SEPARATION OF VARIABLES FOR INTEGRABLE CASE OF THE KOWALEVSKI TOP IN A NON-EUCLIDEAN SPACE

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Original Papers

- Kowalevski S. Sur le problème de la rotation d'un corps solide autour d'un point fixe // Acta Math., 1889, vol. 12, p. 177–232.
- Borisov A. V., Mamaev I. S. *Rigid Body Dynamics in Non-Euclidean Spaces* // Russian Journal of Mathematical Physics, 2016, vol. 23, no. 4, pp. 431–454.

Pseudospherical Euler-Poisson equation

The integrable Kowalevski case in a non-Euclidean space is a pseudospherical Euler-Poisson equation [2]

$$\dot{\boldsymbol{m}} = (g\boldsymbol{m}) \times \frac{\partial H}{\partial \boldsymbol{m}} + (g\boldsymbol{\gamma}) \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = (g\boldsymbol{\gamma}) \times \frac{\partial H}{\partial \boldsymbol{m}};$$

- $\boldsymbol{m} = (m_1, m_2, m_3)$ is an angular momentum;
- $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is unit vector of the axis of symmetry;
- $g = \mathrm{diag}(1, 1, -k^2)$ is a diagonal matrix;
- k^2 is a non-Euclidean parameter.

Hamiltonian

$$H = \frac{1}{2}(m_1^2 + m_2^2 - 2k^2 m_3^2) - b_1\gamma_1$$

Poisson structure

$$\Pi_r = \begin{pmatrix} 0 & k^2 m_3 & m_2 & 0 & k^2\gamma_3 & \gamma_2 \\ -k^2 m_3 & 0 & -m_1 & -k^2\gamma_3 & 0 & -\gamma_1 \\ -m_2 & m_1 & 0 & -\gamma_2 & \gamma_1 & 0 \\ 0 & k^2\gamma_3 & \gamma_2 & 0 & 0 & 0 \\ -k^2\gamma_3 & 0 & -\gamma_1 & 0 & 0 & 0 \\ -\gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Casimir functions

- $C = \gamma_1^2 + \gamma_2^2 - k^2\gamma_3^2$;
- $L = m_1\gamma_1 + m_2\gamma_2 - k^2 m_3\gamma_3$.

Additional integral

An additional integral F coincides with Kowalevski classic integral [1]

$$F = \left(\frac{m_1^2 - m_2^2}{2} + b_1\gamma_1\right)^2 + (m_1 m_2 + b_1\gamma_2)^2.$$

Variables of Kowalevski

With Kowalevski [1] we introduce variables

$$x_1 = m_1 + \mathrm{i}\, m_2, \qquad x_2 = m_1 - \mathrm{i}\, m_2$$

and polynomials

$$R_i = k^2\Big[-\frac{1}{4}x_i^4 + h x_i^2 + 2\ell b_1 x_i + c b_1^2 - f\Big],$$

$$R = -\frac{1}{2}\Big[R_1 + R_2 + \frac{1}{4}k^2(x_1^2 - x_2^2)^2\Big].$$

Here c, ℓ, h, f are constants of Casimir functions C, L, Hamiltonian H and additional integral F.

The variables of separation

The variables of separation define by formulas

$$s_1 = \frac{R - \sqrt{R_1 R_2}}{(x_1 - x_2)^2}, \qquad s_2 = \frac{R + \sqrt{R_1 R_2}}{(x_1 - x_2)^2}.$$

The basic equation

$$S(x_1, x_2, s) \equiv (x_1 - x_2)^2 s^2 - 2Rs - G = 0,$$

here

$$G = \tfrac{R_1R_2 - R^2}{(x_1 - x_2)^2} = -\tfrac{1}{4}k^4(cb_1^2 - f + 2\ell b_1 x_1 + h^2)(x_1 + x_2)^2 -$$

$$-\tfrac{1}{4}k^4(4h\ell b_1 - 2x_1^2 \ell b_1)(x_1 + x_2) - k^4 \ell^2 b_1^2.$$

The variables of separation are commute: $\{s_1, s_2\} = 0$!!!

The useful expressions

$$\left(\frac{\partial S}{\partial s}\right)^2 = 4R_1R_2, \qquad \left(\frac{\partial S}{\partial x_1}\right)^2 = 8\varphi(s)R_2, \qquad \left(\frac{\partial S}{\partial x_2}\right)^2 = 8\varphi(s)R_1.$$

Here

$\varphi(s)$ – the resolvent of Euler for the polynomial $R(x)$

$$\varphi(s) = s^3 + hk^2s^2 + \tfrac{1}{8}k^4(2cb_1^2 + 2h^2 - 2f)s + \tfrac{1}{8}k^6\ell^2b_1^2,$$

$$R(x) = x^4 - 4hx^2 - 8\ell b_1 x - 4cb_1^2 + 4f.$$

Then

$$\begin{cases} \dfrac{\dot{x}_1}{\sqrt{R_1}} - \dfrac{\dot{x}_2}{\sqrt{R_2}} = \dfrac{\dot{s}_1}{\sqrt{2\varphi(s_1)}}, \\[2ex] \dfrac{\dot{x}_1}{\sqrt{R_1}} + \dfrac{\dot{x}_2}{\sqrt{R_2}} = \dfrac{\dot{s}_2}{\sqrt{2\varphi(s_2)}}. \end{cases}$$

The Identity

$$\left(\frac{\dot{x}_1}{\sqrt{R_1}} \mp \frac{\dot{x}_2}{\sqrt{R_2}}\right)^2 = \left(1 \mp \frac{\dot{x}_1\dot{x}_2}{\sqrt{R_1R_2}}\right)^2 - \frac{(R_1 - \dot{x}_1^2)(R_2 - \dot{x}_2^2)}{R_1R_2} \equiv f_{1,2}.$$

Then

The basic relations

$$\dot{x}_{1,2}^2 - R_{1,2} - \tfrac{1}{2}k^2(x_1 - x_2)^2\xi_{1,2} = 0,$$

$$\dot{x}_1\dot{x}_2 - R - \tfrac{1}{2}k^2h(x_1 - x_2)^2 = 0,$$

$$\xi_{1,2} = \tfrac{1}{2}(m_1^2 - m_2^2) + b_1\gamma_1 \pm \mathrm{i}[m_1m_2 + b_1\gamma_2],$$

$$F \equiv \xi_1 \cdot \xi_2 = f,$$

$$R = \tfrac{1}{2}(s_1 + s_2)(x_1 - x_2)^2, \qquad \sqrt{R_1R_2} = \tfrac{1}{2}(s_2 - s_1)(x_1 - x_2)^2$$

The Substitution

$$\left(\frac{\dot{x}_1}{\sqrt{R_1}} \mp \frac{\dot{x}_2}{\sqrt{R_2}}\right)^2 = \frac{P(s_{1,2})}{(s_2 - s_1)^2},$$

$$P(s) = (2s + k^2 h)^2 - k^4 f.$$

The system of separation equations for integrable Kowalevski case in a non-Euclidean space

$$\begin{cases} (s_1 - s_2)^2 \dot{s}_1^2 = 2\varphi(s_1) P(s_1), \\ (s_1 - s_2)^2 \dot{s}_2^2 = 2\varphi(s_2) P(s_2). \end{cases}$$

Here $\varphi(s)$ and $P(s)$ are the polynomials

$$\varphi(s) = s^3 + k^2 h s^2 + \frac{1}{8} k^4 (2cb_1^2 + 2h^2 - 2f)s + \frac{1}{8} k^6 \ell^2 b_1^2,$$
$$P(s) = (2s + k^2 h)^2 - k^4 f.$$

Thus integration reduce to the hyperelliptic quadrature.

The plan of the further research

- The discriminant set and the bifurcation diagram;
- Parametrization of integral manifolds: Algebraic solution and real separation of variables;
- Phase topology of the integrable Kowalevski case in a non-Euclidean space.

Thank you very much
for your attention!