrcles represent reflexive worlds.

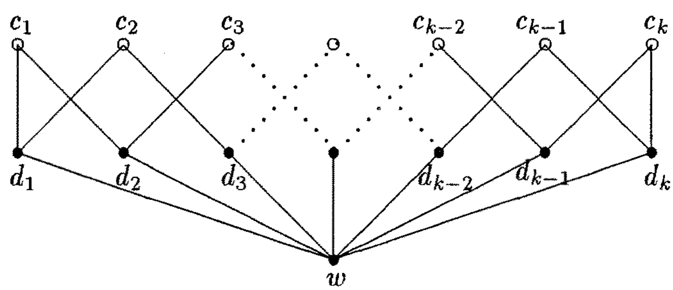

Figure 10.

LEMMA 3.1. [9]. *If $k, m \geqslant 2^n$, then $PZ_k \equiv_n PZ_m$.*

If we now take the McKinsey Formula $\Box\Diamond p \to \Diamond\Box p$, then it is known (cf. [11]) that this formula is true in PZ_k iff k is odd. If one then supposes that this formula has a first-order equivalent (global or local), say φ of quantifier depth n, then we get that both $PZ_{2^n} \models \varphi$ and $PZ_{2^n+1} \models \varphi$, although $PZ_{2^n} \not\models \Box\Diamond p \to \Diamond\Box p$, $PZ_{2^n+1} \models \Box\Diamond p \to \Diamond\Box p$: which is a contradiction. Thus, the McKinsey Formula is not first-order definable (locally, globally) on finite frames ([9]). We shall modify the formula and frames employed in this argument in a manner suitable for our purposes. Call the frame in Figure 11 $FENCE_k$.

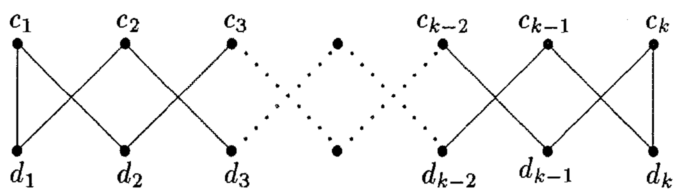

Figure 11.

LEMMA 3.2. *If $k, m \geqslant 2^n$, then $FENCE_k \equiv_n FENCE_m$.*

PROOF. Use the winning strategy for player II from the proof of Lemma 3.1. ■

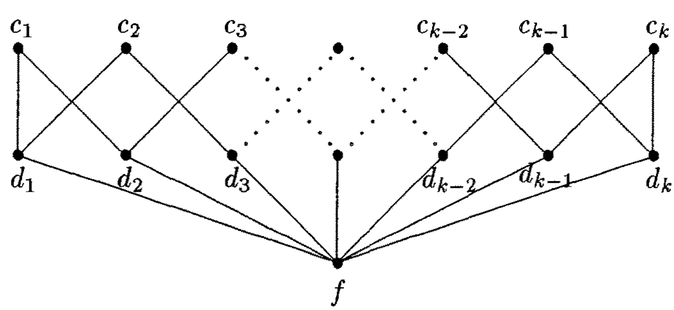

Figure 12.

If we take the frames from the Figure 12, denoting them by $FENCE_k^*$ for future references, then Lemma 3.2 implies that, if $k, m \geqslant 2^n$, then $FENCE_k^* \equiv_n FENCE_m^*$. Indeed the strategy from the proof of Lemma 3.2 is still suitable, with the following stipulation added:

> "if player I chooses f in one model, then II chooses f in the other model".

Next, we use a modified McKinsey Formula of the form

$$\alpha = \underline{\Diamond\Diamond\top\wedge}\Box(\underline{\Diamond\top\to}\Diamond p) \to \Diamond(\underline{\Diamond\top\wedge}\Box p).$$

The underlined parts of α represent changes to the original McKinsey Formula. It is easy to see that α is refutable in $FENCE_k^*$ iff k is odd, and the refutation can occur only in the world f. Thus α is not first-order definable on the class of finite frames, and not even on the class of finite transitive irreflexive frames. These observations establish the following

PROPOSITION 3.3. *The formula* $(\Box(\Box p \to p) \to \Box p) \wedge \alpha$ *is not globally first-order definable on the class of finite frames.*

PROPOSITION 3.4. *The formula* $(\Box(\Box p \to p) \to \Box p) \wedge \alpha \wedge \Diamond\top$ *is not locally first-order definable on finite frames, although is globally first-order definable on them.*

PROOF. A global first-order equivalent of this formula is, for example, $\forall x\, x \neq x$. The absence of a local equivalent is proved as above, but now using the frames $FENCE_k^*$, in which f is the 'actual world'. ■

REMARK. Consider the modal formula

$$\beta = \Box^3\bot \wedge (\Box(\Box p \to p) \to \Box p) \wedge \alpha \wedge \Diamond T.$$

It may be proved analogously that β is globally first-order definable on the class of all transitive frames, but it is not locally first-order definable on the class of all transitive frames. This answers a question by van Benthem ([1], p. 129).

The frame sketched in Figure 1 is obtained from $FENCE_t$ by adding some subframe. Let us call it $\mathcal{F}_t(P, \alpha, m, n)$.

LEMMA 3.5. *If* $t_1, t_2 \geqslant 2^l$, *then* $\mathcal{F}_{t_1}(P, \alpha, m, n) \equiv_l \mathcal{F}_{t_2}(P, \alpha, m, n)$.

PROOF. Player II needs the strategy from the proof of Lemma 3.2 with one additional rule: "if I chooses some element out of $FENCE_{t_i}$ in $\mathcal{F}_{t_i}(P, \alpha, m, n)$, then II chooses the same element in $\mathcal{F}_{t_{1-i}}(P, \alpha, m, n)$". ■

Now we explain the modified McKinsey Formula, i.e., the underlined parts of formula α. The part $\Diamond\Diamond\top\wedge$ has been added to make sure that a possible refutation could occur only at world f in the frame $FENCE_k^*$. The parts $\Diamond\top \to$ and $\Diamond\top\wedge$ were added to give some information just about the worlds

$d_1, \ldots, d_k$ in $FENCE_k^*$. Define the following modification of the McKinsey Formula (changes are again underlined):

$$\gamma = \underline{\underline{\neg F \wedge}} \Box (\underline{\underline{\Diamond\Diamond B \rightarrow}} \Diamond (\underline{\underline{\Diamond B \wedge}} q)) \rightarrow \Diamond (\underline{\underline{\Diamond\Diamond B \wedge}} \Box (\underline{\underline{\Diamond B \rightarrow}} q)),$$

(here q is used instead of p, because p was already used in F), where

$$B = \Diamond F_0^7 \wedge \Diamond F_1^7 \wedge \Box^8 \bot.$$

From Lemmas 1.4, 1.5, we then obtain the following fact:

LEMMA 3.6. *The formula $A(P) \wedge (B(\alpha, m, n) \vee \gamma)$ is refutable in the frame $\mathcal{F}_t(P, \alpha, m, n)$ iff t is odd, and the refutation can take place only in the world f.*

As a consequence of Lemmas 3.5 and 3.6, we obtain one final

PROPOSITION 3.7. *If a program P, starting at a configuration (α, m, n) halts in a final state with the number β, then:*

i) the formula $A(P) \wedge (B(\alpha, m, n) \vee \gamma)$ is not first-order definable on finite frames (locally, globally);

ii) the formula $A(P) \wedge (B(\alpha, m, n) \vee \gamma) \wedge \Diamond T$ is globally first-order definable on finite frames, but is not locally first-order definable on them.

4. Main Results

Now we can formulate the main results of this paper. Here, $A(P)$ is defined as in Section 2.

LEMMA 4.1. *The following three conditions are equivalent for any program P and configuration (α, m, n):*

i) program P, starting from configuration (α, m, n), cannot reach a final state (with number β),

ii) the formula $A(P) \wedge (B(\alpha, m, n) \vee \gamma)$ is locally first-order definable on finite frames,

iii) the formula $A(P) \wedge (B(\alpha, m, n) \vee \gamma)$ is globally first-order definable on finite frames.

PROOF. i) $\Rightarrow$ ii). If i) holds, then by Lemma 1.2, Proposition 2.14 and the Remark preceding Theorem 1.10, in any frame with a designated world where $A(P)$ is true, $B(\alpha, m, n)$ is true as well. So the formula $A(P) \wedge (B(\alpha, m, n) \vee \gamma)$ is locally equivalent to the formula $A(P)$, which is locally first-order definable by Lemma 2.1.
ii) $\Rightarrow$ iii). This direction is trivial.
iii) $\Rightarrow$ i). Let i) be false. Then, by Proposition 3.7.i), iii) is not true either. ∎

Because the Halting Problem for Minsky Machines is undecidable, and $A(P)$ and $B(\alpha, m, n)$ have been constructed effectively from P and (α, m, n), Lemma 4.1 now implies two further results:

THEOREM 4.2. *The problem of recognizing, given any modal formula, whether it is locally first-order definable on finite frames, is algorithmically undecidable.*

THEOREM 4.3. *The problem of recognizing, given any modal formula, whether it is globally first-order definable on finite frames, is algorithmically undecidable.*

Theorems 4.2 and 4.3 are in a sense independent. Neither follows directly from the other, witness the following obserwation:

LEMMA 4.4. *The following conditions are equivalent for any program P and configuration (α, m, n):*

i) program P, starting from configuration (α, m, n), halts in a final state (with number β),

ii) the formula $A(P) \wedge (B(\alpha, m, n) \vee \gamma) \wedge \Diamond T$ is globally first-order definable on finite frames, but is not locally first-order definable on finite frames.

PROOF. i) $\Rightarrow$ ii). By Proposition 3.7.ii). ii) $\Rightarrow$ i). Analogous to the proof of i) $\Rightarrow$ ii), but now using Lemma 2.13 instead of Lemma 2.1. ∎

From Lemma 4.4, we obtain our next result:

THEOREM 4.5. *The set of modal formulas which are globally, but not locally, first-order definable on finite frames, is algorithmically undecidable.*

We conclude with one natural question. The formula LF is first-order definable on finite frames, but is not first-order definable even on all countable frames.

THEOREM 4.6.

i) The set of modal formulas which are first-order definable on the class of finite frames, but not first-order definable on the class of all frames, is algorithmically undecidable.

ii) The set of modal formulas which are first-order definable on the class of finite frames, but not first-order definable on the class of all countable frames, is algorithmically undecidable.

PROOF. An argument for i) and ii) can be given simultaneously. Return to the proof of Lemma 4.1. The proof of the implication iii) $\Rightarrow$ i) showed in fact that, if i) is not true, then $A(P) \wedge (B(\alpha, m, n) \vee \gamma)$ is not a formula which is first-order definable on finite frames, but not first-order definable over all (countable) frames. In the case of the proof i) $\Rightarrow$ ii), we observe that the formula $A(P)$ is not first-order definable on the class of all (countable) frames. This is proved via the usual counter-example to first-order definability for LF, using linear frames in which all conjuncts of $A(P)$ except LF are true. ∎

REMARK. The fact that Theorem 4.6.i) and Theorem 4.6.ii) are independent follows from the undecidability (announced in [6]) of a set of formulas which are first-order definable on countable frames, but not first-order definable in general.

QUESTION: Whether any of the sets of formulas mentioned in Theorems 4.2, 4.3, 4.5, 4.6, or their complements, is recursive enumerable.

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