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Modal Companions of Intermediate Propositional Logics

Abstract. This paper is a survey of results concerning embeddings of intuitionistic propositional logic and its extensions into various classical modal systems.

0.0. What is intuitionistic logic from the viewpoint of classical logic? It is this question that Kolmogorov and Gödel appear to have had in mind while trying to give a classical interpretation of the intuitionistic propositional calculus **Int** with the help of modal operators such as "provable", "solvable". Kolmogoroff [1932] treated **Int** as a certain "logic of problems", whereas Gödel [1933] suggested a more formal interpretation of **Int** by "embedding" it in the Lewis modal system **S4** and treating the modality $\Box$ ("necessarily") as "provable". In the fifties, similar ideas were developed by Novikov in his lectures on constructive mathematical logic which afterwards were published in book form Novikov [1977] under the title "Constructive mathematical logic from the point of view of classical logic".

It seems to us, however, that this history will remain incomplete if we do not mention Orlov's paper published in 1928. Orlov [1928] explicitly introduced the provability operator Φ, described the axioms of provability, which in fact were the same as Gödel's axioms for **S4**, and the intuitionistic validity of a proposition A was understood by him as $\Phi(A)$.[1]

0.1. Afterwards a number of papers appeared in which embeddings of **Int** and its extensions (i.e. intermediate or superintuitionistic logics) in various modal systems were investigated. The present paper is a brief survey of results obtained up to now in this direction.

It should be emphasized that here we deal with only propositional logics. It is for the propositional case that the problem of embedding intermediate logics in modal ones is most developed. Naturally, richer languages (say, first-order ones) are of more importance for applications, but though there are interesting results and problems in the case of intermediate predicate logics, we cannot present them as a whole picture yet.

0.2. A few words about our notation. If L is a (modal or intermediate)

[1] Besides, Orlov's paper is remarkable for introducing the first system of relevant logic (for historical and critical comments consult Popov [1986] and Došen [1990a]).

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logic then by $L + \{A_i\}_{i\in I}$ we denote the smallest logic containing L, the set of formulas $\{A_i\}_{i\in I}$, and closed under substitution and modus ponens. For modal logics, when we want to take the closure under necessitation too, we use $\oplus$ instead of $+$ and write $L \oplus \{A_i\}_{i\in I}$.

The lattice of all extensions of a logic L is denoted by $\mathcal{L}L$, while the lattice of *normal* extensions of a modal logic L is denoted by $\mathcal{L}NL$.

1.0. The theorem on the embedding of **Int** in **S4** stated by Gödel [1933] as a conjecture was proved by McKinsey and Tarski [1948] on the basis of their previous topological and algebraic study of these logics (another topological proof can be found in Novikov [1977]; for syntactical proofs using Gentzen-style techniques see Maehara [1954], Hacking [1963], Schüte [1968], Prawitz and Malmnäs [1968]). The formulation of the theorem is as follows:

For any intuitionistic formula A,

$$\mathbf{Int} \vdash A \textit{ iff } \mathbf{S4} \vdash T(A)$$

where $T(A)$ — "translation" of A —is the modal formula obtained by prefixing the necessity operator $\Box$ to every subformula of A.

(Actually, the authors mentioned above considered several different forms of translation (see, e.g., Section 5.1); however, differences between them can be neglected as far as **S4** and its normal extensions are concerned.) Under the translation T the intuitionistic connectives turn into the corresponding classical ones, but every subformula of A is understood now in the context of its "provability".

1.1. With each intermediate logic

$$L = \mathbf{Int} + \{A_i\}_{i\in I}$$

Dummett and Lemmon [1959] associated the normal modal logic

$$\tau L = \mathbf{S4} \oplus \{T(A_i)\}_{i\in I}$$

and proved that L can be embedded in τL by T, i.e. the equivalence in Section 1.0 can be extended to the equivalence

$$L \vdash A \text{ iff } \tau L \vdash T(A).$$

(The notation τL was introduced later by Maksimova and Rybakov [1974]; see Section 1.3.) This result made it possible to solve some problems in modal logic using known intuitionistic facts.

Dummett and Lemmon observed that

(i) for the classical logic **Cl**

$$\tau\mathbf{Cl} = \mathbf{S5} = \mathbf{S4} \oplus \Diamond p \supset \Box\Diamond p;$$

(ii) for the logic $\mathbf{KC} = \mathbf{Int} + \neg p \vee \neg\neg p$ of the Weak Law of Excluded Middle

$$\tau\mathbf{KC} = \mathbf{S4.2} = \mathbf{S4} \oplus \Diamond\Box p \supset \Box\Diamond p;$$

(iii) for the Dummett logic $\mathbf{LC} = \mathbf{Int} + (p \supset q) \vee (q \supset p)$

$$\tau\mathbf{LC} = \mathbf{S4.3} = \mathbf{S4} \oplus \Box(\Box p \supset \Box q) \vee \Box(\Box q \supset \Box p).$$

1.2. Grzegorczyk [1967] discovered that **Int** can be embedded via T not only in **S4** but also in a proper extension of **S4** that is now known as the Grzegorczyk logic

$$\mathbf{S4Grz} = \mathbf{S4} \oplus \Box(\Box(p \supset \Box p) \supset p) \supset p.$$ [2]

Thus in the equivalence from Section 1.0 **S4** can be replaced by **S4Grz**:

$$\mathbf{Int} \vdash A \text{ iff } \mathbf{S4Grz} \vdash T(A).$$

1.3. A systematic study of the relationship between the lattice $\mathcal{L}\mathbf{Int}$ of extensions of **Int** and the lattice $\mathcal{LN}\mathbf{S4}$ of normal extensions of **S4** which is given by the translation T was started by Maksimova and Rybakov [1974], Blok [1976], and Esakia [1979, 1979a].

Maksimova and Rybakov [1974] introduced the mapping $\rho : \mathcal{LN}\mathbf{S4} \to \mathcal{L}\mathbf{Int}$ that with each modal logic M associates the intermediate logic $\rho M = \{A \mid M \vdash T(A)\}$ which is embedded in M by T. The logic ρM was called by Esakia [1979a] the *superintuitionistic fragment of* M, while M itself was called a *modal companion of* ρM.

Besides the mapping ρ which is a lattice homomorphism from $\mathcal{LN}\mathbf{S4}$ onto $\mathcal{L}\mathbf{Int}$ preserving infinite unions and intersections of logics, two more mappings were considered by Maksimova and Rybakov [1974]: the mapping τ (mentioned in Section 1.1) which turns out to be a lattice isomorphism from $\mathcal{L}\mathbf{Int}$ into $\mathcal{LN}\mathbf{S4}$ preserving infinite unions of logics and mapping $\sigma : \mathcal{L}\mathbf{Int} \to \mathcal{LN}\mathbf{S4}$ defined in algebraic terms (see Section 3.3). It was proved that the set $\rho^{-1}L$ of all modal companions of an arbitrary intermediate logic

[2] Actually, Grzegorczyk [1967] used another axiom:

$$\Box(\Box(\Box(p \supset \Box q) \supset \Box q) \& \Box(\Box(\neg p \supset \Box q) \supset \Box q) \supset \Box q).$$

The axiom above is due to Sobociński [1964].

L is infinite (more exactly, it contains an infinite descending chain of logics) and has the least element τL and the greatest element σL. Thus, $\rho^{-1}L$ is the infinite interval of logics

$$\rho^{-1}L = \{M \mid \tau L \subseteq M \subseteq \rho L\} = [\tau L, \sigma L].$$

Blok [1976] and Esakia [1979, 1979a][3] found that, for each intermediate logic L,

$$\sigma L = \tau L \oplus \Box(\Box(p \supset \Box p) \supset p) \supset p = \tau L \oplus \mathbf{S4Grz},$$

i.e. the greatest companion of L is obtained from the least one by adding to it the Grzegorczyk formula. (In particular, **S4Grz** is the greatest companion of **Int** in $\mathcal{L}\mathcal{N}$**S4**.) According to the Blok-Esakia theorem, the mapping σ turns out to be an isomorphism from $\mathcal{L}$**Int** onto the lattice $\mathcal{L}\mathcal{N}$**S4Grz** of normal extensions of **S4Grz**.

Note that from the results above and the decidability of **S4Grz** (see Segerberg [1971]) it follows that there exists an algorithm which is capable of deciding, for a modal formula A, whether **S4** $\oplus$ A is a modal companion of **Int**.

1.4. Thus we have the following picture shown in Figure 1: the lattice $\mathcal{L}\mathcal{N}$**S4** (without the inconsistent logic) is divided by ρ into the intervals $\rho^{-1}L$ (or "Augean stables" as they were appropriately called by Esakia [1979a]; for a justification see Sections 1.5, 1.6) which are in 1–1 correspondence with the logics in $\mathcal{L}$**Int** (without the inconsistent logic as well), the "ends" of these intervals lying in the intervals [**S4**,**S5**] and [**S4Grz**,**S5Grz**].

Of standard logics, ρ^{-1}**Int** contains the McKinsey system **S4.1** = **S4** $\oplus$ $\Box\Diamond p \supset \Diamond\Box p$. The interval ρ^{-1}**Cl** corresponding to the classical propositional calculus **Cl** consists of all consistent extensions of **S5**; as was shown by Scroggs [1951], it contains only countably many logics which are linearly ordered by inclusion.

[3]It is to be noted that Blok and Esakia obtained these results independently. Esakia announced them at the VII USSR Symposium for Logic and Methodology of Science in 1976.

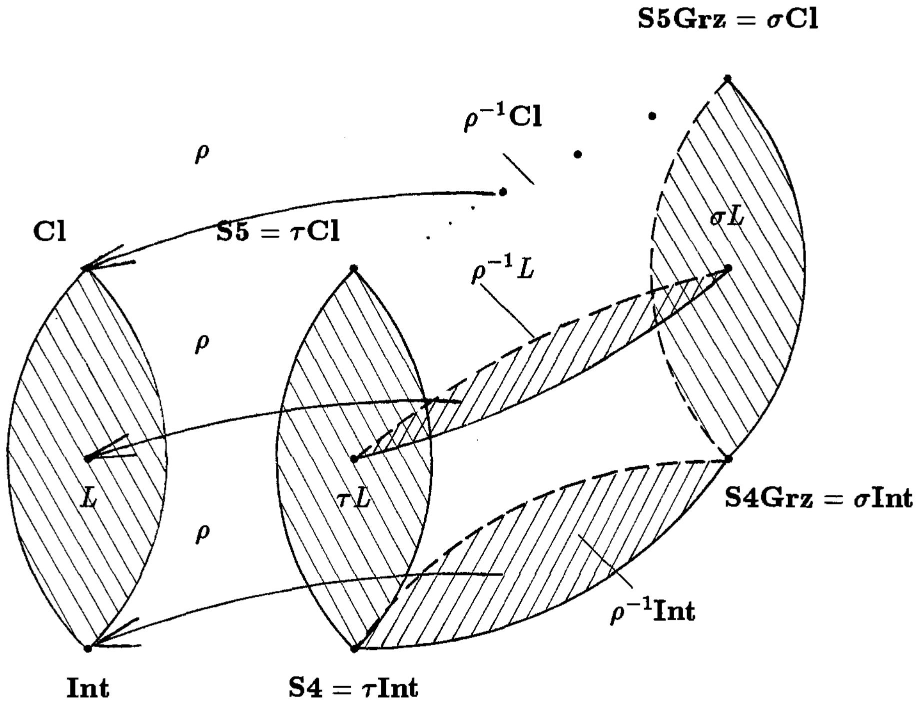


Figure 1

1.5. Rybakov [1976] proved that the lattice $\mathcal{L}L$ of extensions of an intermediate logic L is isomorphically embedded in the interval $\rho^{-1}L$. It follows, in particular, that there are a continuum of modal companions of **Int**. However, the intervals corresponding to the tabular and pretabular intermediate logics contain, as was shown by Rybakov [1976], only countably many logics. For (pretabular) Dummett's logic **LC** this fact is also a consequence of Fine's [1971] result, according to which there are $\aleph_0$ logics containing **S4.3**. A semantic description of the intervals for extensions of **LC** was given by Meskhi [1974].

1.6. To describe the structure of the interval $\rho^{-1}L$ corresponding to an arbitrary intermediate logic L seems to be as difficult as to cleanse the Augean stables. However, one can try to compare the structure of the interval $\rho^{-1}L$ with the structure of the lattice $\mathcal{L}L$, and, in particular, to determine whether the complexity of $\rho^{-1}L$ (including the existence of undecidable, incomplete, etc. logics in this interval) mainly depends on the complexity of $\mathcal{L}L$ (and the existence of logics with negative properties in it). In any case, the isomorphic embeddings of Rybakov [1976] make it possible to construct undecidable and incomplete calculi in $\rho^{-1}\mathbf{Int}$ using the known undecidable and incomplete intermediate calculi of Shekhtman [1977, 1978]. We conjecture that:

(1) There exists an undecidable logic (calculus) in $\mathcal{L}L$ iff there exists an undecidable logic (calculus) in $\rho^{-1}L$.

(2) There exists an incomplete logic in $\mathcal{L}L$ iff there exists an incomplete logic in $\rho^{-1}L$.

(3) There exists a logic without the finite model property in $\mathcal{L}L$ iff there exists a logic without this property in $\rho^{-1}L$.

(4) An interval $\rho^{-1}L$ has a continuum of logics iff there are a continuum of logics in the lattice $\mathcal{L}L$.

Rybakov [1976] showed that there exist a continuum of isomorphisms from $\mathcal{L}\mathbf{Int}$ into $\rho^{-1}\mathbf{Int}$. How many isomorphic embeddings of $\mathcal{L}L$ into $\rho^{-1}L$ do there exist depending on L?

2.0. A useful frame-theoretic characterization of the relation "an intermediate logic $\leftrightarrow$ its modal companion" was obtained by Zakharyashchev [1984, 1989] who exploited his apparatus of so-called canonical formulas. These formulas may be regarded as a generalization of frame and subframe formulas of Fine [1974, 1985], but, unlike the latter, the canonical formulas can axiomatize all logics in $\mathcal{L}\mathbf{Int}$ and $\mathcal{L}\mathbf{S4}$. The definitions of intuitionistic and modal canonical formulas and their basic properties are presented below in Sections 2.1 and 2.2.

2.1. Let $\mathfrak{F} = \langle W, R\rangle$ be a rooted partially ordered finite frame, $a_0, \ldots, a_n$ be all the distinct points in $\mathfrak{F}$ and a_0 be the origin. A pair $\delta = (\overline{a}, \overline{b})$ of antichains in W is called a *disjunctive domain* (or simply *d-domain*) in $\mathfrak{F}$ if

(i) $\overline{a}$ has at least two points;

(ii) $\forall a \in \overline{a}\, \forall b \in \overline{b}\, \neg aRb$;

(iii) $\forall c \in W (\forall a \in \overline{a}\, cRa \Rightarrow \exists b \in \overline{b}\, cRb)$.

Let $\mathfrak{D}$ be some (possibly empty) set of d-domains in $\mathfrak{F}$. With $\mathfrak{F}$ and $\mathfrak{D}$ we associate the formula

$$\mathrm{X}(\mathfrak{F}, \mathfrak{D}, \bot) = \&_{a_i R a_j} A_{ij} \ \&\ \&_{\delta \in \mathfrak{D}} B_\delta \ \&\ C \supset p_0$$

where

$$A_{ij} = (\&_{\neg a_j R a_k} p_k \supset p_j) \supset p_i,$$
$$C = \&_{i=0}^{n}(\&_{\neg a_i R a_k} p_k \supset p_i) \supset \bot$$

and, for $\delta = (\overline{a}, \overline{b})$,

$$B_\delta = \&_{a_j \in \overline{b}}(\&_{\neg a_j R a_k} p_k \supset p_j) \supset \bigvee_{a_i \in \overline{a}} p_i.$$

By $\mathrm{X}(\mathfrak{F}, \mathfrak{D})$ we denote the formula which is obtained from $\mathrm{X}(\mathfrak{F}, \mathfrak{D}, \bot)$ by deleting the conjunct C. Formulas of the form $\mathrm{X}(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\mathrm{X}(\mathfrak{F}, \mathfrak{D})$ are called *canonical formulas* and *positive canonical formulas for* **Int**, respectively.

Zakharyashchev [1983, 1989] gave a refutability criterion for canonical formulas and described an algorithm which, for any formula A, constructs a finite number of canonical formulas $\mathrm{X}(\mathfrak{F}_1, \mathfrak{D}_1, \bot), \ldots, \mathrm{X}(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$ such that

$$\mathbf{Int} + A = \mathbf{Int} + \mathrm{X}(\mathfrak{F}_1, \mathfrak{D}_1, \bot) + \ldots + \mathrm{X}(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

If A is positive then the algorithm may use only positive canonical formulas and if A has no occurrences of $\vee$ then $\mathfrak{D}_i = \emptyset$, for all $i = 1, \ldots, n$.

Thus, each intermediate logic L can be represented in the form

$$L = \mathbf{Int} + \{\mathrm{X}(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in I}.$$

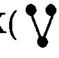
Besides, if all additional axioms of L are positive it can be also represented in the form

$$L = \mathbf{Int} + \{\mathrm{X}(\mathfrak{F}_i, \mathfrak{D}_i)\}_{i \in I}.$$

For example:

Cl = **Int** + X( , $\emptyset$),

KC = **Int** + X( , $\emptyset$, $\bot$),

LC = **Int** + X( , $\emptyset$).

2.2. The canonical formulas for **S4** are defined similarly to those for **Int**. The only difference of real importance is that they are associated with quasi-ordered frames which may contain *proper clusters*, i.e. non-trivial equivalence classes under the equivalence relation $\equiv$: $a \equiv b$ iff aRb & bRa.

Let $\mathfrak{F} = \langle W, R \rangle$ be a rooted quasi-ordered finite frame, $a_0, \ldots, a_n$ be all its points and a_0 be the origin. With $\mathfrak{F}$ and a set $\mathfrak{D}$ of some d-domains in $\mathfrak{F}$

we associate the formula

$$\Upsilon(\mathfrak{F},\mathfrak{D},\bot) = \&_{a_iRa_j} A_{ij} \ \& \ \&_{i=0}^{n} A_i \ \& \ \&_{\delta\in\mathfrak{D}} B_\delta \ \& \ C \supset p_0$$

where

$$A_{ij} = \Box(\Box p_j \supset p_i),$$
$$A_i = \Box((\&\Gamma_i \supset p_i) \supset p_i),$$
$$\Gamma_i = \{p_k, \Box p_l \mid k \neq i, \neg a_i R a_l\},$$
$$C = \Box(\&_{i=0}^{n} \Box p_i \supset \bot)$$

and, for $\delta = (\bar{a}, \bar{b})$,

$$B_\delta = \Box(\&_{a_j\in\bar{b}} \Box p_j \supset \bigvee_{a_i\in\bar{a}} \Box p_i).$$

The formula $\Upsilon(\mathfrak{F},\mathfrak{D})$ is obtained from $\Upsilon(\mathfrak{F},\mathfrak{D},\bot)$ by deleting the conjunct C. Formulas of the form $\Upsilon(\mathfrak{F},\mathfrak{D},\bot)$ and $\Upsilon(\mathfrak{F},\mathfrak{D})$ are called *canonical* and *positive canonical formulas for* **S4**, respectively.

Zakharyashchev [1984, 1988] obtained a refutability criterion for modal canonical formulas and gave an algorithm which, for any modal formula A, constructs canonical formulas $\Upsilon(\mathfrak{F}_1,\mathfrak{D}_1,\bot),\ldots,(\mathfrak{F}_n,\mathfrak{D}_n,\bot)$ such that

$$\mathbf{S4} \oplus A = \mathbf{S4} \oplus \Upsilon(\mathfrak{F}_1,\mathfrak{D}_1,\bot) \oplus \ldots \oplus \Upsilon(\mathfrak{F}_n,\mathfrak{D}_n,\bot).$$

If A is positive (i.e. contains only $\supset, \&, \vee, \Box$) then the algorithm may use only positive canonical formulas.

Thus, each normal modal logic M containing **S4** can be represented in the form

$$M = \mathbf{S4} \oplus \{\Upsilon(\mathfrak{F}_i,\mathfrak{D}_i,\bot)\}_{i\in I}$$

and if all additional axioms of M are positive it can be also represented in the form

$$M = \mathbf{S4} \oplus \{\Upsilon(\mathfrak{F}_i,\mathfrak{D}_i)\}_{i\in I}.$$

For example:

S5 = **S4** ⊕ Υ( , ∅),

S4Grz = **S4** ⊕ Υ(•↔•, ∅),

S4.1 = **S4** ⊕ Υ(•↔•, ∅, ⊥),

S4.2 = **S4** ⊕ Υ( , ∅, ⊥),

S4.3 = **S4** ⊕ Υ( , ∅).

2.3. Zakharyashchev [1984, 1989] gave the following characterization of modal companions of an intermediate logic:

A logic M is a modal companion of an intermediate logic

$$L = \mathbf{Int} + \{X(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in I}$$

iff M can be represented in the form

$$M = \mathbf{S4} \oplus \{Y(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in I} \oplus \{Y(\mathfrak{F}_j, \mathfrak{D}_j, \bot)\}_{j \in J}$$

where each of the frames $\mathfrak{F}_j$, for $j \in J$, contains at least one proper cluster.

Using this theorem, it is not difficult to obtain practically all results presented above. For instance, since

$$\mathbf{S4} \oplus Y(\bullet\!\leftrightarrow\!\bullet, \emptyset) \vdash Y(\mathfrak{F}, \mathfrak{D}, \bot)$$

for every frame $\mathfrak{F}$ with a proper cluster, each intermediate logic $L = \mathbf{Int} + \{X(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in I}$ has the greatest modal companion

$$\sigma L = \mathbf{S4} \oplus \{Y(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in I} \oplus Y(\bullet\!\leftrightarrow\!\bullet, \emptyset).$$

2.4. With the help of the refutability criteria for the canonical formulas Zakharyashchev [1989a] found a subtler deductive characterization of the mappings τ and σ: for any intermediate logic L,

(i) $\tau L \vdash Y(\mathfrak{F}, \mathfrak{D}, \bot)$ iff $L \vdash X(\mathfrak{F}^c, \mathfrak{D}, \bot)$;

(ii) $\sigma L \vdash Y(\mathfrak{F}, \mathfrak{D}, \bot)$ iff $L \vdash X(\mathfrak{F}^c, \mathfrak{D}, \bot)$ or $\mathfrak{F}$ contains a proper cluster.

Here by $\mathfrak{F}^c$ we denote the *skeleton* of the frame $\mathfrak{F}$, i.e. the quotient frame of $\mathfrak{F}$ with respect to clusters (or the equivalence relation $\equiv$; see Section 2.2).

As a consequence of this characterization we obtain that an intermediate logic L is decidable iff τL is decidable iff σL is decidable.

2.5. The modal companion theorem in Section 2.3 together with the algorithms mentioned in Section 2.1 and 2.2 enable us not only to describe all modal companions of an intermediate logic but also, given a finite set of axioms of a modal logic M, to find effectively canonical axioms of the intermediate logic ρM. Indeed, if $M = \mathbf{S4} \oplus A_1 \oplus \ldots \oplus A_n$ then first we can effectively find a canonical representation

$$M = \mathbf{S4} \oplus \Upsilon(\mathfrak{F}_1, \mathfrak{D}_1, \bot) \oplus \ldots \oplus \Upsilon(\mathfrak{F}_l, \mathfrak{D}_l, \bot) \oplus$$

$$\Upsilon(\mathfrak{F}_{l+1}, \mathfrak{D}_{l+1}, \bot) \oplus \ldots \oplus \Upsilon(\mathfrak{F}_m, \mathfrak{D}_m, \bot)$$

where $\mathfrak{F}_i$, for $i = 1, \ldots, l$, is partially ordered and $\mathfrak{F}_j$, for $j = l+1, \ldots, m$, contains a proper cluster, and after that we obtain

$$\rho M = \mathbf{Int} + \mathrm{X}(\mathfrak{F}_1, \mathfrak{D}_1, \bot) + \ldots + \mathrm{X}(\mathfrak{F}_l, \mathfrak{D}_l, \bot).$$

2.6. In Section 1.4 we have seen that there is an algorithm which can recognize, given a modal formula B, whether or not $\rho(\mathbf{S4} \oplus B) = \mathbf{Int}$. In other words, the property "to be a modal companion of **Int**" is decidable in the class of finitely axiomatizable normal extensions of **S4**. What if we replace here **Int** by some other intermediate calculus $\mathbf{Int} + A$?

We say an intuitionistic formula A is *decidable* if there is an algorithm which is capable of deciding, given a formula C, whether or not $\mathbf{Int} + C \vdash A$. The examples of decidable formulas are all formulas containing only one variable, as follows from Anderson [1972], and the Dummett formula $(p \supset q) \vee (q \supset p)$. The shortest undecidable formula known to us is

$$F_4 = \neg(p \& q) \vee \neg(\neg p \& q) \vee \neg(p \& \neg q) \vee \neg(\neg p \& \neg q).$$

There are also undecidable implicative formulas.

Using the result of Section 2.5 it is not difficult to show that the property "to be a modal companion of $\mathbf{Int} + A$" is decidable iff both the logic $\mathbf{Int} + A$ and the formula A are decidable. It follows in particular that there is no algorithm which can recognize, for a modal formula B, whether or not $\mathbf{S4} \oplus B$ is a modal companion of $\mathbf{Int} + F_4$.

Note also that a formula A is decidable in the class of intermediate logics iff $T(A)$ is decidable in the class of normal extensions of **S4**. This is another consequence of the use of the canonical formulas.

3.0. Now, it is natural to clarify the relationship between the semantics of logics connected by the mappings ρ, τ and σ.

Recall that pseudo-Boolean (or Heyting) algebras and topological Boolean (or interior) algebras yield the algebraic semantics for **Int** and **S4**, respectively (for definitions consult Rasiowa and Sikorski [1963]). Kripke models for these logics are built on frames $\langle W, R\rangle$ where, for **Int**, R is a partial order on a set of worlds W and, for **S4**, R is a quasi-order. General frames for **Int** and **S4** (see Goldblatt [1976]), being actually relational representations of pseudo-Boolean and topological Boolean algebras, have the form $\mathfrak{F} = \langle W, R, S\rangle$ where $\langle W, R\rangle$ is an ordinary Kripke frame and $S \subseteq 2^W$ is a non-empty set of subsets of W which is closed under intuitionistic or modal operations and used as a restriction of possible valuations.

3.1. McKinsey and Tarski [1946] proved that the algebra $\mathfrak{B}$ of open elements of each topological Boolean algebra $\mathfrak{A}$ is a pseudo-Boolean one (and, of course, an intuitionistic formula A is valid in $\mathfrak{B}$ iff $T(A)$ is valid in $\mathfrak{A}$); conversely, each pseudo-Boolean algebra is isomorphic to the algebra of open elements of some topological Boolean algebra. This is an algebraic counterpart of the Embedding Theorem in Section 1.0.

In terms of general frames it can be reformulated as follows: if $\mathfrak{F} = \langle W, R, S\rangle$ is a general frame for **S4** then $\rho\mathfrak{F} = \langle W^c, R^c, S^{co}\rangle$ is a general frame for **Int**, where $\langle W^c, R^c\rangle$ is the skeleton of $\langle W, R\rangle$ and S^{co} is the set of skeletons of open (i.e. upwards closed) sets in S; moreover, if $\mathfrak{F} = \langle W, R, S\rangle$ is a general frame for **Int** then, forming the Boolean closure S' of S, we obtain the general frame $\sigma\mathfrak{F} = \langle W, R, S'\rangle$ for **S4** with $\rho\sigma\mathfrak{F} = \mathfrak{F}$. The semantic operator ρ satisfies the following fundamental relation: for any intuitionistic formula A and any general frame $\mathfrak{F}$ for **S4**,

$$\rho\mathfrak{F} \models A \text{ iff } \mathfrak{F} \models T(A).$$

3.2. Using the algebraic results of McKinsey and Tarski from Section 3.1, Dummett and Lemmon [1959] showed that, given a characteristic (topological Boolean) algebra $\mathfrak{A}$ for τL, we can construct a characteristic (pseudo-Boolean) algebra for L simply by taking the algebra of open elements of $\mathfrak{A}$. In other words, in order to construct a characteristic general frame for L we may apply the operator ρ to a characteristic general frame for τL. (This result is certainly true not only for τL but for all modal companions of L; see Section 3.3.)

Dummett and Lemmon [1959] made also a conjecture as to the solution of the converse problem: if L is characterized by a Kripke frame $\langle W, R\rangle$ then τL is characterized by the Kripke frame $\langle \omega W, \omega R\rangle$ which is obtained from $\langle W, R\rangle$ by replacing each of its points with a cluster containing ω points (in other words $\langle \omega W, \omega R\rangle$ is the direct product of $\langle W, R\rangle$ and the cluster with ω points). They justified this conjecture by the observation that it holds for **Int**, **Cl**, **KC** and **LC**. (It is worth noting that in those early days of relational semantics Dummett and Lemmon did not suspect the existence of Kripke incomplete logics; moreover, they, as well as some others, even believed that all intermediate logics and their least modal companions have the finite model property.)

3.3 Maksimova and Rybakov [1974] obtained an algebraic characterization of the mappings ρ and σ. They proved in fact that if $\mathfrak{F}$ is a characteristic general frame for M then $\rho\mathfrak{F}$ is a characteristic frame for ρM and if $\mathfrak{F}$ is a characteristic frame for L then $\sigma\mathfrak{F}$ is a characteristic frame for σL.

Esakia [1979a] gave a semantic characterization of τ and σ in terms of so called *perfect Kripke models* which were introduced by Esakia [1974] and are closely connected with descriptive general frames of Goldblatt [1976]. This characterization yields a positive solution to a version of Dummett-Lemmon's conjecture formulated by Esakia [1979a] for this kind of semantics.

3.4. Using the apparatus of canonical formulas (see Section 2) Zakharyashchev [1989a] found a semantic characterization of the mapping τ in terms of general frames: if an intermediate logic L is characterized by a general frame $\mathfrak{F} = \langle W, R, S\rangle$ then τL is characterized by the class of general frames of the form $\tau_k\mathfrak{F} = \langle kW, kR, kS\rangle$ where $1 \le k < \omega, \langle kW, kR\rangle$ is the direct product of $\langle W, R\rangle$ and the cluster with k points $0, \ldots, k-1$ (i.e. each point in W is replaced by the cluster with k points) and kS is an arbitrary set of subsets of kW such that $\rho\tau_k\mathfrak{F} = \mathfrak{F}$ and $\{i\} \times G \in kS$, for each i $(0 \le i < k)$ and G belonging to the Boolean closure of S. Moreover, in this statement as a characteristic general frame for τL we may take the frame $\tau_\omega\mathfrak{F} = \langle \omega W, \omega R, \omega S\rangle$ instead of the infinite class of frames $\{\tau_k\mathfrak{F} \mid 1 \le k < \omega\}$.

This result yields at once a positive solution to the original Dummett and Lemmon conjecture from Section 3.2 (see also Zakharyashchev [1989]).

4.0. The mappings ρ, τ and σ not only give some structural correspondences between the lattices $\mathcal{L}$**Int** and $\mathcal{L}N$**S4**, but sometimes they allow to transfer various properties of intermediate logics to their modal companions and vice versa.

For example, Rybakov [1986], having first observed that an inference rule A/B is admissible in an intermediate logic L iff the rule $T(A)/T(B)$ is admissible in σL, proved then the decidability of the admissibility problem in **Int** by reducing it to the corresponding problem in **S4Grz** which is easier to cope with because the connectives become classical. Is it true that an inference rule A/B is admissible in L iff $T(A)/T(B)$ is admissible in τL? Is it true that a rule A/B is admissible in L iff $T(A)/T(B)$ is admissible in each normal modal companion of L?

4.1. Usually there are no difficulties with the preservation of properties while passing from a modal logic M to its superintuitionistic fragment ρM. In the following table we present all preservation theorems (that are known to us) for the passage from an intermediate logic L to τL and σL.

Property of L	Preservation of the property while passing to	
	τL	σL
1. Decidability	Yes	Yes
	Zakharyashchev [1989a]	Zakharyashchev [1989a]
2. Tabularity	No	Yes
	Dummett and Lemmon [1959] + Scroggs [1951]	Maksimova and Rybakov [1974]
3. Pretabularity	No	Yes
	Maksimova [1975], Esakia and Meskhi [1977]	Maksimova [1975], Esakia and Meskhi [1977]
4. Finite model property	Yes	Yes
	Esakia [1979a], Zakharyashchev [1989, 1989a]	Maksimova and Rybakov [1974]
5. Local tabularity	No	No
	Makinson [1966]	Makinson [1966]
6. Kripke completeness	Yes	No
	Zakharyashchev [1989a]	Shekhtman [1980]

7. Compactness (in the sense of Thomason [1972])	Yes	?
	Follows from Zakharyashchev [1989a]	
8. Topological completeness[4]	?	No
		Shekhtman [1980]
9. Disjunction property	Yes	Yes
	Zakharyashchev [1989a]	Gudovshchikov and Rybakov [1982]
10. Hallden completeness	No	No
	Chagrov and Zakharyashchev [1990, 1991]	Chagrov and Zakharyashchev [1990, 1991]
11. Interpolation property	No	No
	Maksimova [1982]	Maksimova [1982]
12. Polynomial finite model property	?	?
13. First-order definability	Yes	No
	Chagrova [1990]	Chagrova [1990]

The last result in the table needs some refinement. Actually, Chagrova [1990] proves that the class of Kripke frames for σL is first-order definable iff L is a logic of a finite slice (in the sense of Hosoi [1967]) and the class of frames for L is definable. She conjectures that the class of frames for σL is first-order definable in the class of frames having no infinite ascending chains iff the class of frames for L is definable.

4.2. None of the properties in the table above is preserved while passing from an arbitrary intermediate logic to its arbitrary modal companion.

[4]The following problem due to Kuznetsov is open still: whether there exists an intermediate logic which is not characterized by any topological space.

Rybakov [1977] proved that there are a continuum of noncompact modal companions of **Int** (which, of course, are incomplete and do not have the finite model property) and there are undecidable recursively axiomatizable modal companions of **Int**. The fact that the disjunction property is not in general preserved was pointed out by Gudovshchikov and Rybakov [1982]. It follows from the results of Zakharyashchev [1987] that there are a continuum of **Int**'s companions without the disjunction property. We conjecture that the same is true for every intermediate logic having the disjunction property.

However, there are several positive results for the tabular and pretabular intermediate logics. Maksimova and Rybakov [1974] showed that if L is tabular then each of L's modal companions has the finite model property; moreover, as Rybakov [1976] proved afterwards, all of them are finitely axiomatizable and hence decidable. The same was proved by Rybakov [1976] for the pretabular intermediate logics. Chagrov [1983] strengthened these results: all modal companions of tabular and pretabular intermediate logics have the polynomial finite model property, i.e. the number of elements in refutation Kripke frames for the logic is bounded by some polynomial of the length of a refuted formula.

Chagrov [1985] showed that, for any function φ, there is a modal logic M such that M has the finite model property, the number of elements in refutation frames for M grows more rapidly than φ, but ρM has the linear finite model property. For other results concerning complexity problems in modal and intermediate logics consult Chagrov and Zakharyashchev [1991a].

4.3. There is a property, viz. the pre-local-tabularity, that turns out to be more intricate in the class of intermediate logics than in the class of normal extensions of **S4**. A logic L is called *pre-local-tabular* if it is not local-tabular but every one of its proper extensions L' is (i.e. for each n, there exists only a finite number of pairwise non-equivalent in L' formulas having n variables). There is only one pre-local-tabular logic among the normal extensions of **S4** and the problem of recognizing the local-tabularity is decidable (see Maksimova [1975a]). However, Mardaev [1984] constructed a continuum of pre-local-tabular intermediate logics, and the decidability problems for the local-tabularity and pre-local-tabularity are open.

4.4. Some preservation theorems in the table were used in Chagrov and Zakharyashchev [1991] for proving the undecidability of the decidability, the finite model property, the disjunction property and some other properties simultaneously in the class of intermediate calculi and in the class of normal calculi containing **S4Grz**.

5.0. The results discussed in the previous sections could be attributed to the "classical (or conventional) theory" of embeddings of **Int** and its extensions in modal logics: (i) only normal extensions of **S4** are considered, in which (ii) differences between various translations can be neglected.

However, if we return to the original problem, viz. classical interpretation of **Int**, then it turns out that the choice of **S4** as a modal basis cannot be considered as completely justified. First, why should we connect the modality "provable", i.e. $\Box$, with the provability, that is the deducibility, in the modal system itself; in other words, is it correct to postulate the rule of necessitation $A/\Box A$? Second, recent intuitionistic investigations (see, for example, de Swart [1977]) have led to the development of a model apparatus that could not be directly interpreted in **S4** semantics — we mean models with "exploding" or "strange" worlds at which all intuitionistic formulas are simultaneously true (or false). In modal logic semantics, as an analogue of such "exploding" worlds one could take worlds that "see nothing", i.e. final irreflexive worlds at which "everything is necessary"; but their existence contradicts the **S4**-axiom $\Box p \supset p$. Another analogue of equal value (from the technical point of view) could be Kripke's [1965] non-normal worlds at which "everything is possible"; but this contradicts the **S4**-axiom $\Box p \supset \Box\Box p$. Third, one can try to interpret intuitionistic formulas as statements on provability in formal Peano arithmetic or on validity in the standard arithmetic model (see, for example, Goldblatt [1978], Boolos [1980], Kuznetsov and Muravitsky [1980, 1986], Artemov [1986]), but then by Löb's theorem, the modal logic simulating Gödel's provability predicate in formal arithmetic will be essentially different from **S4**. It should be noted that this approach may be considered as an exact formalization of both Kolmogorov's and Gödel's ideas. Finally, one could feel doubt about the translation: which translation is most "correct", i.e. which translation grasps most precisely our conception of intuitionistic formulas?

5.1. In the sequel we will use the following translations:

	prefixes $\Box$ to every
S_i	subformula save conjunctions and disjunctions
T_i	subformula
S_i'	proper subformula save conjunctions and disjunctions
T_i'	proper subformula

Here $i = 1, 2$: by the subscripts 1 and 2 we mark the translations of formulas with the primitive connectives $\{\&, \vee, \supset, \neg\}$ and $\{\&, \vee, \supset, \bot\}$, respectively.

We will omit subscripts when results do not depend on the choice of the primitives.

Other translations, which are used for embedding **Int** in provability and tense logics, will be introduced in Sections 9.0 and 11.

6.0. The search for minimal modal companions of **Int** led to the following logics.

6.1. Došen [1981] proved that

$$\mathbf{K} \oplus \Box(\Box p \supset \Diamond p) \oplus \Box(\Box p \supset \Box\Box p) \oplus \Box(\Box\Box p \supset \Box p)$$

and

$$\mathbf{K} \oplus \Box(\Box p \supset \Diamond p) \oplus \Box(\Box p \supset \Box\Box p) \oplus \Box(\Box(\Box p \vee \Box p) \supset \Box p \vee \Box q)$$

are the minimal normal logics in which **Int** can be eebedded by T_1 and S_1, respectively. Note that Došen used the primitives $\{\&, \vee, \supset, \neg, \bot, \top\}$ but the restriction of this set to $\{\&, \vee, \supset, \neg\}$ changes nothing. (It is interesting that the axiom $\Box(\Box p \vee \Box q) \supset \Box p \vee \Box q$ was used by Dzhaparidze for constructing a modal logic having provability interpretation; see Beklemishev [1989].)

The minimal normal S_2- and T_2-companions of **Int** can be obtained from the minimal S_1- and T_1-companions by deleting the axiom $\Box(\Box A \supset \Diamond A)$ which is equivalent to the translation of the formula $\neg\neg(p \supset p)$ (for S_2 this result was formulated by Chagrov [1983a]).

The axioms for the minimal normal S_i'- and T_i'-companions of **Int** are those for the minimal S_i- and T_i-companions with the outermost $\Box$ deleted (for S_1' this was observed by Došen [1990]).

6.2. The question about minimal companions of **Int** without the postulated rule $A/\Box A$ is open. More exactly, we do not know any suitable (such as in Section 6.1) axiomatization of the logic with the modal axioms $\Box(p \supset q) \supset (\Box p \supset \Box q)$ and $tr(\mathbf{Int})$, where tr is one of our translations. What about relational semantics for such logics?

Hacking [1963][5] and Chagrov [1981] proved that **S3** is a modal companion of **Int** for the translations S_1 and S_2. As far as the translations T_i are concerned one can easily show that for $i = 1, 2$,

$$\mathbf{S2} + T_i(\mathbf{Int}) = \mathbf{S3} + T_i(\mathbf{Int}) = \mathbf{S3} + \Box\Box(p \supset p) = \mathbf{S3} \oplus (p \supset p) = \mathbf{S4}.$$

The same fact for Novikov's [1977] D-translation was observed by Chagrov [1983a]. Note by the way that $\mathbf{S2} + S_i(\mathbf{Int}) = \mathbf{S3}$, i.e. **S3** is the minimal

[5] We are grateful to K. Došen for pointing out the paper Hacking [1963].

S_i-companion of **Int** among the extensions of **S2** (see Chagrov [1983a]).

6.3.0. Now, let us consider the boundary case: what formulas are provable in the modal logics whose modal axioms are the translations of the formulas provable in **Int**? Some results in this direction are listed below (see Chagrov [1989]). By **Cl**($\Box$) we will denote the set of classical tautologies and their substitution-instances in the language with $\Box$, i.e. the "smallest modal logic"; $\Box^*M = \{\Box^n A \mid A \in M, n \geq 0\}$.

6.3.1. For an intermediate logic L,

$$\mathbf{Cl}(\Box) + S_2(L) \vdash \Box(p \supset q) \supset (\Box p \supset \Box q) \text{ iff } L \vdash \bot.$$

Indeed, if $L \nvdash \bot$, i.e. $\mathbf{Int} \supseteq L \supseteq \mathbf{Cl}$, then $\mathbf{Cl}(\Box)+S_2(L) \nvdash \Box(p \supset q) \supset (\Box p \supset \Box q)$, as we shall see in Section 6.3.2. If $L \vdash \bot$ then $\mathbf{Cl}(\Box) + S_2(L) \vdash \Box q$, and so $\mathbf{Cl}(\Box) + S_2(L) \vdash \Box(p \supset q) \supset (\Box p \supset \Box q)$.

Another fact concerning $\mathbf{Cl}(\Box) + S_2(L)$: if $\mathbf{Int} \nvdash A$ then $\mathbf{Cl}(\Box) + S_2(\mathbf{Int}) \nvdash \Box A$. (It follows, in particular, that $\mathbf{Cl}(\Box) + S_2(\mathbf{Int})$ is not closed under necessitation $A/\Box A$.) To prove this fact, say, for $A = p \vee \neg p$, one can use the frame shown in Figure 2 with two accessibility relations $R_1 = \{\langle a,a\rangle, \langle b,b\rangle, \langle b,c\rangle, \langle c,c\rangle\}$, $R_2 = R_1 \cup \{\langle a,b\rangle, \langle a,c\rangle\}$ and the actual (or distinguished) world a. The truth-relation $\models$ for $\&, \vee$ and $\bot$ is defined as usual, while

$$\begin{array}{lll} x \models A \supset B & \text{iff} & \forall y(xR_1y \,\&\, y \models A \Rightarrow y \models B), \\ x \models \Box A & \text{iff} & \forall y(xR_2y \Rightarrow y \models A). \end{array}$$

To refute $\Box(p \vee \neg p)$ at a we may take $x \models p$ iff $x = c$.

6.3.2. $\Box^*\mathbf{Cl}(\Box) + \Box^* S_2(\mathbf{Cl}) + \Box(\Box(p \supset q) \supset (\Box p \supset \Box q)) \nvdash \Box(p \supset q) \supset (\Box p \supset \Box q)$. To prove this we can use the frame shown in Figure 3 (a is the only irreflexive point); $\models$ is standard for the Boolean connectives and $x \models \Box A$ iff $\exists y(xRy \,\&\, y \models A)$.

$$\Box^*\mathbf{Cl}(\Box) + \Box^* S_2(\mathbf{Int}) + \Box(p \supset q) \supset (\Box p \supset \Box q) \nvdash \Box(\Box(p \supset q) \supset (\Box p \supset \Box q)).$$

This may be proved with the help of the frame shown in Figure 4 where a is the only irreflexive point, d is the actual world and

$$x \models \Box A \text{ iff } \left\{ \begin{array}{l} x \in \{d,b,c\} \Rightarrow \forall y(xRy \Rightarrow y \models A) \\ x \in \{a,b,c\} \Rightarrow \exists y(xRy \,\&\, y \models A). \end{array} \right.$$

6.3.3. Unrestricted "boxing" of $\Box(p \supset q) \supset (\Box p \supset \Box q)$ yields

$$\Box^*\mathbf{Cl}(\Box) + \Box^* S_2(\mathbf{Int}) + \Box^*(\Box(p \supset q) \supset (\Box p \supset \Box q)) =$$
$$\mathbf{K} + \Box(\Box p \supset \Box\Box p) + \Box(\Box\Box p \supset \Box p) + \Box\Box(\Box p \supset \Box\Box p) + \Box\Box(\Box\Box p \supset \Box p) =$$
$$\mathbf{K} \oplus \Box(\Box p \supset \Box\Box p) \oplus \Box(\Box\Box p \supset \Box p).$$

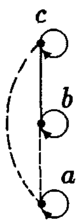


Figure 2

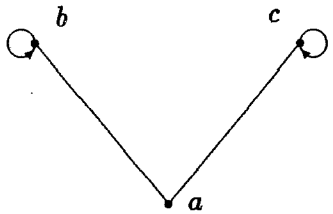


Figure 3

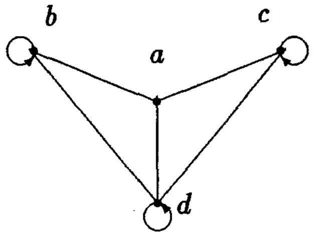


Figure 4

7.0. Having found new classes of modal companions of **Int** and other intermediate logics, we are facing the problem of describing the set of modal companions of a given intermediate logic once again. According to the available results, we will consider here mainly two questions: about maximal modal companions of **Int** and about the decidability of the property "to be a modal companion of **Int**".

7.1. Došen [1990] proved that the set of normal S_1'-companions of **Int** forms the interval [**K4N**, **S4Grz**] where

$$\mathbf{K4N} = \mathbf{K} \oplus \Box p \supset \Diamond p \oplus \Box p \supset \Box\Box p \oplus \Box(\Box p \vee \Box q) \supset \Box p \vee \Box q.$$

Using this result, one can easily show that the set of S_1'-companions of an intermediate logic L is $[\mathbf{K4N} \oplus S_1'(L), \mathbf{S4Grz} \oplus S_1'(L)]$. The property "$\mathbf{K4N} \oplus A$ is a S_1'-companion of **Int**" is decidable. However, the question on the decidability of the property "to be a S_1'-companion of **Int**" for the class of normal extensions of **K** or even **K4** is open.

We know nothing of maximal normal S_1- and T_1-companions of **Int** and of the decidability of the property "to be a S_1- (T_1)-companion of **Int**".

7.2.0. The translations S_2 and T_2 unlike S_1' do not require to use the axiom of the form $\Box p \supset \Diamond p$ (see Section 6.1). As a result we obtain an absolutely different situation.

7.2.1. Chagrov [1989] showed that there are a continuum of maximal logics within the set of all normal S_2-companions (T_2-companions) of **Int**. The properties "to be a S_2-companion of **Int**" and "to be a T_2-companion of **Int**" turn out to be undecidable in the class of normal extensions of **K4**.

(This may be proved using the technique of Chagrov [1990a].) Will these results remain true if we take some other intermediate logic different from **Int**?

7.2.2. The situation becomes even more complicated if we consider arbitrary (not necessarily normal) S_2- and T_2-companions of **Int**. Chagrov [1989] proved that there is a continual set of pairwise inconsistent S_2- (T_2-) companions of **Int** such that among the extensions of each of them there exist a continuum of maximal (pairwise consistent) S_2- (T_2-)companions of **Int**. In addition to the undecidability result from Section 7.2.1 (that is true for the present case too) we can say that the property "to be a normal logic" is undecidable in many classes of logics (for example, in the class of extensions of **S4Grz**).

7.2.3. The cardinality results in Section 7.2.1, 7.2.2 and 8.1 are proved with the help of the constructions of Chagrov [1992]. Similar technique may be used for obtaining the cardinality results in Section 9.4.

Here we will prove in outline that there are a continuum of pairwise inconsistent S_2- (T_2-)companions of **Int**. Let $\mathfrak{F}(Q)$, for $Q \subseteq \omega$, be the frame shown in Figure 5, in which a_1 is irreflexive and, for $i \in \omega$, a_{3+2i} is reflexive iff $i \in Q$. The points in $\mathfrak{F}(Q)$ are evidently definable by the following variable-free formulas:

$$A_1 = \Box\bot, B_1 = \Diamond\top\&\Box\Diamond\top, A_2 = \Diamond A_1\&\neg\Diamond B_1,$$
$$B_2 = \Diamond A_1\&\Diamond B_1\&\Diamond(\Diamond A_1\&\Diamond B_1)\&\neg\Diamond A_2,$$
$$A_3^r = \Diamond B_1\&\Diamond A_2\&\Diamond(\Diamond B_1\&\Diamond A_2)\&\neg\Diamond B_2, \text{ if } a_3 \text{ is reflexive},$$
$$A_3^{ir} = \Diamond B_1\&\Diamond A_2\&\neg\Diamond(\Diamond B_1\&\Diamond A_2)\&\neg\Diamond B_2, \text{ if } a_3 \text{ is irreflexive}$$

and so on.

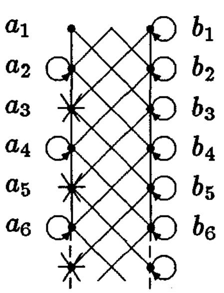


Figure 5

Now we take the sequence of the transitive and reflexive Jaśkowski frames $\mathfrak{F}_i$ (i.e. the relational counterparts of Jaśkowski's [1936] matrices) shown in

Figure 6. (To construct $\mathfrak{F}_{i+1}$ we take $i+1$ copies of $\mathfrak{F}_i$ and join to them the least point). Each intuitionistic non-thesis and its S_2-translation can be refuted in some of $\mathfrak{F}_i$.

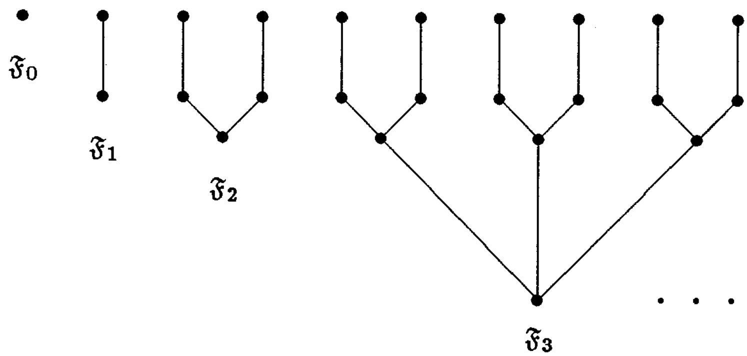


Figure 6

Define $\mathfrak{F}_i(Q)$ by joining a copy of $\mathfrak{F}(Q)$ to every maximal point in $\mathfrak{F}_i$ (see Figure 7); the points from $\mathfrak{F}_i$ are considered to be the only actual worlds in $\mathfrak{F}_i(Q)$. By $M(Q)$ we denote the modal logic of all frames $\mathfrak{F}_i(Q), i = 0, 1, \ldots$. Since **Int** $\nvdash A$ implies $M(Q) \nvdash S_2(A)$ and **K4** $\oplus \Box(\Box\Box p \supset \Box p) \subseteq M(Q), M(Q)$ is a S_2-companion of **Int**. Besides, we have

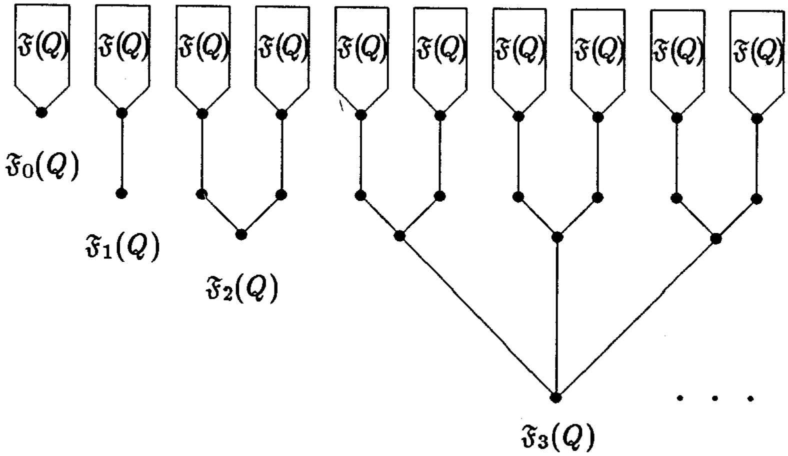


Figure 7

$$M(Q) \vdash \Diamond A^{\tau}_{3+2i} \quad \text{if} \quad i \in Q,$$
$$M(Q) \vdash \neg\Diamond A^{\tau}_{3+2i} \quad \text{if} \quad i \notin Q.$$

Therefore, if $Q_1 \neq Q_2$ then $M(Q_1)$ and $M(Q_2)$ are mutually inconsistent.

A slight change of this construction gives us a continuum of logics mentioned in Section 7.2.1. Let $M'(Q)$ be the logic of all frames $\mathfrak{F}_i(Q)$ in which, this time, all worlds are considered to be actual. $M'(Q)$ is now a normal S_2-companion of **Int**,

$$\begin{array}{ll} M'(Q) \vdash \neg\Diamond A^{\tau}_{3+2i} & \text{if} \quad i \notin Q, \\ M'(Q) \vdash \neg\Diamond A^{\tau}_{3+2i} \supset S_2(F_4) & \text{if} \quad i \in Q. \end{array}$$

(F_4 was defined in Section 2.6; it may be refuted in $\mathfrak{F}_i(Q)$, for $i \geq 3$, only at points from $\mathfrak{F}_i$ where $\Diamond A^{\tau}_{3+2j}$ is true iff $j \in Q$).

If $Q_1 \neq Q_2$ then, by modus ponens, $M'(Q_1) \oplus M'(Q_2) \vdash S_2(F_4)$, and so $M'(Q_1)$ and $M'(Q_2)$ have no common extension which is a S_2-companion of **Int**. At the same time, by Zorn's lemma, each of $M'(Q)$ is contained in some maximal S_2-companion of **Int**.

8.0. As we have seen in Section 6.2, **S3** is a modal S-companion of **Int**. What about extensions of **S3**?

8.1. Chagrov [1982] showed that there are infinitely many pairwise inconsistent S-companions of **Int** among the extensions of **S3**; for example, $\mathbf{S3}_i = \mathbf{S3} + \mathrm{X}_i (o < i < \omega)$, $\mathbf{S3}_\omega = \mathbf{S3} + \{\Diamond \mathrm{X}_i \mid i < \omega\}$ where $\mathrm{X}_0 = \Diamond\bot, \mathrm{X}_1 = \Box\Box\top, \mathrm{X}_2 = \Box\Diamond\Diamond\bot, \mathrm{X}_{i+3} = \Diamond \mathrm{X}_i \ \&\ \Diamond \mathrm{X}_{i+1} \ \&\ \neg\Diamond \mathrm{X}_{i+2}$. Each of consistent extensions of **S3** is consistent with some of $\mathbf{S3}_i$, for $1 \leq i \leq \omega$, (see Segerberg [1976]).

Chagrov [1987a] discovered that there exist a continuum of maximal S-companions of **Int** among the extensions of $\mathbf{S3}_i$, for each $i, 3 \leq i \leq \omega$. The properties "$\mathbf{S3}_i + A$ is a A-companion of **Int**" are undecidable for $3 \leq i \leq \omega$. For $i = 1, 2$ see Section 8.2.

Is it true that every maximal S-companion of **Int** among the extensions of **S3** is, for some i, a maximal S-companion among the extensions of $\mathbf{S3}_i$? The equivalent question: is it true that each maximal S-companion of **Int** among the extensions of **S3** has only one Post-complete extension?

The most curious logic in the sequence of $\mathbf{S3}_i$ is $\mathbf{S3}_\omega$. It is anti-tabular, i.e. it is consistent but has no finite models. Each of its consistent extensions is an S-companion of the one-variable fragment of **Int**. For these and other properties of $\mathbf{S3}_\omega$ consult Chagrov [1982].

8.2. Chagrov [1985a] observed that there is a proper (non-normal, of course) extension of **S4Grz** in which **Int** can be embedded by T (or S).

Zakharyashchev [1990] gave a characterization (in terms of canonical formulas which can axiomatize not only normal but all extensions of **S4**) of T-companions of **Int** among the extensions of $\mathbf{S3}_1 = \mathbf{S4}$. Somewhat simplified, it can be formulated as follows: a logic $M \supseteq \mathbf{S4}$ is a T-companion of **Int** iff it can be represented in the form

$$M = \mathbf{S4} + \{\Upsilon(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in I} + \{\Upsilon(\mathfrak{F}_j, \mathfrak{D}_j.\bot)\}_{j \in J}$$

where $\mathfrak{F}_j$, for $j \in J$, has a proper cluster, $\mathfrak{F}_i$, for $i \in I$, is partially ordered and there is a d-domain $(\overline{a}, \overline{b}) \in \mathfrak{D}_i$ such that $\overline{a}$ is not the set of all immediate successors of any point in $\mathfrak{F}_i$. It follows at once that the union $\mathbf{M}^*$ of all such T-companions is the greatest T-companion of **Int** containing **S4**.

$\mathbf{M}^*$ is characterized by the Kripke frame $\mathfrak{F}_\omega = \langle W_\omega, R_\omega \rangle$ with the actual world o which are defined as follows. Let $\langle W_0, R_0 \rangle$ be the disjoint union of all the Jaśkowski frames and Σ_i, for $i \geq 1$, be the set of all finite antichains in $\langle W_{i-1}, R_{i-1} \rangle$; we then let $W_i = W_{i-1} \cup \{c_{\overline{a}} \mid \overline{a} \in \Sigma_i\}$, R_i be the reflexive and transitive closure of $R_{i-1} \cup \{\langle c_{\overline{a}}, a \rangle \mid a \in \overline{a}\}$ and, finally,

$$W_\omega = \bigcup_{i<\omega} W_i \cup \{o\}, \; R_\omega = \bigcup_{i<\omega} R_i \cup \{\langle o, a \rangle \mid a \in W_\omega\}.$$

$\mathbf{M}^*$ is decidable, Halldén-complete, has the disjunction property, but does not have the finite model property. We do not know whether $\mathbf{M}^*$ is finitely axiomatizable.

There are a continuum of T-companions of **Int** among the extensions of **S4Grz**. The property "$\mathbf{S4} + A$ is a T-companion of **Int**" is decidable. However, the analogue of the Blok-Esakia theorem (see Section 1.3) does not hold for the lattice $\mathcal{L}\mathbf{M}^*$; more exactly, there is an intermediate logic which cannot be embedded by T in any extension of $\mathbf{M}^*$.

Similar results seem to hold for the extensions of $\mathbf{S3}_2 = \mathbf{S8}$.

8.3. A modal logic M is said to be *a strong (normal) Tr-companion* of **Int** if

$$M + Tr(\Gamma) \vdash Tr(A) \text{ iff } \mathbf{Int} + \Gamma \vdash A$$

($M \oplus Tr(\Gamma) \vdash Tr(A)$ iff $\mathbf{Int} + \Gamma \vdash A$ in the normal case).

What will change if we consider strong companions instead of companions?

The logics $\mathbf{S3}_i$, for $3 \leq i \leq \omega$, are not strong S-companions of **Int**, while **S3**, $\mathbf{S3}_1 = \mathbf{S4}$ and $\mathbf{S3}_2 = \mathbf{S8}$ are (see Chagrov [1981, 1982]). The greatest companion of **Int** among the extensions of **S4** is not a strong companion, as we have just seen in Section 8.2.

Do there exist weakest Tr-companions of **Int**? Here by *a weakest Tr-companion* we mean a logic M such that M is **Int**'s Tr-companion and if **Int** $\nvdash A$ then $M + Tr(A)$ is inconsistent.

9.0. So far have not attached any precise mathematical meaning to the modality $\Box$ which might be intuitively understood as "provable". However, if we interpret $\Box$ as Gödel's provability predicate Pr in the formal Peano arithmetic then, as was shown by Solovay [1976], we shall arrive at the propositional Gödel–Löb logic $\mathbf{GL} = \mathbf{K4} \oplus \Box(\Box p \supset p) \supset \Box p$ of arithmetic provability and the logic $\mathbf{S} = \mathbf{GL} + \Box p \supset p$ of arithmetic truth.

Recall that **GL** is characterized by irreflexive Kripke frames having no infinite ascending chains. "Reflexivizing" them we get frames for **S4Grz**, and hence **S4Grz** can be embedded into **GL** via the translation $^\circ$: $(A)^\circ$ is the result of replacing each of A's subformulas of the form $\Box B$ with $\Box B \& B$. It follows that **Int** is embedded into **GL** by the translation $T^\circ : T^\circ(A) = (T(A))^\circ$.

9.1. These facts concerning $^\circ$ and T° were independently observed by Kuznetsov and Muravitsky [1977, 1980], Goldblatt [1978] and Boolos [1980]. Moreover, Boolos [1980] showed that **S4Grz** can be also embedded into **S** by $^\circ$, and so **S**, as well as **GL**, is a T°-companion of **Int**. Artemov [1987] discovered that **S4Grz** is embedded in a proper normal extension of **GL**.

Now, it is natural to try to define "provability" analogues of the mappings ρ, τ and σ from Section 1.3.

Kuznetsov and Muravitsky [1986] defined a mapping μ from $\mathcal{L}N\mathbf{GL}$ into $\mathcal{L}N\mathbf{Grz}$ by taking $\mu M = \{A \mid M \vdash A^\circ\}$; the logic μM was called by them the *modal fragment* of M. They showed that μ is a semilattice $\cap$-homomorphism from $\mathcal{L}N\mathbf{GL}$ onto $\mathcal{L}N\mathbf{Grz}$ but unlike ρ it is not a lattice homomorphism, i.e. in general $\mu(M_1 \oplus M_2) \neq \mu(M_1) \oplus \mu(M_2)$ (see Section 9.2).

Muravitsky [1988], using the algebraic semantics for **S4Grz** and **GL**, proved that, for a set Γ of formulas,

$$\mathbf{S4Grz} \oplus \Gamma \vdash A \text{ iff } \mathbf{GL} \oplus \Gamma^\circ \vdash A^\circ$$

where $\Gamma^\circ = \{B^\circ \mid B \in \Gamma\}$. So $\mathbf{GL} \oplus \Gamma^\circ$ is the least normal extension of **GL** having $\mathbf{S4Grz} \oplus \Gamma$ as its modal fragment, and we may define an analogue of τ, viz. the mapping ν from $\mathcal{L}N\mathbf{S4Grz}$ into $\mathcal{L}N\mathbf{GL}$, by taking $\nu(\mathbf{S4Grz} \oplus \Gamma) = \mathbf{GL} \oplus \Gamma^\circ$. This mapping is certainly a lattice isomorphism from $\mathcal{L}N\mathbf{S4Grz}$ into $\mathcal{L}N\mathbf{GL}$.

What properties are preserved under μ and ν? We know only that μ does not preserve the interpolation property (this is a consequence of Maksimova [1979], Boolos [1980], Rautenberg [1983]) and ν does not preserve Halldén-completeness.

It is not difficult to see that, for every consistent extension M of **S**, $M \vdash A^\circ$ implies **S4.3Grz** $\vdash A$. We conjecture that, for $\Gamma \subseteq$ **S4.3Grz**, **S4Grz** $\oplus\, \Gamma \vdash A$ iff **S** $+ (\Box\Gamma^\circ) \vdash A^\circ$.

9.2. What about analogues of σ? Shavrukov [1990] proved that

$$\mathbf{A}^* = \mathbf{GL} \oplus \Box\Box p \supset \Box(p \& \Box p \supset q) \vee \Box(q \& \Box q \supset p)$$

and

$$\mathbf{A} = \mathbf{S} + \mathbf{A}^*$$

are the greatest logics among the normal and arbitrary extensions of **GL**, respectively, in which **S4Grz** can be embedded by $^\circ$. He also showed that both these logics are decidable (and so the property "**S4Grz** is embedded in **GL** $+ A$ by $^\circ$" is decidable too) but do not have the finite model property.

However, there are normal extensions of **S4Grz** which cannot be embedded by $^\circ$ into any extension of $\mathbf{A}^*$. For example, let $\mathfrak{F}$ be the frame shown in Figure 8; then

$$\mathbf{S4} \oplus \Box \Upsilon(\mathfrak{F}, \emptyset) \nvdash \Box(\Box p \supset \Box q) \vee \Box(\Box q \supset \Box p),$$

but

$$\mathbf{A}^* \oplus (\Box \Upsilon(\mathfrak{F}, \emptyset))^\circ \vdash (\Box(\Box p \supset \Box q) \vee \Box(\Box q \supset \Box p))^\circ.$$

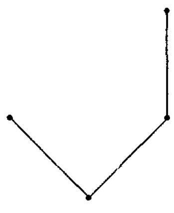


Figure 8

Thus, here we have the same situation as for the greatest non-normal companion $\mathbf{M}^*$ of **Int** considered in Section 8.2. Note also that $\mathbf{M}^*$ and $\mathbf{A}^*$ have similar semantics: Shavrukov [1990] proved that $\mathbf{A}^*$ is characterized by the class of frames which are obtained from finite tree frames for **GL** by inserting an infinite descending chain between each point and its immediate successor.

All non-trivial intervals of the form [**GL**, **GL**$\oplus\Gamma$], [**S**, **S**$+\Gamma$], in particular [**GL**, $\mathbf{A}^*$], [**S**, **A**], have continuously many logics.

9.3. It is easy to show (with the help of the Blok-Esakia theorem; see Section 1.3) that a normal extension M of **GL** is a T°-companion of an intermediate logic L iff σL is embedded into M by $^\circ$. So we may define a semilattice $\cap$-homomorphism ρ° from $\mathcal{L}N\mathbf{GL}$ onto $\mathcal{L}\mathbf{Int}$ and isomorphism τ° from $\mathcal{L}\mathbf{Int}$ into $\mathcal{L}N\mathbf{GL}$ by taking $\rho^\circ M = \rho\mu M = \sigma^{-1}\mu M$ and $\tau^\circ L = \nu\sigma L$.

By Shavrukov's results of Section 9.2, $\mathbf{A}^*$ is the greatest T°-companion of **Int** among the normal extensions of **GL** and the property "$\mathbf{GL} \oplus A$ is T°-companion of **Int**" is decidable. However, it is impossible to define an analogue σ° of σ, since there are intermediate logics having no T°-companions among the normal extensions of $\mathbf{A}^*$.

What properties of logics are preserved under σ° and τ°?

9.4. Chagrov [1990a, part II] proved that among the extensions of **S** there are a continuum of logics M_α which are T°-companions of **Int** but, for different α and β, $M_\alpha + M_\beta = \mathbf{S} + T^\circ(\neg p \vee \neg\neg p)$. It follows that there are a continuum of maximal T°-companions of **Int** among the extensions of **S**. Chagrov also proved that the property "$\mathbf{S} + A$ is a T°-companion of **Int**" is undecidable. The formula $T^\circ(\neg p \vee \neg\neg p)$ is undecidable in the class of extensions of **S** despite the decidability of $\neg p \vee \neg\neg p$ (see Section 2.6). We do not know whether $T^\circ(\neg p \vee \neg\neg p)$ is decidable in the class of normal extensions of **GL**.

9.5. **K4** is the least normal T°-companion of **Int**; moreover, it is a strong T°-companion of **Int**.

9.6. What are "superintuitionistic" fragments of extensions of **GL** for the translations defined in Section 5.1? Visser [1981] considered a logic **FPL** which is characterized by finite transitive irreflexive frames, found an axiomatization for it (the clue is the Löb rule $(\top \supset A) \supset A/\top \supset A$ where $\top = \bot \supset \bot$) and showed that **GL** is a strong companion of **FPL** with respect to two translations, S_2 being one of them.

10. **Int** can be embedded in many logics with rich languages. One of the most interesting examples is the propositional dynamic logic (see, for instance, Parikh [1978] and Segerberg [1982]).

Let α be a program and α^* be its iteration. Then the fragment of the propositional dynamic logic with only one modality $[\alpha^*]$ coincides with **S4**. Thus, we obtain an embedding of **Int** in dynamic logic if in the Gödel translation T we replace $\Box$ by $[\alpha^*]$.

Chagrov [1987] introduced an alternation propositional dynamic logic which is related to alternation computations (see Chandra and Stockmeyer [1976]) as ordinary dynamic logic is related to non-deterministic computations. The main feature of this new logic is the absence of theses of the form $[\alpha](p \supset q) \supset ([\alpha]p \supset [\alpha]q)$ (cf. Section 6.3). So **S4** is not its fragment in the sense we have just described. Nevertheless, **Int** can be embedded in the alternation propositional dynamic logic by the translation T in which $\Box$ is replaced by $[\alpha^*]$ (see Chagrov [1990]).

Embeddings of **Int** in dynamic logics may be regarded as another realization of Kolmogorov's and Gödel's ideas.

11. The embeddings considered in all previous sections reflected some similarity between Kripke semantics for **Int** and its modal companions. Bowen and de Jongh [1986] showed that **Int** can be embedded in the tense logic **WR**–FG (with two modalities F and G) which is characterized by frames with reflexive well-founded partial orders $<$ satisfying the following (branching) conditions:

$$\forall x, y(x \neq y \,\&\, \neg x < y \,\&\, \neg y < x \Rightarrow \neg\exists z(x < z \,\&\, y < z)),$$

$$\forall x(\exists y(x < y \,\&\, x \neq y) \Rightarrow \exists y, z(x < y \,\&\, x < z \,\&\, y \neq z \,\&\, \neg y < z \,\&\, \neg z < y)).$$

The truth-relation $\models$ for G and F is defined as follows:

$x \models \mathrm{G}A$ iff $\forall y > x\; y \models A$,

$x \models \mathrm{F}A$ iff in each maximal chain containing x there is $y > x$ such that $y \models A$.

To embed **Int** into **WR**-FG Bowen and de Jongh used the translation $A^{\Box\mathrm{FG}}$ which is obtained from $S_2(A)$ by replacing $\Box$ with FG. First they embedded **Int** into $\mathbf{K4} \oplus \Box\Box p \supset \Box p \oplus \Diamond\top$ by S_2 and then showed that this modal logic is the $\Box$-fragment of **WR**–FG with $\Box = \mathrm{FG}$. The second result was obtained by proving a completeness theorem for the modal logic with respect to reflexive well-founded branches models with the following definition of $\models$ for $\Box$:

$x \models \Box A$ iff for each complete chain containing x there is $y > x$ such that, for all $z > y, z \models A$.

This definition resembles Beth semantics for **Int** rather than that due to Kripke.

12. Embeddings of **Int** and some of its sublogics (such as relevant logics, the **BCK** logic, etc.) into modal logics on non-classical bases and into multimodal logics were considered, for example by Kuznetsov and Muravitsky [1986], Chagrov [1987], Girard [1987], Došen [1990b].

Added in proofs. Professor Hiroakira Ono kindly informed us that our previous survey *The disjunction property of intermediate propositional logics* (Studia Logica 50, 2 (1991), pp. 189–216) contains a lacuna: it was A. Wroński who first proved that **Int** $+ F_{13}$ has the disjunction property (see Reports on Mathematical Logic 2 (1974), pp. 63–75).

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