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Topological Atlas of the Kowalevski – Sokolov Top

Mikhail P. Kharlamov[1*], Pavel E. Ryabov[2], and Alexander Yu. Savushkin[1***]**

[1]*Russian Academy of National Economy and Public Administration*
ul. Gagarina 8, Volgograd, 400131 Russia

[2]*Financial University under the Government of Russian Federation*
Leningradskiy pr. 49, Moscow, 125993 Russia

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Abstract—We investigate the phase topology of the integrable Hamiltonian system on $e(3)$ found by V. V. Sokolov (2001) and generalizing the Kowalevski case. This generalization contains, along with a homogeneous potential force field, gyroscopic forces depending on the configurational variables. The relative equilibria are classified, their type is calculated and the character of stability is defined. The Smale diagrams of the case are found and the isoenergy manifolds of the reduced systems with two degrees of freedom are classified. The set of critical points of the momentum map is represented as a union of critical subsystems; each critical subsystem is a one-parameter family of almost Hamiltonian systems with one degree of freedom. For all critical points we explicitly calculate the characteristic values defining their type. We obtain the equations of the diagram of the momentum map and give a classification of isoenergy and isomomentum diagrams equipped with the description of regular integral manifolds and their bifurcations. We construct the Smale – Fomenko diagrams which, when considered in the enhanced space of the energy-momentum constants and the essential physical parameters, separate 25 different types of topological invariants called the *Fomenko graphs*. We find all marked loop molecules of rank 0 nondegenerate critical points and of rank 1 degenerate periodic trajectories. Analyzing the cross-sections of the isointegral equipped diagrams, we get a complete list of the Fomenko graphs. The marks on them producing the exact topological invariants of Fomenko – Zieschang can be found from previous investigations of two partial cases with some additions obtained from the loop molecules or by a straightforward calculation using the separation of variables.

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1. PRELIMINARIES

The coalgebra $\mathfrak{g}_0 = e(3)^*$ can be realized as $\mathbb{R}^6(\mathbf{M}, \boldsymbol{\alpha})$ with the Poisson bracket

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \alpha_j\} = \varepsilon_{ijk} \alpha_k, \quad \{\alpha_i, \alpha_j\} = 0. \tag{1.1}$$

For a given function H of $\mathbf{M}, \boldsymbol{\alpha}$ (called the *Hamiltonian*), the related Hamilton equations

$$\dot{x} = \{H, x\} \tag{1.2}$$

written in the variables M_i, α_j are called the *Euler – Poisson equations*. We note that the nonstandard arguments order in the bracket appearing on the right-hand side of (1.2) must coincide with the classical analogs.

*E-mail: mikeh@inbox.ru
**E-mail: peryabov@fa.ru
***E-mail: a_savushkin@inbox.ru

The bracket (1.1) has two Casimir functions

$$L = \frac{1}{2}\mathbf{M} \cdot \boldsymbol{\alpha}, \qquad \Gamma = \boldsymbol{\alpha}^2. \tag{1.3}$$

The dot here stands for the scalar product in $\mathbb{R}^3$, the coefficient in L is traditionally introduced for the problems of rigid body dynamics dealing with Kowalevski type configurations.

On any common level

$$\mathcal{P}^4_{a,\ell} = \{L = \ell, \Gamma = a^2\}$$

the induced Poisson bracket is nondegenerate and the restriction of system (1.2) becomes a Hamiltonian system with two degrees of freedom. However, it is sometimes convenient to take for the phase space of (1.2) the 5-dimensional manifold $\mathcal{P}^5 = \mathbb{R}^3(\mathbf{M}) \times S^2(\boldsymbol{\alpha})$ given by one equation

$$\boldsymbol{\alpha}^2 = a^2 \qquad (a > 0).$$

This relation in mechanics is called the *geometric integral* and the sphere defined by it is called the *Poisson sphere*. Both the function L and the generated integral relation $L = \ell$ are called the *area integral*. In this approach, fixing $a > 0$, we consider a one-parameter family in $\mathcal{P}^5$ of the systems on $\mathcal{P}^4_{a,\ell}$ with the parameter $\ell \in \mathbb{R}$. From now on we omit a in the notation of $\mathcal{P}^4_{a,\ell}$ and denote this manifold by $\mathcal{P}^4_\ell$. It is known that for all $a > 0$ it is diffeomorphic to the tangent bundle of the 2-sphere.

In [1], a simultaneous generalization was found of the integrable Kowalevski gyrostat [2] and the integrable Sokolov system for the Kirchhoff equations [3]. Both of these problems are described by systems of type (1.1), (1.2). The case discovered in [1] is naturally called the *Kowalevski – Sokolov gyrostat*. In this article we consider the problem with the following Hamiltonian

$$H = \frac{1}{4}(M_1^2 + M_2^2 + 2M_3^2) + \varepsilon_1(\alpha_3 M_2 - \alpha_2 M_3) - \varepsilon_0 \alpha_1. \tag{1.4}$$

In comparison with the general Hamiltonian of [1], the linear term of type $c\,M_3$ is missing here. This term is characteristic of the problems of gyrostat motion; the constant c is then known as the gyrostatic momentum. So we call the system defined by the Hamiltonian (1.4) the *integrable Kowalevski – Sokolov top*. The problem is defined by three parameters which can be considered as physical ones. These are a, ε_0, and ε_1. Let us note that in the generic case $a\varepsilon_0\varepsilon_1 \neq 0$ this triple is redundant. By introducing the appropriate measurement units we can make two out of three values equal to 1. The exception is the pair $\varepsilon_0, \varepsilon_1$, in which the ratio $\varepsilon_1/\varepsilon_0$ is an essential parameter. Nevertheless, we are going to keep all three parameters to have the possibility of obtaining the known limiting cases as $\varepsilon_1 \to 0$ (the classical Kowalevski case), and as $\varepsilon_0 \to 0$ with an arbitrary $a > 0$ (the Sokolov case for the Kirchhoff equations [3] and, after introducing one more parameter into the Poisson bracket, the case of Borisov – Mamaev – Sokolov on $so(4)$ [4]). Due to the existence of such limits, the Hamiltonian (1.4) is sometimes called a *deformation of the Kowalevski case* (see, e.g., [5], where different generalizations of the Kowalevski problem are discussed from the bi-Hamiltonian point of view). By obvious combinations of reflections in $\mathbb{R}^6$ and time reversal we can obtain the inequalities

$$\varepsilon_0 > 0, \qquad \varepsilon_1 > 0.$$

Indeed, the rotation by π of the moving frame around the third axis changes both signs of $\varepsilon_0, \varepsilon_1$, and the substitution $(M_1, \alpha_2, \alpha_3, t) \to (-M_1, -\alpha_2, -\alpha_3, -t)$ is equivalent to the change of the sign of ε_1.

The first integral additional to Γ, L, and H found in [1] proves the integrability of system (1.2), i.e., the complete Liouville integrability of the family of Hamiltonian systems on $\mathcal{P}^4_\ell$. This integral can be written in the form

$$\begin{aligned} K = {} & \left[\frac{1}{4}(M_1^2 - M_2^2) + \varepsilon_1(\alpha_2 M_3 - \alpha_3 M_2) - \varepsilon_1^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + \varepsilon_0 \alpha_1\right]^2 \\ & + \left[\frac{1}{2}M_1 M_2 + \varepsilon_1(\alpha_3 M_1 - \alpha_1 M_3) + \varepsilon_0 \alpha_2\right]^2. \end{aligned}$$