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BIFURCATIONS OF FIRST INTEGRALS IN THE SOKOLOV CASE

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We study the phase topology of a new Liouville integrable Hamiltonian system with an additional quartic integral (the Sokolov case).

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1. Introduction

We consider the system of Kirchhoff equations

$$\begin{aligned} \dot{s}_1 &= (-s_2 + \alpha r_3)(\alpha r_2 + s_3), & \dot{r}_1 &= r_3 s_2 - 2r_2 s_3 - \alpha r_2^2, \\ \dot{s}_2 &= s_3 s_1 + \alpha s_1 r_2 - \alpha^2 r_1 r_3, & \dot{r}_2 &= -s_1 r_3 + 2r_1 s_3 + \alpha r_1 r_2, \\ \dot{s}_3 &= -\alpha r_1 s_3, & \dot{r}_3 &= s_1 r_2 - s_2 r_1, \end{aligned} \tag{1}$$

where α is a parameter. System (1) can be written on the space $\mathbb{R}^6$ in the Hamiltonian form

$$\dot{\mu}_i = \{\mu_i, H\},$$

where $\boldsymbol{\mu} = (\mathbf{s}, \mathbf{r}) \in \mathbb{R}^6$ and the quantity

$$H = \frac{1}{2}(s_1^2 + s_2^2 + 2s_3^2) + \alpha s_3 r_2 - \frac{1}{2}\alpha^2 r_3^2$$

is the energy of the (Hamiltonian) "solid–liquid" system. Here, the vectors $\mathbf{s}$ and $\mathbf{r}$ are respectively called the impulsive angular momentum and the impulsive force. For differentiable functions G and K on $\mathbb{R}^6$, the Poisson bracket is defined by

$$\{G, K\}(\boldsymbol{\mu}) = \nabla G\, P(\boldsymbol{\mu}) \nabla K.$$

The matrix

$$P(\boldsymbol{\mu}) = \begin{pmatrix} S & R \\ R & 0 \end{pmatrix},$$

where

$$S = \begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix},$$

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determines the Poisson structure on $\mathbb{R}^6$. Two geometric first integrals of system (1),

$$F_1 = r_1^2 + r_2^2 + r_3^2, \qquad F_2 = s_1 r_1 + s_2 r_2 + s_3 r_3,$$

are the Casimir functions with respect to this structure.

Vector field (1) restricted to the four-dimensional manifold

$$M^4 = \big\{(\mathbf{s}, \mathbf{r}) \in \mathbb{R}^6 \colon F_1 = f_1,\ f_1 > 0,\ F_2 = g\big\} \cong T^*\mathbb{S}^2$$

determines a Hamiltonian system with two degrees of freedom. The Liouville integrability therefore implies the presence of an extra integral in addition to the Casimir functions F_1 and F_2 and the Hamiltonian H. As shown in [1], [2], the function

$$F = s_3^2\{s_1^2 + s_2^2 + s_3^2 + 2\alpha(r_2 s_3 - r_3 s_2) + \alpha^2(r_2^2 + r_3^2)\} +$$
$$+ 2\alpha s_3(s_2 - \alpha r_3)(s_1 r_1 + s_2 r_2 + s_3 r_3)$$

is a first integral of system (1), i.e., we have $\{H, F\} = 0$.

Without loss of generality, we hereafter set $f_1 = 1$ and $\alpha = 1$.

In this paper, we study the topology of the isoenergetic surfaces $Q_h^3 = \{H = h\}$, construct the bifurcation set, and also describe the bifurcations of the Liouville tori. In the proofs of the main theorems, we use the methods of the topological analysis of concrete systems proposed in [3] and developed in [4]–[6] as well as the general theory of Hamiltonian systems [7], [8]. The methods for computer modeling of the bifurcations of Liouville tori are also useful. Combining these approaches allows proving the most difficult statements about the number and bifurcations of the Liouville tori. The described topology of the Liouville foliation on the energy surfaces in the Sokolov case is new, i.e., this case is not topologically equivalent to any of the previously studied ones. In particular, the presence of the boundary critical torus for a zero area constant is a new element. From the topological standpoint, this case is most closely related to the Steklov integrability case.

2. Some definitions

The *moment mapping* is by definition the mapping $\Phi \colon M^4 \to \mathbb{R}^2(f, h)$ taking a point x of the manifold to the pair $\big(f = F(x), h = H(x)\big)$ of the values of the functions F and H at this point. Obviously, the Liouville torus is mapped to a single point on the plane. The set $K = \{x \in M^4 \colon \operatorname{rank} d\Phi(x) < 2\}$ of the points of the manifold M^4 at which the functions F and H are dependent is called the *singular set* of the moment mapping. The image $\Sigma = \Phi(K)$ of this set is called the *bifurcation set*. In addition to the moment mapping, we consider the mapping $\Phi_1 \colon \mathbb{S}^2 \times \mathbb{R}^3 \to \mathbb{R}^2(g, h)$ defined by $\Phi_1(x) = \big(g = F_2(x), h = H(x)\big)$, where the point x belongs to the space $\mathbb{S}^2 \times \mathbb{R}^3$. The image of the singular set of the mapping Φ_1 is the *bifurcation diagram* (bifurcation set) Σ_1 on the plane $\mathbb{R}^2(g, h)$. The preimage of each point $(g, h) \notin \Sigma_1$ is a regular compact three-dimensional manifold $Q_h^3 = \{x \in M^4 \mid H(x) = h\}$, called the *isoenergetic surface*. The topological type of Q_h^3 is the same for all points (g, h) belonging to the same connected component of $\mathbb{R}^2(g, h) \setminus \Sigma_1$ and can change only after passing through Σ_1. The *Liouville foliation* is the foliation of the isoenergetic surface by the level sets of the additional integral. The fiber of the Liouville foliation is the connected component of the preimage of a point under the moment mapping, and the bifurcation set Σ is the image of the singular fibers of the Liouville foliation.

For regular compact energy level surfaces, there is a theory of classification of integrable Hamiltonian systems [7]. In this theory, the global behavior of a system on the isoenergetic surface can be described by