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Bifurcation Analysis of the Dynamics of Two Vortices in a Bose – Einstein Condensate. The Case of Intensities of Opposite Signs

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Abstract—This paper is concerned with a system two point vortices in a Bose – Einstein condensate enclosed in a trap. The Hamiltonian form of equations of motion is presented and its Liouville integrability is shown. A bifurcation diagram is constructed, analysis of bifurcations of Liouville tori is carried out for the case of opposite-signed vortices, and the types of critical motions are identified.

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1. INTRODUCTION

In this paper the focus is on the important problem of analyzing the dynamics of two vortex filaments in a Bose – Einstein condensate enclosed in a trap (see, e. g., [1] and references therein). Vortices and vortex lattices in the Bose – Einstein condensate, for example, on ultracold atoms, are described in the reviews [2–4]. Isolated vortices and systems of two vortices are being actively investigated from both the theoretical [5, 6] and the experimental [7] point of view.

Vortices in a condensate can mostly be described as solutions of the Gross – Pitaevskii equation, a three-dimensional partial differential equation. In this case, the structure of the flow of a superfluid (condensate) in a neighborhood of a vortex filament can be defined most explicitly. The major drawback of this approach is the absence of analytical solutions of the Gross – Pitaevskii equation with necessary boundary conditions and, as a consequence, we can explore the dynamics of the condensate in general, and vortices in particular, only by numerically solving the partial differential equation.

In many interesting cases, vortex filaments are rectilinear and parallel to each other, and the problem becomes essentially two-dimensional. In this situation, an alternative approach is to describe the dynamics of vortices in a condensate using systems of ordinary differential equations in the coordinates of vortex filaments. This approach is well known in classical hydrodynamics [8], where the problem of the dynamics of vortex filaments in an ideal fluid is studied. The simplest properties of the motion vortex filaments, vortex pairs and polygonal configurations of vortices in

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an infinite ideal fluid have been known since the original publication of the works of Kirchhoff [9] and Helmholtz [10]. Greenhill [11] was the first to consider the motion of point vortices inside a circle. Recent studies continue to investigate the classical problems and to generalize them to the case of rigid bodies interacting with vortices (see, e. g., [13–16]). In this case, integrable Hamiltonian systems or more general chaotic dynamical systems arise which can be explored by applying the well-developed topological and qualitative methods of analysis [17–19].

In this paper, we address the problem of the motion of two rectilinear vortex filaments in a Bose – Einstein condensate enclosed in a cylindrical trap. The filaments are parallel to the generatrix of the circular cylinder, and so it is obvious that the problem is two-dimensional. The main goal of this paper is to analyze the phase topology of this problem.

In this paper, a qualitative analysis of the dynamics of two vortices in a Bose – Einstein condensate is carried out using the techniques of bifurcation complexes as presented in [18]. Various types of bifurcation complexes are constructed in [20–22].

2. MODEL

Following [1], we consider rectilinear vortex filaments in a Bose – Einstein condensate with cylindrical symmetry. Physically, such a situation arises mainly when the condensate is retained in a harmonic trap. However, the general form of the equations remains unchanged in the case of a trap that has the shape of a circular cylinder with rigid walls. Throughout this paper, we will consider a two-dimensional problem in a plane perpendicular to the the generatrices of the cylinder and to the vortex filaments. In this case, it is common to speak of the motion of point vortices in some region of the plane, in our case inside a circle.

The motion of the vortex filament arises as a result of interaction with other vortices. The angular velocity of a homogeneous fluid at distance r from the vortex is defined by the expression

$$\Phi(r) = \frac{\hbar}{mr^2},$$

where $\hbar$ is the Planck constant and m is the atomic mass. The velocity of the flow generated by the first vortex at the point where the second vortex is located determines the velocity of motion of the second vortex. In the same way, the velocity of motion of the first vortex is determined by the flow from the second vortex. If the vortices have the same direction of rotation, then the system of two such vortices rotates relative to the common center. In the case where the jth and kth vortices have opposite directions of rotation (i. e., are of opposite sign), then the vortex pair moves in the direction of the fluid flow between the vortices with velocity

$$v_{jk} = r_{jk}\Phi(r_{jk}), \tag{2.1}$$

where r_{jk} is the distance between the vortex filaments.

The velocity of motion of the kth vortex is formed from the gyroscopic precession of the condensate relative to its center O at distance r_k. The angular frequency of precession is given by the formula

$$\Omega(r_k) = \frac{\Omega_0}{1 - r_k^2/R^2}, \tag{2.2}$$

where R is the radius of the cylinder, and

$$\Omega_0 = \frac{\hbar}{mR^2}.$$

We obtain equations of motion of the system of two vortices.