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MATHEMATICAL PHYSICS

Bifurcation Diagram of the Two Vortices in a Bose–Einstein Condensate with Intensities of the Same Signs[1]

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Abstract—This paper deals with the problem of motion of a system of two point vortices in a Bose–Einstein condensate enclosed in a cylindrical trap. Bifurcation diagram is analytically determined for the intensities of one sign and bifurcations of Liouville tori are investigated. We obtain explicit formulas for determining the type of critical trajectories, which allow us to investigate the stability of the obtained solutions.

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1. INTRODUCTION

Vortices and vortex lattices in the Bose–Einstein condensate which appears on ultracold atoms, and in the modern objects of the condensed matter physics are described in [1]. Isolated vortices and systems of two vortices are being actively investigated from both the theoretical and the experimental point of view [2].

In this paper, we address the problem of the motion of two rectilinear vortex filaments in a Bose–Einstein condensate enclosed in a cylindrical trap (see [3] and papers cited therein). The filaments are parallel to the generatrix of the circular cylinder, and so it is obvious that the problem is two-dimensional. The main goal of this paper is to analyze the phase topology of this problem for the case of the same signs of vortex intensities.

As in [4], the dynamics of a system of two vortices is described by a system of differential equations with respect to the coordinates of vortex filaments. In [4] it is shown that the equations of motion of two point vortices can be represented in Hamiltonian form

$$\dot{\zeta} = \{\zeta, H\} \tag{1}$$

with the Hamiltonian

$$H = \ln[1-(x_1^2+y_1^2)] + a^2\ln[1-(x_2^2+y_2^2)] - a\ln[(x_2-x_1)^2+(y_2-y_1)^2]. \tag{2}$$

Here (x_k, y_k) is a Cartesian coordinates of the kth vortex $(k = 1,2)$, the phase vector ζ has a coordinates $\{x_1, y_1, x_2, y_2\}$, the parameter a is a ratio of intensities $\dfrac{\Gamma_2}{\Gamma_1}$. The phase space $\mathcal{P}$ specified as a direct product of two open circles of radius 1:

$$\mathcal{P} = \{(x_1, y_1, x_2, y_2): x_1^2+y_1^2 < 1, x_2^2+y_2^2 < 1\}.$$

The Poisson structure on the phase space $\mathcal{P}$ is given in the standard form

$$\{x_i, y_j\} = -\frac{1}{\Gamma_i}\delta_{ij},$$

where δ_{ij} is the Kronecker delta.

The system (1) admits one additional first integral of motion – *the momentum of vorticity*

$$F = x_1^2 + y_1^2 + a(x_2^2 + y_2^2).$$

The function F together with Hamiltonian H form on $\mathcal{P}$ a complete involutive set of integrals of the system (1). According to Arnold-Liouville theorem, it can be argued that a compact connected component of an integral manifold $\mathcal{M} = \{H = h, F = f\}$ is diffeomorphic to a two-dimensional torus. We define the integral mapping $\mathcal{F}: \mathcal{P} \to \mathbb{R}^2$ counting $(f, h) = \mathcal{F}(\zeta) = (F(\zeta), H(\zeta))$. The mapping $\mathcal{F}$ is also referred to as the *momentum mapping*. Denote by $\mathcal{C}$ the set of all critical points of the momentum mapping, that is, the points

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at which rank$d\mathcal{F}(x) < 2$. The set of critical values $\Sigma = \mathcal{F}(\mathcal{C} \cap \mathcal{P})$ is called *a bifurcation diagram*.

Topological methods of investigation of stability for various motions of integrable Hamiltonian systems [5, 6] are actively used both in nonholonomic mechanics [7] and in vortex dynamics [8–14]. The main role in topological analysis plays bifurcation diagram Σ of the momentum mapping $\mathcal{F}$. In paper [4] the bifurcation diagram for the problem of motion of a system of two point vortices in a Bose–Einstein condensate (1) in the case of intensities of opposite signs ($a < 0$) is analytically investigated. This publication will analytically derive a bifurcation diagram of the system (1) when the intensity ratio parameter a has a positive sign.

To find the bifurcation diagram we use the method of critical subsystems developed by M.P. Kharlamov for the study of the phase topology of integrable problems of rigid body dynamics. When analyzing the stability of non-degenerate (in the sense of singularity theory) trajectories bifurcation diagram allows to quickly and clearly determine the stability in cases where the use of common standard methods is quite difficult. As an application, we present an analysis of the stability of critical trajectories (i.e., the non-degenerate singularities of rank 1 of the momentum mapping) by defining a trajectory type (elliptic/hyperbolic) for each curve from the bifurcation set.

2. CRITICAL SUBSYSTEMS AND A BIFURCATION DIAGRAM OF THE MOMENTUM MAPPING

We define the following polynomial expressions F_k in phase variables

$$F_1 = x_1y_2 - y_1x_2,$$
$$F_2 = (x_1^2 + x_2^2 - x_1x_2)(x_2^2 + y_2^2) - x_2^2,$$
$$F_3 = x_1(x_2^2 + y_2^2)(x_2 + ax_1) - x_2(ax_2 + x_1)$$

and denote by $\mathcal{N}_1$ the closure of the set of solutions of the system

$$F_1 = 0, \quad F_2 = 0 \tag{3}$$

and by $\mathcal{N}_2$ the closure of the set of solutions of the system

$$F_1 = 0, \quad F_3 = 0. \tag{4}$$

Then the theorem holds.

Theorem 1. *The set* $\mathcal{C}$ *of critical points of the momentum mapping* $\mathcal{F}$ *is exhausted by the set of solutions of the set of systems* (3) *and* (4). *Sets* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are two-dimensional invariant submanifolds of a system* (1) *with Hamiltonian* (2).

Proof. To prove the first statement of the theorem, it is necessary to find the points of the phase space where the rank of the moment map is not maximal. Using a direct calculation, one can verify that the Jacobi matrix of the moment map has second-order zero minors at points $\zeta \in \mathcal{P}$ whose coordinates satisfy the equations of the system (3) and (4), from where $\mathcal{C} = \mathcal{N}_1 \cup \mathcal{N}_2$. The invariance of the relations (3) and (4) can be verified with the help of the following chains

$$\dot{F}_1 = \{F_1, H\}_{F_1=0} = \sigma_1 F_2 F_3, \quad \dot{F}_2 = \{F_2, H\}_{F_1=0} = \sigma_2 F_2,$$
$$\dot{F}_3 = \{F_3, H\}_{F_1=0} = \sigma_3 F_3,$$

where σ_k are some polynomial functions from phase variables.

To determine the bifurcation diagram Σ, it is convenient to go to the polar coordinates using following expressions

$$x_1 = r_1\cos\theta_1, \quad y_1 = r_1\sin\theta_1,$$
$$x_2 = r_2\cos\theta_2, \quad y_2 = r_2\sin\theta_2.$$

The first equation of system (3) and (4) takes the form $\sin(\theta_1 - \theta_2) = 0$, i.e., $\theta_1 - \theta_2 = 0$ and $\theta_1 - \theta_2 = \pi$. The first possibility, unlike the dynamics of two vortices of opposite sign intensities [4], is not realized at any positive value of the intensity ratio parameter a. For the second possibility, i.e., when $\theta_1 = \theta_2 + \pi$, parametrization of the critical set $\mathcal{C}$ can be represented as a union of two critical subsystems $\mathcal{N}_1$ and $\mathcal{N}_2$ in the following form

$$\mathcal{N}_1 : \begin{cases} \theta_1 = \theta_2 + \pi, \\ r_1(s) = \dfrac{1 + 2s}{s^2 + s + 1}, \\ r_2(s) = \dfrac{s^2 - 1}{s^2 + s + 1}, \quad s \in [1, +\infty), \end{cases} \tag{5}$$

and

$$\mathcal{N}_2 : \begin{cases} \theta_1 = \theta_2 + \pi, \\ r_1(t) = \sqrt{\dfrac{at - 1}{t(a - t)}}, \quad r_2(t) = \sqrt{\dfrac{t(at - 1)}{a - t}}, \\ t \in \left(1; \dfrac{1}{a}\right), \quad a \in (0;1); \quad t \in \left(\dfrac{1}{a}; 1\right), \quad a \in (1; +\infty); \\ r_1 = r_2, \quad a = 1. \end{cases} \tag{6}$$

Taking into account the parametric representation of the critical set $\mathcal{C}$ in the form of (5) and (6), we obtain the following theorem.

Theorem 2. *The bifurcation diagram* Σ *of the momentum mapping* $\mathcal{F}$ *in the problem of motion of two vortices in a Bose–Einstein condensate consists of curves* Π_1 *and* Π_2 *which have the following explicit parametric representation*