ISSN 0001-4346, Mathematical Notes, 2016, Vol. 99, No. 6, pp. 834–839. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © A. V. Borisov, P. E. Ryabov, S. V. Sokolov, 2016, published in Matematicheskie Zametki, 2016, Vol. 99, No. 6, pp. 848–854.

Bifurcation Analysis of the Motion of a Cylinder and a Point Vortex in an Ideal Fluid

A. V. Borisov[1,2*], **P. E. Ryabov**[3,4**], **and S. V. Sokolov**[3,4***]

[1] *Udmurt State University, Izhevsk, Russia*
[2] *Kalashnikov Izhevsk State Technical University, Izhevsk, Russia*
[3] *Moscow Institute of Physics and Technology (State University), Dolgoprudny, Moscow oblast, Russia*
[4] *Blagonravov Institute for Machine Science, Russian Academy of Sciences, Moscow, Russia*

Received December 25, 2015

Abstract—We consider an integrable Hamiltonian system describing the motion of a circular cylinder and a vortex filament in an ideal fluid. We construct bifurcation diagrams and bifurcation complexes for the case in which the integral manifold is compact and for various topological structures of the symplectic leaf. The types of motions corresponding to the bifurcation curves and their stability are discussed.

DOI: 10.1134/S0001434616050217

Keywords: *Hamiltonian system, integrability, bifurcation complex.*

We consider the problem of the motion of a circular cylinder and a vortex filament parallel to the generator of the cylinder in an infinite volume of an ideal incompressible fluid. The fluid is assumed to be in permanent circulation motion around the cylinder. This system can be reduced to a system with two degrees of freedom and is Liouville integrable [1]. It is a key problem in the theory of interaction of a rigid body with point vortices in the 2D hydrodynamics of an ideal incompressible fluid, by analogy with the Euler case in rigid body dynamics, the Gröbli–Kirchhoff problem on three vortices, and the Kepler problem in celestial mechanics. This case is distinguished in that the integrability is lost once one tries to consider a more complicated system. As examples, the studies concerning the motion of a circular cylinder and two vortices [2], one vortex and a cylinder of arbitrary shape [3], and a circular cylinder and a vortex in the field of gravity [4] show that the corresponding systems have chaotic properties and are nonintegrable. The system considered in the present paper has the specific feature that the phase manifold is noncompact in the general case, which necessitates a modification of known methods of topological analysis [5]. A generalization of the system considered in [1] to the case of a sphere can be found in [6].

1. EQUATIONS OF MOTION. FIRST INTEGRALS

The equations of motion of a cylinder and a point vortex can be represented in Hamiltonian form [1]

$$\dot{\zeta}_i = \{\zeta_i, H\} = \sum_k \{\zeta_i, \zeta_k\} \frac{\partial H}{\partial \zeta_k}, \tag{1.1}$$

where the ζ_i are the coordinates of the phase vector $\zeta = \{x_1, y_1, v_1, v_2, x_c, y_c\}$, the Hamiltonian has the form

$$H = \frac{1}{2} a v^2 + \frac{1}{2} \lambda_1^2 \ln(r_1^2 - R^2) - \frac{1}{2} \lambda_1 \lambda \ln r_1^2, \tag{1.2}$$

*E-mail: borisov@rcd.ru
**E-mail: orelryabov@mail.ru
***E-mail: sokolovsv72@mail.ru

and the nonzero components of the Poisson structure are given by

$$\begin{aligned} &\{v_1, x_1\} = \frac{1}{a}\frac{r_1^4 - R^2(x_1^2 - y_1^2)}{r_1^4}, && \{v_1, y_1\} = \{v_2, x_1\} = -\frac{1}{a}\frac{2R^2 x_1 y_1}{r_1^4}, \\ &\{v_2, y_1\} = \frac{1}{a}\frac{r_1^4 + R^2(x_1^2 - y_1^2)}{r_1^4}, && \{v_1, v_2\} = \frac{\lambda}{a^2} - \frac{\lambda_1}{a^2}\frac{r_1^4 - R^4}{r_1^4}, \\ &\{x_1, y_1\} = -\frac{1}{\lambda_1}, && \{x_c, v_1\} = \{y_c, v_2\} = \frac{1}{a}. \end{aligned} \tag{1.3}$$

Here $\mathbf{r}_c = (x_c, y_c)$ is the position vector of the center of the cylinder in the fixed coordinate system Oxy; $\mathbf{v} = (v_1, v_2)$ is the cylinder velocity; $\mathbf{r}_1 = (x_1, y_1)$ is the vector from the center of the cylinder to the vortex; R is the cylinder radius; a is the cylinder mass (including the adjoined mass); $\lambda = \Gamma/(2\pi)$ and $\lambda_1 = \Gamma_1/(2\pi)$, where Γ and Γ_1 are the fluid circulation around the cylinder and the vortex intensity. The fluid density is taken to be 2π.

Note that the nonlinear Poisson structure (1.3) is everywhere nondegenerate in its domain. (In other words, its rank is equal to the dimension of the phase space.) In contrast to ordinary mechanical systems, it cannot be obtained from the Lagrangian formalism, and its existence is, generally speaking, not obvious (but typical of various systems of hydrodynamic type, as was noted earlier by Kirchhoff). Let us also mention the recent paper [7], where new examples of nontrivial Poisson structures of hydrodynamic origin are indicated.

System (1.1) is invariant with respect to the group $E(2)$ of motions of the plane, but the action of this group is non-Poissonian (non-Hamiltonian). As a result, the commutation formulas for the three independent first integrals

$$\begin{gathered} Q = av_2 + \lambda x_c - \lambda_1 x_1\left(\frac{R^2}{\mathbf{r}_1^2} - 1\right), \qquad P = av_1 - \lambda y_c + \lambda_1 y_1\left(\frac{R^2}{\mathbf{r}_1^2} - 1\right), \\ K = a(v_1 y_c - v_2 x_c) - \frac{1}{2}\lambda \mathbf{r}_c^2 - \frac{1}{2}\lambda_1 \mathbf{r}_1^2 + \frac{1}{2}\left(\frac{R^2}{\mathbf{r}_1^2} - 1\right)(\mathbf{r}_1, \mathbf{r}_c) \end{gathered} \tag{1.4}$$

generated by this action are nonstandard,

$$\{Q, P\} = \lambda, \qquad \{K, Q\} = P, \qquad \{K, P\} = -Q.$$

The integrals Q and P correspond to translational invariance, and K corresponds to rotational invariance. In principle, one can use these integrals to reduce the system (decrease its order) by two degrees of freedom, that is, find closed-form quadratures for the description of the motion. This reduction, for which the general considerations go back to Lie himself, and a modification to the case of non-Poissonian actions can be found in [8], is given in constructive form in [1] and [2]. Note that the paper [2] also discusses whether it is possible to apply Dirac reduction when restricting the bracket (1.3) to the non-Poissonian manifold given by the equations

$$P = Q = 0.$$

We do not use closed-form quadratures in the subsequent analysis, and so we restrict ourselves to an elementary reduction of system (1.2), (1.3) to two degrees of freedom.

To this end, we indicate the integral (see [1])

$$F = a\mathbf{v}^2 + \lambda_1\left[2a\left(1 - \frac{R^2}{\mathbf{r}_1^2}\right)(x_1 v_2 - y_1 v_1) + (\lambda_1 - \lambda)\mathbf{r}_1^2 + \lambda_1\frac{R^4}{\mathbf{r}_1^2}\right], \tag{1.5}$$

which can be functionally expressed via P, Q, and K,

$$F = 2\lambda K + P^2 + Q^2 + 2R^2\lambda_1^2,$$

but does not contain the coordinates (x_c, y_c) of the center of the cylinder. As a result, we have a completely Liouville integrable reduced system for $\xi = \{x_1, y_1, v_1, v_2\}$ with Hamiltonian (1.2), with Poisson structure obtained from (1.3) by eliminating the commutation relations for x_c and y_c, and with the additional integral (1.5), which is in involution with the Hamiltonian. This situation is similar to the