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O. E. OREL, P. E. RYABOV

Department of Mathematical Modelling
Moscow State Technical University after N. E. Bauman
2-nd Baumanskaja, 5, Moscow, 107005, Russia
E-mail: orel@pop.transit.ru

BIFURCATION SETS IN A PROBLEM ON MOTION OF A RIGID BODY IN FLUID AND IN THE GENERALIZATION OF THIS PROBLEM

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In the paper, topology of energy surfaces is described and bifurcation sets is constructed for the classical Chaplygin problem and its generalization. We also describe bifurcations of Liouville tori and calculate the Fomenko invariant (for the classical case this result is obtained analytically and for the generalized case it is obtained with the help of computer modeling). Topological analysis shows that some topological characteristics (such as the form of the bifurcation set) change continuously and some of them (such as topology of energy surfaces) change drastically as $g \to 0$.

1. Introduction

The classical Chaplygin problem describes a particular case of motion of a rigid body in fluid. The fluid is in irrotational motion, at rest at infinity, and unbounded in all directions. In [2] Chaplygin described this motion by the following system of the Kirchhoff equations:

$$\begin{aligned} \dot{s}_1 &= -s_2 s_3 - c r_2 r_3, & \dot{r}_1 &= s_2 r_3 - 2 s_3 r_2, \\ \dot{s}_2 &= s_1 s_3 - c r_1 r_3, & \dot{r}_2 &= 2 s_3 r_1 - s_1 r_3, \\ \dot{s}_3 &= 2 c r_1 r_2, & \dot{r}_3 &= s_1 r_2 - s_2 r_1, \end{aligned} \tag{1}$$

where c is a certain constant, describing characteristics of the body.

The first integrals of these equations have the form

$$\begin{aligned} f_1 &= r_1^2 + r_2^2 + r_3^2 = f && \text{(geometric integral)}, \\ f_2 &= s_1 r_1 + s_2 r_2 + s_3 r_3 = g && \text{(area integral)}, \\ H &= \frac{1}{2}(s_1^2 + s_2^2 + 2 s_3^2) + \frac{c}{2}(r_1^2 - r_2^2) && \text{(Hamiltonian)}. \end{aligned}$$

In the case $g = 0$ Chaplygin found the additional integral of system (1):

$$K = (s_1^2 - s_2^2 + c r_3^2)^2 + 4 s_1^2 s_2^2.$$

REMARK 1. *System (1) also appears in the problem on motion of a rigid body about a fixed point. Distribution of masses of the rigid body is subjected to the Kowalevskii conditions ($A = B = 2C$), and* *the potential has the form* $\frac{c}{2}(r_1^2 - r_2^2)$. *This problem for the potential energy of the more general form was considered by Goryachev [5]. The additional integral, which generalizes the Kowalevskii case, is also mentioned in [5].*

Note that after certain linear changes of coordinates and time parameter we can make the constant f of the geometric integral and the constant c to be equal to 1. These changes of variables do not influence topological analysis of the problem. In what follows, we assume $f = 1$, $c = 1$.

System (1) can be considered as a Hamiltonian system on an orbit of the co-adjoint representation of the group of motions of three-dimensional Euclidean space in its co-algebra $e(3)^*$. The variables $s_1, s_2, s_3, r_1, r_2, r_3$ are the standard coordinates in $e(3)^*$ endowed with the Poisson structure

$$\{s_i, s_j\} = \varepsilon_{ijk}s_k, \quad \{r_i, r_j\} = 0, \quad \{s_i, r_j\} = \{r_i, s_j\} = \varepsilon_{ijk}r_k,$$

and the orbit M^4 is represented by the equations $\{f_1 = 1, f_2 = 0\}$. The additional integral K is almost everywhere independent of H; therefore, the system is completely integrable in the sense of Liouville.

In [2] separation of variables is found. We set

$$\lambda_1 = \frac{s_1^2 + s_2^2 + K}{r_3^2}, \qquad \lambda_2 = \frac{s_1^2 + s_2^2 - K}{r_3^2}.$$

Then the dynamical system can be written in the new variables λ_1, λ_2 as

$$\begin{aligned} \dot\lambda_1 &= \sqrt{P_1(\lambda)}, \\ \dot\lambda_2 &= \sqrt{P_2(\lambda)}, \end{aligned} \tag{2}$$

where

$$P_1(\lambda) = 2(\lambda^2 - 1)(\lambda - \alpha), \quad P_2(\lambda) = 2(1 - \lambda^2)(\beta - \lambda),$$
$$\alpha = 2H + \sqrt{K}, \quad \beta = 2H - \sqrt{K}.$$

Note that this transition is degenerated for $K = 0$.

The generalization of the Chaplygin problem was obtained by Borisov and Mamaev in [1]. They consider the Lie algebra endowed with the following Poisson structure:

$$\{s_1, s_2\} = s_3 + \frac{g}{r_3^2}, \quad \{s_1, s_3\} = -s_2, \quad \{s_2, s_3\} = s_1,$$
$$\{r_i, r_j\} = 0, \quad \{s_i, r_j\} = \{r_i, s_j\} = \varepsilon_{ijk}r_k.$$

This Poisson structure is nondegenerate on the orbit $M^4 = \{f_1 = 1, f_2 = g\}$, where

$$\begin{aligned} f_1 &= r_1^2 + r_2^2 + r_3^2, \\ f_2 &= r_3(s_1r_1 + s_2r_2 + s_3r_3), \end{aligned}$$

and the Hamiltonian

$$H = \frac{1}{2}(s_1^2 + s_2^2 + 2s_3^2) + \frac{c}{2}(r_1^2 - r_2^2)$$

defines the Hamiltonian system

$$\begin{aligned} \dot s_1 &= -s_2s_3 - cr_2r_3 + \frac{gs_2}{r_3^2}, & \dot r_1 &= s_2r_3 - 2s_3r_2, \\ \dot s_2 &= s_1s_3 - cr_1r_3 - \frac{gs_1}{r_3^2}, & \dot r_2 &= 2s_3r_1 - s_1r_3, \\ \dot s_3 &= 2cr_1r_2, & \dot r_3 &= s_1r_2 - s_2r_1, \end{aligned} \tag{3}$$

on this orbit. This system is integrable in the sense of Liouville by means of the same additional integral

$$K = (s_1^2 - s_2^2 + cr_3^2)^2 + 4s_1^2 s_2^2.$$

From the physical point of view this generalization describes the motion of an axially symmetric rigid body ($A = B = 2C$) about a fixed point in superposition of two uniform force fields. The centers of these fields lie in the equatorial plane of the inertia ellipsoid at equal distances from the fixed point. This analogue of the Kowalevskii case was considered by Yehia [9], who however described it in terms of other variables. As before we set $c = 1$. For $g = 0$ we get the classical Chaplygin problem.

In the present paper, topology of energy surfaces is described and bifurcation sets is constructed for the classical Chaplygin problem and its generalization. We also describe bifurcations of Liouville tori and calculate the Fomenko invariant (for the classical case this result is obtained analytically and for the generalized case it is obtained with the help of computer modeling). Topological analysis shows that some topological characteristics (such as the form of the bifurcation set) change continuously and some of them (such as topology of energy surfaces) change drastically as $g \to 0$.

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2. Necessary definitions

Let (M^4, ω) be a symplectic manifold and let $v = \operatorname{sgrad} H$ be a Hamiltonian system on it. Suppose that the system is completely integrable by means of an additional Morse–Bott integral K. By the Liouville theorem, each nonsingular compact level surface of the first integrals H and K is a disconnected union of two-dimensional tori (Liouville tori) with quasiperiodic motion on them.

The *moment map* $\Phi : M^4 \to \mathbb{R}^2$ assigns to a point x on the manifold the pair of values $H(x)$ and $K(x)$: $x \to (H(x), K(x))$. Clearly, a Liouville torus is mapped to a single point in the plane (h, k). The *set of singularities* F of the moment map is the set of points of M^4 at which the functions H and K are dependent: $F = \{x \in M^4 : \operatorname{rank} d\Phi(x) < 2\}$. The image $\Sigma = \Phi(F)$ of this set is called a *bifurcation set*. The surface $Q_h^3 = \{x \in M^4 | H(x) = h\}$ is called an *energy surface*. In what follows, we assume that this surface is nonsingular and compact.

Consider two systems v and v' on manifolds M and M', respectively. We restrict them on energy surfaces Q and Q'. Decomposition of these manifold by connected components of level surfaces of the additional integrals is called their *Liouville foliations*. We also consider a manifold Q'' that is constructed from Q, using several operations of the following type: the manifold Q is cut along a Liouville torus, and then the boundaries are glued again by a diffeomorphism of boundary tori. This operation is called *twisting along a Liouville torus*.

Definition 1. Two systems v and v' on energy surfaces Q and Q' are called *roughly Liouville equivalent*, if there exists a manifold Q'' obtained from Q with the aid of twistings along tori such that Q' and Q'' are fiberwise homeomorphic. This means that there exists a homeomorphism (preserving orientation) that takes Q' onto Q'' and preserves the Liouville foliations of these manifolds.

In [3] Fomenko constructed the topological invariant for the systems with two degrees of freedom. This invariant classifies integrable Hamiltonian systems on energy surfaces up to rough Liouville equivalence. According to this theory, the *Fomenko invariant*, or the *molecule*, $W(Q_h^3, v)$ is assigned to each nondegenerate integrable Hamiltonian system v restricted to the energy surface Q_h^3. This invariant is a graph, describing the Liouville foliation of the energy surface Q_h^3. The edges of the graph correspond to one-parameter families of nonsingular Liouville tori, and its vertices (=atoms) describe bifurcations of these tori on singular levels of the integral K. One of the main results of the theory is the following: two nondegenerate integrable systems are roughly Liouville equivalent if and only if their molecules coincide.

The simplest bifurcations (atoms) are denoted by A and B: the bifurcation A characterizes degeneration of a torus to a circle, and B denotes bifurcation of one tori to two ones (or, conversely, gluing of two tori to one torus). Besides, in problems, possessing some symmetry, the atom C_2 often appears; it describes symmetrical bifurcation of two tori to other two ones.

In our problem the manifold M^4 is given by the equations $\{f_1 = 1, f_2 = g\}$ in the space $\mathbb{R}^6$, and the symplectic structure is defined through the Poisson structure. The Hamiltonian system (3) is Liouville integrable, and we will prove that the energy surface Q^3_h is compact for any h. We will denote the bifurcation set for $f_2 = g$ by $\Sigma(g)$. Thus, $\Sigma(0)$ is the bifurcation set for the classical Chaplygin problem.

3. Topology of energy surfaces

Theorem 1. *In the classical case the surface Q^3_h has the following topological type:*

1. $\varnothing$ *for* $h < -1/2$,
2. $2S^3$ *for* $-1/2 < h < 0$,
3. $S^1 \times S^2$ *for* $0 < h < 1/2$,
4. $\mathbb{RP}^3$ *for* $1/2 < h$.

Proof.

Topological type of the energy surface $Q^3_h = \{H = h\}$ can be studied with the help of the projection π to the Poisson sphere $S^2 = \{r_1^2 + r_2^2 + r_3^2 = 1\}$ (see [8, 6]). In our problem the projection maps the surface Q^3_h onto the domain determined by the condition

$$\varphi(r) \leq h, \tag{4}$$

where

$$\varphi(r) = \frac{1}{2}(r_1^2 - r_2^2).$$

The function $\varphi(r)$ is the Morse function on the sphere. Having studied its singularities, we obtain that for various values h the domain has the following form: the domain (4) is empty for $h < -1/2$, it has the form of two discs for $-1/2 < h < 0$ and an annulus for $0 < h < 1/2$, it coincides with the sphere for $h > 1/2$. This means that for $h < -1/2$ the surface Q^3_h is empty, for $-1/2 < h < 0$ it consists of two S^3. For $0 < h < 1/2$ the energy surface is homeomorphic to $S^1 \times S^2$, and for $h > 1/2$ it is homeomorphic to $\mathbb{RP}^3$. Theorem is proved. ∎

Theorem 2. *In the generalized case the surface Q^3_h has the following topological type:*

1. $\varnothing$ *for* $h < h(g)$,
2. $4S^3$ *for* $h(g) < h < g^2$,
3. $2S^3$ *for* $g^2 < h$.

Here the separating curve $h(g)$ is given parametrically:

$$h = \frac{1 - 3t^4}{4t^2}, \quad g = \frac{1 - t^4}{2t}, \quad t \in [-1, 0) \cup (0, 1]. \tag{5}$$

Topological type of Q^h_3 and the separating curves $h = g^2$ and $h = h(g)$ are represented in Figure 1.

Proof.

We show that in the generalized case the projection π to the Poisson sphere $S^2 = \{r_1^2+r_2^2+r_3^2 = 1\}$ maps the surface Q_h^3 onto the domain determined by the condition

$$\varphi_g(r) \leq h, \tag{6}$$

where

$$\varphi_g(r) = \frac{1}{2}\left(r_1^2 - r_2^2 + \frac{g^2}{2-r_3^2} + \frac{g^2}{r_3^2}\right).$$

The surface Q_h^3 is stratified over this domain with the circle fiber contracted to the point over the boundary.

To prove this fact we introduce new variables w_i by the formulae:

$$s_i = \frac{g}{r_3} r_i + w_i.$$

Then in variables (r, w) the equations of the orbit $M^4 = \{f_1 = 1,\ f_2 = g\}$ take the form

$$r_1^2 + r_2^2 + r_3^2,$$
$$r_1 w_1 + r_2 w_2 + r_3 w_3 = 0,$$

and the Hamiltonian is written down as

$$H = \frac{1}{2}\left(w_1^2 + w_2^2 + 2\left(w_3 + \frac{g}{2}\right)^2\right) + \frac{1}{2}\left(\frac{g^2}{2} + \frac{g^2}{r_3^2} + r_1^2 - r_2^2\right).$$

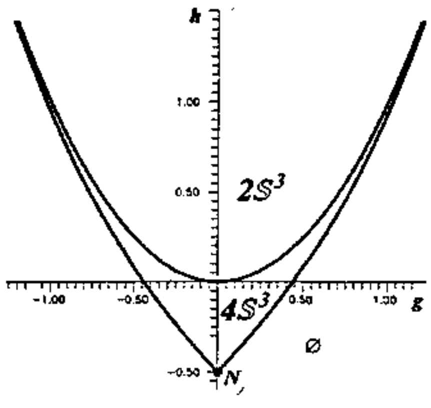

Fig. 1.

We fix a value h of the energy and a point $r = (r_1, r_2, r_3)$ on the Poisson sphere and try to find the preimage $\pi^{-1}(r)$ of this point on the surface Q_h^3. Obviously, the preimage of this point is the intersection of the ellipsoid

$$\frac{1}{2}\left(w_1^2 + w_2^2 + 2\left(w_3 + \frac{g}{2}\right)^2\right) = h - \frac{1}{2}\left(\frac{g^2}{2} + \frac{g^2}{r_3^2} + r_1^2 - r_2^2\right) \tag{7}$$

with the center $(0, 0, -g/2)$ and the plane

$$r_1 w_1 + r_2 w_2 + r_3 w_3 = 0$$

in the space (w_1, w_2, w_3). For simplicity, we denote the right-hand side of equation (7) by $\rho_g(h, r)$. It can be easily shown that for $\rho_g(h,r) < g^2 r_3^2/(8 - 4r_3^2)$ the plane does not intersect the ellipsoid, for $\rho_g(h,r) = g^2 r_3^2/(8 - 4r_3^2)$ it touches the ellipsoid, and for $\rho_g(h,r) > g^2 r_3^2/(8-4r_3^2)$ the plane intersects the ellipsoid along the circle.

Replacing $\rho_g(h,r)$ by its expression in terms of g, h, and r, we obtain that the preimage of the point (r_1, r_2, r_3) is not empty if and only if

$$h \geq \frac{1}{2}\left(r_1^2 - r_2^2 + \frac{g^2}{2 - r_3^2} + \frac{g^2}{r_3^2}\right). \tag{8}$$

It is easy to see that for $g = 0$ the function $\varphi_g(r)$ gives the function $\varphi(r)$. Since there is the variable r_3 in the denominator of (8) for $g \neq 0$, the domain (6) does not contain the circle $r_3 = 0$ for any h. Thus, the surface Q_h^3 always consists of at least two disconnected parts.

For $g \neq 0$ the function $\varphi_g(r)$ is the Morse function and has four singular points of index 0 (two points in each domain $r_3 > 0$ and $r_3 < 0$) on the level $h = h(g)$ (where $h(g)$ is determined by formula (5)) and two singular points of index 1 (one point in each domain $r_3 > 0$ and $r_3 < 0$) on the level $h = g^2$. Thus, the domain (6) is empty for $h < h(g)$, it has the form of four discs (two discs in the domain $r_3 > 0$ and two ones in the domain $r_3 < 0$) for $h(g) < h < g^2$ or two discs (one disc in each domain $r_3 > 0$ and $r_3 < 0$) for $g^2 < h$. Theorem is proved. ■

4. Bifurcation set

To determine the bifurcation set we need some simple propositions based on the analysis of right-hand sides of systems (1), (3).

Proposition 1. *Any fixed point of the flow* sgrad H *lies on the hyperplane* $r_1 = 0$ *or* $r_2 = 0$. *Any closed trajectory of the flow* sgrad H *intersects the hyperplane* $r_1 = 0$ *or* $r_2 = 0$.

Proof.
We have $\dot{s}_i = 0$, $\dot{r}_i = 0$, $i = 1, 2, 3$, at any fixed point of the flow. In particular, the third equation of systems (1) and (3) gives us $\dot{s}_3 = 2r_1r_2 = 0$, whence $r_1 = 0$ or $r_2 = 0$. We now consider an arbitrary closed trajectory. Since the coordinate s_3 depends on time periodically on this trajectory, there exists an instant of time t_0 such that $\dot{s}_3(t_0) = 0$. Therefore, by virtue of the third equation of the system we have $r_1(t_0) = 0$ or $r_2(t_0) = 0$.
■

Proposition 2. *Any Liouville torus intersects the hyperplane* $r_1 = 0$ *or* $r_2 = 0$.

Proof.
We take an arbitrary point on the Liouville torus and issue the trajectory from it. This trajectory is either closed or everywhere dense on the torus. The coordinate s_3 depends on time periodically or almost periodically on this trajectory. Then there exists t_0 such that $\dot{s}_3(t_0) = 0$. Therefore, the trajectory (and, consequently, the Liouville torus) intersects the hyperplane $r_1 = 0$ or $r_2 = 0$. ■

Corollary 1. *Singular fibers of the Liouville foliation intersect the hyperplane* $r_1 = 0$ *or* $r_2 = 0$.

Proof.
A singular fiber of the Liouville foliation has a fixed point or a closed trajectory. Therefore, it intersects the hyperplane $r_1 = 0$ or $r_2 = 0$. ■

Theorem 3. *The bifurcation set* $\Sigma(0)$ *is a union of the curves* γ_1, γ_2, γ_3, *where*

$$\begin{aligned} \gamma_1 &: \quad k = 0, \quad h \geq -\frac{1}{2}, \\ \gamma_2 &: \quad k = (2h+1)^2, \quad h \geq -\frac{1}{2}, \\ \gamma_3 &: \quad k = (2h-1)^2, \quad h \geq 0. \end{aligned}$$

The bifurcation set is illustrated in Figure 2.

Proof.
In order to calculate $\Sigma(0)$ we study singularities of the system of the first integrals f_1, f_2, H, and K. It is convenient to determine critical points of the moment map from the condition

$$\operatorname{rank} J < 4,$$

where

$$J = \begin{pmatrix} s_1 & s_2 & 2s_3 & r_1 & -r_2 & 0 \\ s_1\eta + 2s_1s_2^2 & -s_2\eta + 2s_1^2s_2 & 0 & 0 & 0 & r_3\eta \\ r_1 & r_2 & r_3 & s_1 & s_2 & s_3 \\ 0 & 0 & 0 & r_1 & r_2 & r_3 \end{pmatrix} \tag{9}$$

is the Jacobi matrix of the map $H \times K \times f_1 \times f_2$. We introduce the notation $\eta = s_1^2 - s_2^2 + r_3^2$ into formula (9) .

According to Corollary 1, it is sufficient to consider the cases $r_1 = 0$ or $r_2 = 0$. First we suppose

$$r_1 = 0.$$

In this case the condition $\operatorname{rank} J < 4$ is valid if and only if all Δ_{ijkl} are equal to zero. Here Δ_{ijkl} are determinants of matrices consisting of columns of the Jacobi matrix (9) with the numbers $1 \leq i < j < k < l \leq 6$.

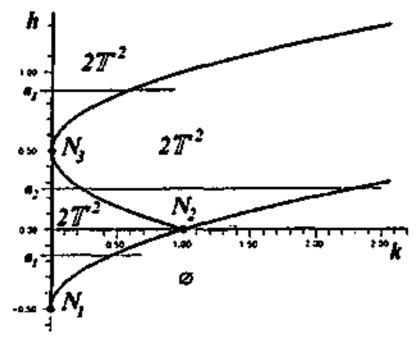

Fig. 2.

We solve the system of equations $\Delta_{ijkl} = 0$ for $r_1 = 0$ in the case

$$s_1 = 0.$$

In this case the relation

$$(-s_2^2 + r_3^2)(r_3 s_2 - r_2 s_3)(r_2 r_3 + s_2 s_3) = 0$$

is valid at the critical points. From this relation and the system of the first integrals we find the bifurcation curves γ_1 and γ_2.

Now suppose

$$r_2 = 0.$$

We consider the system of equations $\Delta_{ijkl} = 0$ for $r_2 = 0$ in the case

$$s_2 = 0.$$

With these assumptions we obtain the following relation for the critical points

$$(s_1^2 + r_3^2)(r_3 s_1 - r_1 s_3)(r_1 r_3 - s_1 s_3) = 0.$$

The corresponding critical values determine the curve γ_3.

Analysis of other conditions $r_1 = 0, s_1 \neq 0$ and $r_2 = 0, s_2 \neq 0$ shows that there are no bifurcation curves other then γ_i. Theorem is proved. ∎

REMARK 2. *It can be easily shown that in the classical Chaplygin case the bifurcation curves γ_i are parts of surfaces of multiple roots of the polynomials $P_i(\lambda)$, $i = 1, 2$, which appear in separation of variables. This fact is not accidental. If in a problem the separating variables are found, then we often seek the bifurcation set in the form of a surface of multiple roots of the corresponding polynomials (see, for example, [4]). However, in some problems separating variables have not known yet. Analysis of rank of the Jacobi matrix for the map $H \times K \times f_1 \times f_2$ allows us to obtain the equation of the bifurcation set and even to find polynomials, whose multiple roots give these equations, in all cases without separating variables (see [7]). Polynomials of this type are found in the generalized Chaplygin case as shown in the theorem below. Probably, these polynomials (or similar polynomials) take part in separation of variables.*

Theorem 4. *For nonzero area constant the bifurcation set $\Sigma(g)$ is a union of the curves γ_4, γ_5, γ_6. Here the curve γ_4 is a half-line*

$$k = 0, \quad h(s) = \frac{1}{2} + \frac{g^2 \pm 2gs(1 - s^2)}{(1 - s^2)^2}, \quad s \in (-1, 1),$$

and the curves γ_5, γ_6 are parts of surfaces of multiple roots of the polynomials

$$R_1(\lambda) = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$
$$R_2(\lambda) = b_4\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0,$$

where

$$
\begin{aligned}
a_4 &= (1+\alpha)^2,\\
a_3 &= -2(\beta\alpha+\alpha^2+\beta+3\alpha-8g^2+2),\\
a_2 &= \beta^2+4\beta\alpha+\alpha^2+6(\beta+\alpha+1)+4g^2(\alpha-2\beta-9),\\
a_1 &= -2(\beta^2+\beta\alpha+3\beta+\alpha+2g^2(\alpha-3\beta-6)+2),\\
a_0 &= (1+\beta)^2+4g^2(g^2-\beta-1),\\
b_4 &= (1-\beta)^2,\\
b_3 &= -2(\beta^2+\beta\alpha-3\beta-\alpha+8g^2+2),\\
b_2 &= \beta^2+4\beta\alpha+\alpha^2-6(\beta+\alpha-1)+4g^2(\beta-2\alpha+9),\\
b_1 &= -2(\beta\alpha+\alpha^2-\beta-3\alpha+2g^2(\beta-3\alpha+6)+2),\\
b_0 &= (1-\alpha)^2+4g^2(g^2-\alpha+1).
\end{aligned}
$$

Here, as before, $\alpha = 2H+\sqrt{K}$, $\beta = 2H-\sqrt{K}$.

The bifurcation set is illustrated in Figure 3.

Proof.

Similarly to the previous theorem, it is convenient to determine the critical points of the moment map from the condition

$$\operatorname{rank} J < 4,$$

where

$$J = \begin{pmatrix} s_1 & s_2 & 2s_3 & r_1 & -r_2 & 0\\ s_1\eta+2s_1s_2^2 & -s_2\eta+2s_1^2s_2 & 0 & 0 & 0 & r_3\eta\\ r_1 & r_2 & r_3 & s_1 & s_2 & \dfrac{g+s_3r_3^2}{r_3^2}\cdot\\ 0 & 0 & 0 & r_1 & r_2 & r_3 \end{pmatrix}$$

is the Jacobi matrix of the map $H\times K\times f_1'\times f_2$, $f_1' = (f_1-g)/r_3$. We introduce the notation $\eta = s_1^2-s_2^2+r_3^2$ into formula (9).

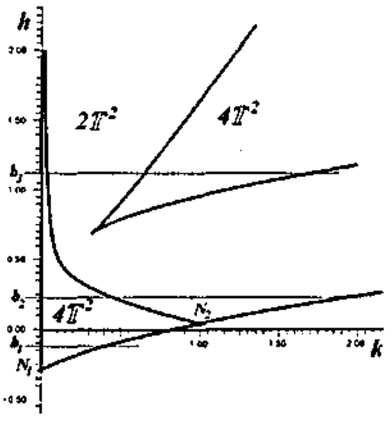

Fig. 3.

According to Corollary 1, it is sufficient to consider the cases $r_1=0$ and $r_2=0$. Suppose

$$r_1 = 0.$$

We solve the system of equations $\Delta_{ijkl}=0$ for $r_1=0$ in the case

$$s_1 = 0.$$

Then the relation

$$(-s_2^2+r_3^2)\{r_3^2(s_2s_3+r_2r_3)v+gs_2s_3r_2\} = 0$$

takes place at the critical points. Here we put

$$v = s_3r_2 - s_2r_3. \tag{10}$$

From the system of the first integrals we obtain the curve γ_4 as well as the curve

$$\left\{\begin{aligned} h &= \frac{1}{2}\left(\frac{g^2}{1-r_2^2}+v^2+(r_2v+g)^2-r_2^2\right),\\ k &= (1-r_2^2)^2\left\{1-\left(\frac{gr_2}{1-r_2^2}-v\right)^2\right\}^2, \end{aligned}\right. \tag{11}$$

where the variable v (with regard to relation (10)) satisfies the equation

$$r_2(1-r_2^2)^2v^3+g(1-r_2^2)^2v^2-r_2\{(1-r_2^2)^2+g^2r_2^2\}v-g^3r_2^2=0$$

for $r_2\in(-1,1)$.

If we eliminate the variable v, then system (11) can be represented in the form $R_1(r_2)=0$, $R_1'(r_2)=0$. Thus, we obtain the curve γ_5.

Consider the case $r_2=0$, $s_2=0$. Similarly, we find the system of equations, which determines the other bifurcation curve:

$$\begin{cases} h=\dfrac{1}{2}\left(\dfrac{g^2}{1-r_1^2}+u^2+(r_1u+g)^2+r_1^2\right), \\ k=(1-r_1^2)^2\left\{1+\left(\dfrac{gr_1}{1-r_1^2}-u\right)^2\right\}^2, \\ r_1(1-r_1^2)^2u^3+g(1-r_1^2)^2u^2- \\ -r_1\{(1-r_1^2)^2-g^2r_1^2\}u-g^3r_1^2=0 \end{cases}$$

for $r_1\in(-1,1)$ and $u=r_1s_3-r_3s_1$.

Eliminating the variable u, we represent this system in the form $R_2(r_1)=0$, $R_2'(r_1)=0$. Therefore, we obtain the curve γ_6.

It can be shown that the cases $s_1\neq0$ and $s_2\neq0$ do not give new curves. Theorem is proved. ■

In conclusion, we formulate the results, concerning the number of Liouville tori in the preimage of the moment map and analysis of their bifurcations.

Theorem 5. *In the Chaplygin problem the preimage of any nonsingular value of the moment map consists of two tori. All saddle bifurcations have the type C_2. The Fomenko invariant is given in Table 1 for various energy levels.*

Proof.

To prove this theorem we analyze dynamical system (2) in terms of separating variables (λ_1,λ_2) and the formulae of transition from the initial variables (s,r) to the separating variables. ■

In the generalized case, we find the number of tori in the preimage of the moment map and describe their bifurcations with the help of computer modeling. Number of tori in various domains of the image of the moment map are shown in Figure 3. The Fomenko invariant for the generalized case is given in Table 2. Note that the bifurcation set for the generalized case continuously transforms to the bifurcation set for the classical case as $g\to0$. However, the number of tori in some domains is doubled. This can be explained by the fact that tori, intersecting the circle $r_3=0$ for $g=0$ in the projection to the Poisson sphere, are "cut" by this circle into two tori for $g\neq0$. Besides, we can observe decomposition of the singularity C_2 (in the classical case) into more simple bifurcations of type B (in the generalized case). This fact very often takes place in perturbations of integrable systems, when systems lose their symmetry.

Topological analysis of the generalized Chaplygin problem shows that this problem is a substantially different generalization of the Kowalevskii case (in formulation [9]): comparison of topological invariants shows that the Kowalevskii case and the case [9] are not Liouville equivalent, and, moreover, are not orbitally equivalent.

h	Q^3_h	$W(Q^3_h)$
a_1	$2S^3$	$A — A \quad A — A$
a_2	$S^1 \times S^2$	$A, A — C_2 — A, A$
a_3	$\mathbb{R}P^3$	$A, A — C_2 — A, A$

Table 1.

h	Q^3_h	$W(Q^3_h)$
b_1	$4S^3$	$A — A \quad A — A$ $A — A \quad A — A$
b_2	$2S^3$	$A, A — B — A$ $A, A — B — A$
b_3	$2S^3$	$A, A — B — B — A$; $A — B$ $A, A — B — B — A$; $A — B$

Table 2.

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О. Е. ОРЕЛ, П. Е. РЯБОВ

БИФУРКАЦИОННЫЕ МНОЖЕСТВА В ОДНОЙ ЗАДАЧЕ О ДВИЖЕНИИ ТВЕРДОГО ТЕЛА В ЖИДКОСТИ И ЕЕ ОБОБЩЕНИИ

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В статье описана топология энергетических поверхностей и построены бифуркационные множества для классической задачи Чаплыгина и ее обобщений. Мы также описываем бифуркации торов Лиувилля и вычисляем инварианты Фоменко (для классического случая эти результаты получены аналитически, а для обобщенного при помощи компьютерного моделирования). Топологический анализ показывает, что некоторые топологические характеристики (такие как форма бифуркационного множества) изменяются непрерывно, а другие (как топология энергетических поверхностей) терпят разрыв при $g \to 0$.