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MECHANICS

Numerical Simulation of Unsteady Capillary-Gravity Waves

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Abstract—In this work, plane problems of numerical simulation of wave motion are studied. Potential flows of a perfect incompressible fluid are considered. A numerical algorithm for calculating the shape of a free boundary is proposed. The algorithm is based on the boundary element method with the use of quadrature formulas with no saturation. The algorithm is used for studying the breaking of capillary gravity waves and calculating thin cumulative jets. The stability of the scheme and high accuracy in calculations of sharp cumulative jets are achieved due to special control for the distribution of grid points and a decrease in the grid step in the neighborhood of the forward end of the cumulative jet with an ultimately rapid growth of the curvature.

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INTRODUCTION

This work is devoted to numerical simulation of a fluid free surface by use of an algorithm based on the boundary element method. The algorithm involves a nonuniform grid on the boundary contour and approximations with no saturation, which makes it possible to increase considerably the calculation accuracy and to decrease the number of grid points in the calculations. We consider problems related to the evolution of instability and breaking of capillary-gravity waves, as well as an analog of Pokrovskii's experiment demonstrating the efficiency of the numerical algorithm in calculations of thin cumulative fluid jets.

MATHEMATICAL FORMULATION OF THE PROBLEM

The problem under consideration is to calculate the evolution of the wave shape with time at a given velocity field at the initial time instant. Let us introduce a Cartesian coordinate system Oxy where the y axis is directed vertically upward. Let S denote the fluid portion between two vertical straight lines $x = 0$ and $x = 2\pi$, bottom $y = -h$, and the curvilinear fragment of the free boundary L.

The stream function Ψ of the potential flow of a perfect incompressible fluid is periodic in the coordinate x and satisfies the Laplace equation in the domain S:

$$\Psi(x, y, t) = \Psi(x + 2\pi, y, t), \quad \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0. \quad (1)$$

On the bottom $y = -h$, the impermeability condition $\Psi|_{y=-h} = 0$ is satisfied.

This implies the following integral identity [1]:

$$\pi\Psi(M, t) = \left[A\left(-\frac{\partial\Phi}{\partial s}\right) + B\Psi\right](M, t), \quad \frac{\partial\Phi}{\partial s} = -\frac{\partial\Psi}{\partial n}, \quad (2)$$

where A and B are integral operators that are expressed in terms of the Green function G:

$$[AF](M) := -\int_L G(M, M')F(M')ds', \quad (3)$$

$$[BF](M) := \int_L \frac{\partial G}{\partial n'}(M, M')(F(M') - F(M))ds', \quad (4)$$

$$G(M, M') = \frac{1}{2}\ln\left[\frac{2(\cosh(y' - y) - \cos(x' - x))}{1 - 2E\cos(x' - x) + E^2}\right], \quad (5)$$

$$E = e^{-2h-y-y'}, \quad M, M' \in L. \quad (6)$$

Equation (2) allows one to calculate the values of Ψ at the free boundary L by known values of the potential Φ.

As the boundary is displaced, we require fulfillment of the kinematic condition: the projection of the boundary velocity to the normal must coincide with the normal velocity of the fluid at the boundary. The tangential velocity of the boundary has no effect on

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the shape; therefore, it can differ from that of the fluid at the boundary. We choose it from the condition according to which the proportions of distances between neighboring points of the grid must be preserved. The fulfillment of this condition provides stability of the numerical scheme.

From the condition of zero pressure on the wave surface we find the change in the potential on the free surface:

$$\left.\frac{\partial\Phi}{\partial t}\right|_{x,y}+\frac{(\mathbf{v},\mathbf{v})}{2}+gy-\sigma k=0, \qquad (7)$$

where the liquid density ρ is assumed to be the unit, g characterizes the force of gravity, σ is the surface tension coefficient, and k is the curvature.

Equations (2–7) form a system sufficient for constructing the numerical algorithm of calculating the wave motion in a fluid.

NUMERICAL ALGORITHM

The algorithm was constructed using our work [2] in which the evolution of cylindrical cavities in a plane-parallel potential flow of a perfect incompressible fluid was calculated. Integration and differentiation involve approximations [4, 5] the order of which unboundedly increases with an increase in the number of grid points. In this case, approximations are said to be with no saturation [3]. As in the case of the algorithm for calculating the shape of cylindrical cavities, a smooth parameterization of the boundary by use of an auxiliary parameter $\zeta\in[0,1]$ is introduced and a uniform grid $\zeta_i=i/N$, where $i=1,2,\dots,N$, is specified. Approximation of the operator A has the form

$$\left[A\left(-\frac{\partial\Phi}{\partial s}\right)\right](\zeta_i,t)\approx\frac{1}{N}\sum_{j=1}^{N}(\beta(|i-j|)+G_{ij})\frac{\partial\Phi}{\partial\zeta}(\zeta_j,t),$$

$$\beta(0)=\alpha(0),$$

$$\beta(m)=-\ln\left|\sin\frac{\pi m}{N}\right|+\alpha(m),$$

$$\alpha(m)=-\left(\ln 2+\frac{(-1)^m}{N}+\sum_{k=1}^{\frac{N}{2}-1}\frac{1}{k}\cos\frac{2\pi km}{N}\right),$$

$$G_{ij}=G(\zeta_i,\zeta_j)\quad i\neq j,$$

$$G_{ii}=\lim_{\zeta\to\zeta_i}(G(\zeta_i,\zeta)-\ln|\sin(\pi(\zeta_i-\zeta))|).$$

Approximation of the operator B is defined by the formula

$$[B\Psi](\zeta_i,t)\approx\frac{1}{N}\sum_{j=1}^{N}G_{ij}^n(\Psi(\zeta_j,t)-\Psi(\zeta_i,t)),$$

$$G_{ij}^n=\frac{\partial G}{\partial x'}(\zeta_i,\zeta_j)\frac{\partial y}{\partial\zeta}(\zeta_j,t)$$

$$-\frac{\partial G}{\partial y'}(\zeta_i,\zeta_j)\frac{\partial x}{\partial\zeta}(\zeta_j,t)\quad i\neq j,\quad G_{ii}^n=0.$$

As in [2], the free boundary L was parameterized using the distribution function of grid points $f(\zeta)$ by the formula

$$ds=l(t)f(\zeta)d\zeta,\quad 0\le\zeta\le 1,\quad \int_0^1 f(\zeta)d\zeta=1,$$

where $l(t)$ is the length of one period of the wave L at the time instant t.

The normal velocity of the boundary, changes in the spatial coordinates x and y of the wave surface, and the values of the potential Φ at the boundary are calculated similarly to the algorithm from [2].

CALCULATION RESULTS

We begin the description of the obtained numerical results with simulation of the wave breaking under the action of the force of gravity. A survey of works on this subject was presented in [6]. For comparison of the results, we use [7], in which a numerical algorithm for solving the problem under consideration was also constructed.

For the initial shape for numerical experiments with breaking, a progressive wave for an infinitely deep fluid ($h=-\infty$, $\sigma=0$) was used. The wave shape was found by the algorithm [8] from the extremum condition for the functional $E_c-E_g-E_\sigma$, where

$$E_c=\frac{c^2}{2}\int_0^l\bar{\Phi}\frac{\partial\bar{\Phi}}{\partial n}ds,$$

$$E_g=\frac{g}{2}\int_0^l y^2\frac{\partial x}{\partial s}ds,\quad E_\sigma=\sigma l,$$

c is the wave speed, and $c\bar{\Phi}$ is the velocity field potential.

In [7], a progressive wave with amplitude $a=0.406$ (approximately 90% of the limit amplitude) was considered. The dependence of pressure on time was specified by the formula $p_s=p_0\sin t\sin(x-ct)$ at $0\le t\le\pi$, where $p_0=0.146$, and additional pressure was absent for all $t\ge\pi$, i.e., $p_s=0$. Then, the process of wave breaking was studied. The algorithm from [7] turned out to be unstable: the free surface acquired a saw-tooth shape. To suppress the instability, the authors of [7] smoothed the boundary by the formula

$$\bar{f}_j=(-f_{j-2}+4f_{j-1}+10f_j+4f_{j+1}-f_{j+2})/16.$$

This artificially decreased the boundary curvature and resulted in the fact that only the initial stage of

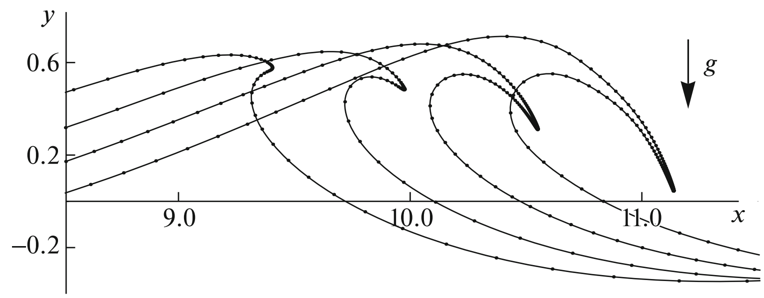

Fig. 1. Breaking of a gravity wave ($N = 192$).

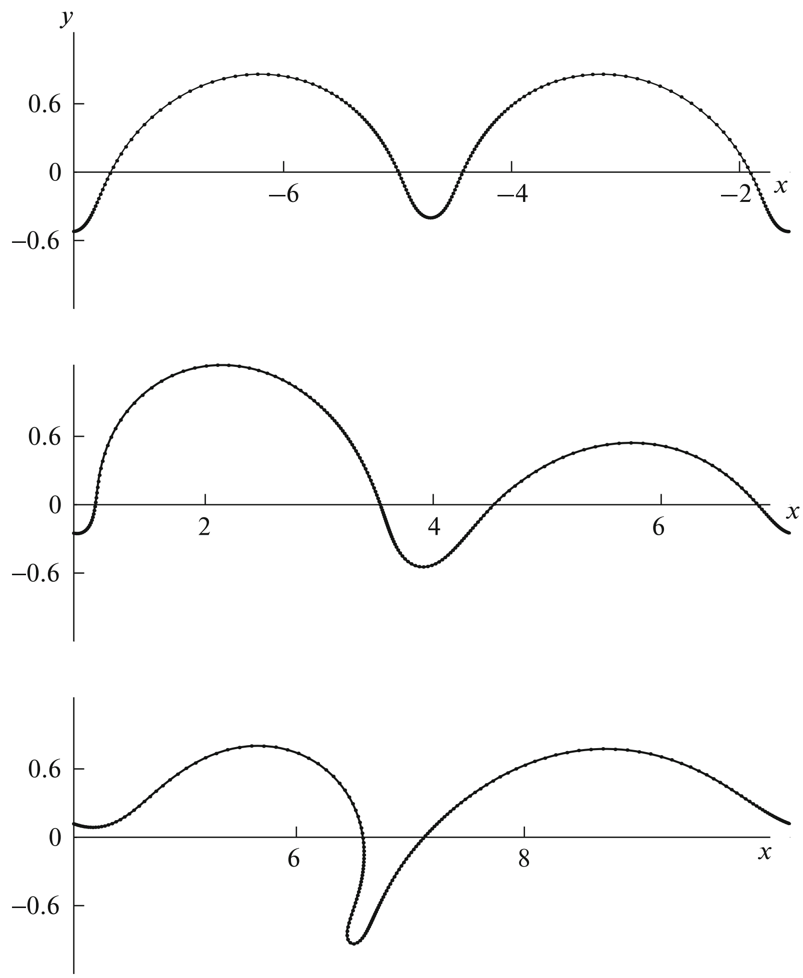

Fig. 2. Deformation of a perturbed Crapper wave ($N = 192$).

breaking up to the time instant $t = 4.928$ could be calculated.

The algorithm proposed in this paper was also applied to solving the problem from [7]. The advantages of the new algorithm are clearly demonstrated in Fig. 1. The figure depicts four wave profiles at time instants $t = 4.8$, 5.1, 5.4, and 5.7. The first two of them were obtained with the same number of grid points $N = 60$ as in [7]. They correspond to a considerably later stage of breaking. The points were condensed in a neighborhood of the forming cumulative jet. The wave shapes for next time instants were obtained by an

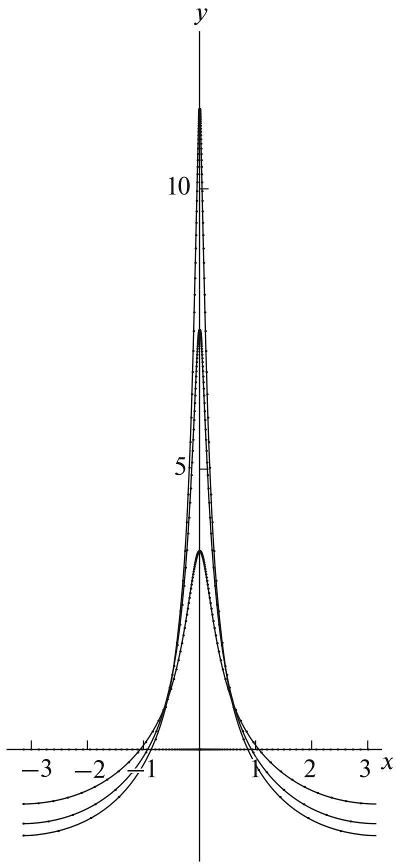

Fig. 3. Analog of Pokrovskii's experiment ($N = 128$).

increase in the number of grid points to $N = 192$. The last wave profile at $t = 5.7$ is presented with the use of smoothing.

The numerical algorithm developed was used for calculating the evolution of instability of a Crapper perturbed capillary wave. The parametric equations of the Crapper wave have the following form [9]:

$$x(\alpha) = -\frac{\lambda}{2\pi}\left(\alpha + \frac{4b\sin\alpha}{1 - 2b\cos\alpha + b^2}\right),$$
$$y(\alpha) = \frac{2\lambda}{\pi}\frac{b\cos\alpha - b^2}{1 - 2b\cos\alpha + b^2},$$

where λ is the wavelength and b is a parameter related to the wave amplitude a and dimensionless speed c:

$$a = 4b\lambda/(2\pi(1 - b^2)), \quad c^2 = (1 - b^2)/(1 + b^2).$$

The propagation rate of the wave is determined in terms of c by the formula $u^2 = c^2 2\pi\sigma/(\rho\lambda)$.

In [10], Lyapunov's stability of the exact Crapper solution with respect to perturbations with a period equal to the wavelength was proved. For perturbations the period of which exceeds the wavelength, stability is not preserved. The numerical experiments involved perturbations of the form $\delta y = \varepsilon \sin x$ at the Crapper wave length $\lambda = \pi$. The calculations demonstrate that the perturbations cause finite-amplitude oscillations of the free surface at small values of the parameter ε and lead to wave breaking at sufficiently large values of the parameter b. Figure 2 depicts the shape evolution of one period of a perturbed wave at $b = 0.3$, $\sigma = 0.05$, and $\varepsilon = 0.03$. The wave profiles are in correspondence with time instants $t = 0$, 30.0, and 40.5.

Let us consider an analog of Pokrovskii's experiment as a very simple demonstration of the cumulative jet formation on a free surface of a fluid. In the original experiment described in [11] (pp. 253–254), a test tube filled with water falls from a height of 10 cm and, after the impact on the horizontal plane, a thin water jet with a height exceeding 1 m water is pushed out of the test tube.

According to [11], at the instant of the impact, the edge of the free surface of the fluid acquires a finite velocity directed downwards; the central part, a velocity directed upwards. For the sake of simplicity of the numerical simulation, we assume that the fluid at the initial time instant fills a half-strip $-\pi < x < \pi$, $y < 0$; the normal velocity at the boundary $y = 0$ varies by the law $v_y = \cos x$. This corresponds to the initial fluid velocity potential $\Phi = e^y \cos x$. The force of gravity was assumed to be zero, which admitted unbounded expansion of the jet. The free jet shapes calculated on a grid at $N = 128$ for time instants $t = 2.05$, 4.1, and 6.15 are depicted in Fig. 3 and demonstrate that the scheme is able to calculate superthin jets without loss of stability. The curvature of the forward end of the jet at the time instant $t = 6.15$ reaches the value $k = 1143.5$.

The numerical algorithm with no saturation demonstrates that the curvature of the forward end of the jet can increase to very large values without loss of smoothness of the free boundary. The latest investigations [12, 13] in which semianalytical solutions were constructed for describing cumulative jet formation lead to the same conclusion.

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