Russ. J. Numer. Anal. Math. Modelling, Vol. 27, No. 1, pp. 5–31 (2012)
DOI 10.1515/ rnam-2012-0002

Numerical simulation of the Black Sea hydrothermodynamics taking into account tide-forming forces

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Abstract — A mathematical model of the Black Sea dynamics taking into account tide-forming forces is considered. Based on the variational assimilation of satellite altimetry data, an algorithm solving the inverse problem of the reconstruction of 'self-attraction' forces and potential forces influencing the formation of the mean level is proposed. A method of approximate solution of this problem is presented. The influence of the tide-forming forces on the dynamics of the Black Sea is investigated numerically.

Nowadays, with the development of modern numerical algorithms and computing complexes, the description of tidal motions in seas and oceans still remains a challenge. There are many reasons for that. Here are some of them: the physical processes lying at the base of the theory of tides are complicated and not completely studied yet; there are difficulties in the description of tide-forming forces (specification of phases at each point of the ocean surface, consideration of elastic properties of the Earth, earth tides, and self-attraction effects); mathematical models of tidal motion are, generally speaking, integro-differential, which essentially complicates the numerical solution of the equations of the dynamic theory of tides and motivates the use of simplified models in practical calculations. A way to solve the problems indicated above was proposed in [23] and consists in the following. In order to construct an adequate theory of tidal oscillations in the ocean, it is proposed to use the general equations of its dynamics and add the gradient of the tide-forming forces into the right-hand sides of the motion equations. One of the first papers in this direction was [8]. The mathematical model of the general World Ocean circulation with tide-forming forces defined as the gradient of the total potential of tidal forces was considered in that paper. In addition, based on observation data concerning the ocean level, the inverse problem of the determination of the potential of forces influencing the mean ocean level was formulated and approximately solved

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The work was supported by the Russian Foundation for Basic Research (10–01–00806, 11–01–12046-ofi-m-2011) and the Program of the Presidium of the Russian Academy of Sciences No. 21.1 'The Black Sea as a simulation ocean model'.

in that paper. Concerning the Black Sea, as far as the authors of this paper know, the modern scientific community has no mathematical model of the general circulation of the Black Sea constructed on the base of 'basic' ('primitive') systems of equations with total consideration of tide-forming forces and effects of various potential forces deforming the geoid surface and the mean sea level. This paper deals with the construction of such a mathematical model.

Below we consider the mathematical model of the Black Sea dynamics taking into account the tide-forming forces. We discuss the forms of specification of tide-forming forces and the influence of these forces on the form of the geoid surface or the mean level as the reference surfaces for the sea level deviations. Based on variational assimilation of satellite altimetry data, an algorithm solving the inverse problem of the reconstruction of 'self-attraction' forces and potential forces influencing the formation of the mean level is proposed. A method of approximate solution of this problem is presented. The numerical study of the influence of the forces introduced into the model on the dynamics of the Black Sea is performed, this primarily relates to the 'level surface' in this basin. The numerical implementation of the general sea circulation model used in this paper is based on the numerical model of the Black Sea dynamics developed at the Institute of Numerical Mathematics of the Russian Academy of Sciences (see [30]) and also the methods presented in [8].

1. Mathematical model of the Black Sea dynamics

In this section we formulate the mathematical model of the Black Sea dynamics and describe the numerical mathematics methods used for the approximation of this model. Reviewing these methods, we generally follow [2, 3, 6, 10, 11, 16, 19, 20, 22, 28].

Introduce the basic definitions and equations used below.

1.1. Consider the geographic (geodesic) system of coordinates (λ, θ, z), where λ is the geographic longitude increasing from the west to the east, θ is the geographic latitude increasing from the south to the north, z is the distance of a point from the surface of the terrestrial ellipsoid of revolution (TE), the geographic system of coordinates is related to this ellipsoid. We also assume the so-called 'spherical approximation' widely used in higher geodesy (see [24]). In this connection, the sphere S_R of the radius R is introduced, where R is the 'mean Earth radius' and the center of the sphere coincides with the center of mass of the Earth or with the center of the terrestrial ellipsoid. The longitude λ varies from 0 to 2π, the latitude θ varies from $-\pi/2$ (South Pole) to $\pi/2$ (North Pole). Instead of z, one may introduce the coordinate $r = R - z$ on the axis Or directed along the outer normal to the surface of the sphere S_R of the radius R. The unit vectors in the λ-, θ-, and z-directions are denoted by e_λ, e_θ, and e_z, respectively. The velocity vector in the ocean is written in the form $\mathbf{U} = ue_\lambda + ve_\theta + we_z \equiv (\mathbf{u}, w)$, or in the coordinate form as $\mathbf{U} = (u, v, w) \equiv (\mathbf{u}, w)$, where $\mathbf{u} = (u, v)$ is the 'horizontal velocity vector' in the coordinate form and w is the 'vertical velocity'.

By Ω we denote the part of the surface of the sphere S_R identified with the corresponding part of the geoid, or the 'mean level' surface considered as the reference surfaces for the ocean level deviations. The set Ω is also called the 'reference surface'. The ocean surface is given by the equation $z = \zeta(\lambda, \theta, t)$, or $f_0(\lambda, \theta, z, t) \equiv \zeta(\lambda, \theta, t) - z = 0$, where $(\lambda, \theta, R_3) \in \Omega$, and t is the time variable, $t \in [0, \bar{t}](\bar{t} < \infty)$. The bottom relief function is defined as $z = H(\lambda, \theta)$, or $F_H(\lambda, \theta, z, t) \equiv -H(\lambda, \theta) + z = 0$ for $(\lambda, \theta, R_3) \in \Omega$, where $H(\lambda, \theta) > 0$.

Below we use the following notations:

$$\lambda \equiv x, \quad \theta \equiv y, \quad \mathbf{x} \equiv (x, y, z)$$
$$\mathbf{U} \equiv (u, v, w) \equiv (\mathbf{u}, w), \quad \mathbf{u} \equiv (u_1, u_2) \equiv (u, v), \quad u_3 \equiv w.$$

Let a volume element in the domain $D(t) = \{(x, y, z) : (x, y, R) \in \Omega, \zeta(x, y, t) < z < H(x, y)\}, t \in [0, \bar{t}]$, be given as $\mathrm{d}D = (R - z)^2 \cos y \mathrm{d}x\mathrm{d}y\mathrm{d}z$ and an element of Ω be given as $d\Omega = R^2 \cos y \mathrm{d}x\mathrm{d}y$.

Introduce the following differential operations of the gradient, divergence, and total derivative in a spherical coordinate system for $r \equiv r(z) \equiv R - z \cong R$, $n \equiv 1/r$, $m \equiv 1/(r \cos y)$ (keep the well-known notations from vector analysis for these operations):

$$\mathbf{Grad}\varphi \equiv \left(\mathbf{grad}\varphi, \frac{\partial \varphi}{\partial z}\right), \quad \mathbf{grad}\varphi \equiv \left(m\frac{\partial \varphi}{\partial x}, n\frac{\partial \varphi}{\partial y}\right)$$
$$\mathbf{Div}\mathbf{U} \equiv \mathbf{div}\mathbf{u} + \frac{1}{r^2}\frac{\partial r^2 w}{\partial z}, \quad \mathbf{div}\mathbf{u} \equiv m\frac{\partial u}{\partial x} + m\frac{\partial}{\partial y}\left[\frac{n}{m}v\right]$$
$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial \varphi}{\partial t} + (\mathbf{U}, \nabla\varphi), \quad (\mathbf{U}, \nabla) \equiv (\mathbf{u}, \mathbf{grad}) + w\frac{\partial}{\partial z}$$
$$(\mathbf{u}, \mathbf{grad}) = um\frac{\partial}{\partial x} + vn\frac{\partial}{\partial y}.$$

We also use the following second-order differential operators:

$$A_\varphi \varphi \equiv -\mathbf{Div}(\hat{a}_\varphi \mathbf{Grad}\varphi)$$

where $\hat{a}_\varphi = \mathrm{diag}((a_\varphi)_{ii})$, $(a_\varphi)_{11} = (a_\varphi)_{22} \equiv \mu_\varphi, (a_\varphi)_{33} \equiv \nu_\varphi$, and the index φ can take the values u, v, T, S (i.e., it can denote the components of the horizontal velocity vector, temperature T, and salinity S). We also assume $\mu_u = \mu_v \equiv \mu$, $\nu_u = \nu_v \equiv \nu$ and suppose that μ, ν, μ_T, μ_S, ν_T, ν_S are given positive and bounded functions.

We also consider the fourth-order differential operator $(A_k)^2$, where the second-order operator A_k is introduced above for $A_\varphi = A_k$ and is determined by the matrix $\hat{k} = \mathrm{diag}\{k_{ii}\}$ with nonnegative diagonal elements k_{ii}. Further, by $l = l(y)$ we denote the Coriolis parameter: $l = 2\omega \sin y$, where ω is the angular Earth rotation velocity and $f(u) = l + mu \sin y \equiv l + f_1(u)$.

Note one general simplification used in the description of large-scale ocean dynamics [20]. The level function $\zeta = \zeta(x, y, t)$ is also one of the unknown functions to be determined. Thus, $D(t)$ is a *domain with an unknown boundary* (or a *domain with a moving boundary*). Therefore, after writing down the equations of ocean hydrothermodynamics in the domain $D(t)$ with the corresponding boundary conditions, one passes to some approximate system of equations now considered in the fixed domain $D = \{(x,y,z) : (x,y,R) \in \Omega,\ 0 < z < H(x,y)\}$. We represent the boundary of the domain $\Gamma \equiv \partial D$ as the union of four nonintersecting parts Γ_S, $\Gamma_{w,\mathrm{op}}$, $\Gamma_{w,c}$, Γ_H, where $\Gamma_S \equiv \Omega$ is the 'unperturbed surface', $\Gamma_{w,\mathrm{op}}$ is the liquid (open) part of the vertical lateral boundary, $\Gamma_{w,c}$ is the rigid part of the vertical lateral boundary, Γ_H is the ocean bottom. The characteristic functions of the Γ_S, $\Gamma_{w,\mathrm{op}}$, $\Gamma_{w,c}$, Γ_H-parts of the boundary Γ are denoted by m_S, $m_{w,\mathrm{op}}$, $m_{w,c}$, m_H, respectively.

We also assume that Ω is a connected set on S_R and the boundaries $\partial\Omega$, Γ are piecewise-smooth of the class $C^{(2)}$ and locally satisfying the Lipschitz condition. The unit vector of the outer normal to Γ is denoted by $\mathbf{N} \equiv (N_1, N_2, N_3)$. Note that $\mathbf{N} = (0,0,-1)$ on Γ_S and $\mathbf{N} = (N_1, N_2, 0)$ on $\Gamma_w = \Gamma_{w,\mathrm{op}} \cup \Gamma_{w,c}$, in this case the vector $\mathbf{n} \equiv (N_1, N_2) \equiv (n_1, n_2)$ is the unit vector of the outer normal to $\partial\Omega$. We also assume that $|N_3| > 0$ always on Γ_H. The expression for the components N_1, N_2, N_3 is determined by the used parametric representation of some part of the boundary.

Considering the velocity vector $\mathbf{U} = (u, v, w)$ on the boundary Γ, we denote its normal component by $U_n : U_n = \vec{U} \cdot \mathbf{N} = uN_1 + vN_2 + wN_3$. Let further $U_n^{(+)} \equiv (|U_n| + U_n)/2$, $U_n^{(-)} \equiv (|U_n| - U_n)/2$. Note that $U_n = U_n^{(+)} - U_n^{(-)}$ on Γ.

1.2. Write down the system of hydrothermodynamics equations for the functions u, v, ζ, T, S in the domain D in the variables (x,y,z) in the 'Boussinesq and hydrostatic approximation' [20], but taking the Lame coefficients corresponding to the spherical system of coordinates [2,3,6]:

$$
\begin{gathered}
\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \mathbf{u} - g\,\mathbf{grad}\,\zeta + A_u \mathbf{u} + (A_k)^2 \mathbf{u} \\
= \mathbf{f} - \tfrac{1}{\rho_0}\mathbf{grad} P_a - -\frac{g}{\rho_0}\mathbf{grad} \int_0^z \rho_1(T,S)\,\mathrm{d}z' \\
\frac{\partial \zeta}{\partial t} - m\frac{\partial}{\partial x}\left(\int_0^H \Theta(z) u \,\mathrm{d}z\right) - m\frac{\partial}{\partial y}\left(\int_0^H \Theta(z)\frac{n}{m} v\,\mathrm{d}z\right) = f_3 \\
\frac{\mathrm{d}T}{\mathrm{d}t} + A_T T = f_T, \quad \frac{\mathrm{d}S}{\mathrm{d}t} + A_S S = f_S
\end{gathered}
\tag{1.1}
$$

where $\rho_1(T,S) = \rho_0\beta_T(T - T^{(0)}) + \rho_0\beta_S(S - S^{(0)}) + \gamma\rho_0\beta_{TS}(T,S) + f_P$, $\mathbf{f} = (f_1, f_2)$, f_T, f_S, f_P are given functions of 'internal' sources, $g = \mathrm{const} > 0$, ρ_0, $T^{(0)}$, $S^{(0)}$ are 'unperturbed' values of water density, temperature, and salinity, β_T, β_S are coefficients (assumed to be constant), $\beta_{TS}(T,S)$, P_a, $f_3 \equiv f_3(x,y,\zeta,t) \equiv f_3(x,y,t)$ are given functions, and γ is a numerical parameter. Here and further we use the following weight function:

$$\Theta(z) \equiv \frac{r(z)}{R}.$$

Below we consider the case $\mathbf{f} \equiv g\ \mathbf{grad} G$ with some scalar function $G = G(x,y,t)$, for example, $G \equiv \zeta^+$ is the static tide, which will be described in detail further.

Note that the coordinates (x,y,z) are geodesic in their physical sense, but due to the approximate notation of the Lame coefficients, system (2.1) takes its form in a system of spherical coordinates.

Considering (1.1) in $D\times(0,\bar{t})$, one can pose the following boundary and initial conditions [2,3,6].

The boundary conditions on Γ_S:

$$
\begin{gathered}
\left(\int_0^H \Theta\mathbf{u}dz\right)\mathbf{n}+\beta_0 m_{\mathrm{op}}\sqrt{gH}\,\zeta=m_{\mathrm{op}}\sqrt{gH}\,d_s \quad \text{on } \partial\Omega \\
U_n^{(-)}u-\nu\frac{\partial u}{\partial z}-k_{33}\frac{\partial}{\partial z}A_k u=\tau_x^{(a)}/\rho_0, \quad U_n^{(-)}v-\nu\frac{\partial v}{\partial z}-k_{33}\frac{\partial}{\partial z}A_k v=\tau_y^{(a)}/\rho_0 \\
A_k u=0, \quad A_k v=0 \\
U_n^{(-)}T-\nu_T\frac{\partial T}{\partial z}+\gamma_T(T-T_a)=Q_T+U_n^{(-)}d_T \\
U_n^{(-)}S-\nu_S\frac{\partial S}{\partial z}+\gamma_S(S-S_a)=Q_S+U_n^{(-)}d_S
\end{gathered}
\tag{1.2}
$$

where $\tau_x^{(a)}$ and $\tau_y^{(a)}$ are the components of the tangent wind stress vector along the axes Ox and Oy, respectively, on the surface $z=0$, γ_T, γ_S, T_a, S_a, Q_T, Q_S, d_T, d_S are given functions. In (1.2) we also have $U_n|_{z=0}=-w|_{z=0}$ and $w=w(u,v)$ is introduced according to the formula

$$
w(x,y,z,t)=\frac{1}{r}\left(m\frac{\partial}{\partial x}\left(\int_z^H ru\,dz'\right)+m\frac{\partial}{\partial y}\left(\frac{n}{m}\int_z^H rv\,dz'\right)\right), \quad (x,y,t)\in\Omega\times(0,\bar{t}).
\tag{1.3}
$$

The boundary conditions on $\Gamma_{w,c}$ (*on the 'rigid lateral wall'*):

$$
\begin{gathered}
U_n=0, \quad A_k\tilde{U}=0, \quad \frac{\partial\tilde{U}}{\partial N_u}\cdot\tau_w+(\frac{\partial}{\partial N_u}A_k\tilde{U})\cdot\tau_w=0 \\
\frac{\partial T}{\partial N_T}=0, \quad \frac{\partial S}{\partial N_S}=0
\end{gathered}
\tag{1.4}
$$

where $\tau_w=(-N_2,\,N_1,\,0)$, $\tilde{U}\equiv(u,v,0)\equiv(\mathbf{u},0)$, $\partial\varphi/\partial N_\varphi\equiv\mathbf{N}\cdot\hat{a}_\varphi\cdot\mathbf{Grad}\varphi$, $\varphi=u,T,S$.

The boundary conditions on $\Gamma_{w,\mathrm{op}}$ (*on the 'liquid part of the lateral wall'*):

$$
\begin{gathered}
U_n^{(-)}(\tilde{U}\cdot\mathbf{N})+\frac{\partial\tilde{U}}{\partial N_u}\cdot\mathbf{N}=U_n^{(-)}d, \quad A_k\tilde{U}=0 \\
U_n^{(-)}(\tilde{U}\cdot\tau_w)+\frac{\partial\tilde{U}}{\partial N_u}\cdot\tau_w+\left(\frac{\partial}{\partial N_u}A_k\tilde{U}\right)\cdot\tau_w=0 \\
U_n^{(-)}T+\frac{\partial T}{\partial N_T}=U_n^{(-)}d_T+Q_T, \quad U_n^{(-)}S+\frac{\partial S}{\partial N_S}=U_n^{(-)}d_S+Q_S
\end{gathered}
\tag{1.5}
$$

where d, d_T, d_S, Q_T, Q_S are some functions, which we also assume to be known.

The boundary conditions on Γ_H ('on the bottom'):

$$w = um\frac{\partial H}{\partial x} + vn\frac{\partial H}{\partial y}, \quad A_k\tilde{U} = 0$$
$$\frac{\partial \tilde{U}}{\partial N_u}\cdot\tau_x + \left(\frac{\partial}{\partial N_k}A_k\tilde{U}\right)\cdot\tau_x = \tau_x^{(b)}/\rho_0, \quad \frac{\partial \tilde{U}}{\partial N_u}\cdot\tau_y + \left(\frac{\partial}{\partial N_u}A_k\tilde{U}\right)\cdot\tau_y = \tau_y^{(b)}/\rho_0$$
$$\frac{\partial T}{\partial N_T} = 0, \quad \frac{\partial S}{\partial N_S} = 0 \tag{1.6}$$

where τ_x and τ_y is the system of unit orthogonal vectors on the surface $z = 0$; $\tau_x^{(b)}$ and $\tau_y^{(b)}$ are the projections of the near-bottom friction vector onto the axes Ox, Oy, respectively. Note that the first condition of (1.6) is equivalent to 'U_n= 0 on Γ_H'.

The initial conditions for *u, v, T, S*, ζ:

$$u = u^0, \quad v = v^0, \quad T = T^0, \quad S = S^0, \quad \zeta = \zeta^0, \quad t = 0 \tag{1.7}$$

where u^0, v^0, T^0, S^0, ζ^0 are given functions.

The problem of large-scale sea dynamics is formulated in terms of the functions u, v, ζ, T, S as follows: *determine* u, v, ζ, T, S *satisfying* (1.1)–(1.7).

If the functions u, v, ζ, T, S are determined, then the function w is obtained from formula (1.3) and the function P — from the formula from [2, 3, 6]:

$$P(x,y,z,t) = P_a(x,y,t) + \rho_0 g(z-\zeta) + \int_0^z g\rho_1(T,S)\mathrm{d}z'.$$

Note that the boundary conditions presented above can be modified depending on the particular physical problem.

Note also that the 'diffusive operators' in the equations for u, v do not take into account some differential operators of lower degrees, which are significant near the poles. Therefore, generally speaking, system (2.1) should be considered in a domain with excluded polar points, and we should utilize some additional relations in its numerical approximation. If the polar points are included into the domain D, it is necessary to consider systems of equations of type (2.1) with the 'vector Laplace operator' and use special functional spaces of solutions and the corresponding boundary conditions (see [4, 5]). In this paper, taking into account the specificity of the considered domain D, the authors restrict ourselves to the consideration of system of equations (2.1).

Problem (2.1)–(2.7) (or a similar problem considered for other boundary conditions) is approximated by the method of splitting using the finite difference method for the approximation of the subproblems on all stages of the splitting scheme. Note that in this case the 'σ-system of coordinates' can be used. All these computational methods and the numerical analogues of problem (2.1)–(2.7) are described, e.g., in [6, 7, 10, 22], and we do not present them here.

2. Methods of calculation of tide-forming forces in the mathematical model of the Black Sea dynamics

In this section the main attention is paid to the description of tide-forming forces in the mathematical model of the general dynamics of oceans and seas. Since the model uses the level function ζ (see (2.1)–(2.7)), the notions of the 'geoid' and the 'mean level' naturally appear here as reference surfaces for the determination of the ocean level (see [26]). The presentation of this section is based on the results of [8].

2.1. Geoid and mean sea level

Below we denote the height of the geoid surface over the terrestrial ellipsoid (TE) by $h_{\mathrm{g}} \equiv h_{\mathrm{g}}(x,y)$, where $(x,y) \in \Omega_{(1)}$ is the projection of the unperturbed ocean surface onto the sphere of a unit radius whose center coincides with the center of the TE and the center of the sphere S_R. The height of the mean level ('mean sea level') is denoted by $h_{\mathrm{m.l.}}(x,y)$, and we define its deviation h' as

$$h' = h - \overline{h}, h_{\mathrm{m.l.}}(x,y) \equiv \overline{h} = \frac{1}{T}\int_0^T h(x,y,t)\mathrm{d}t \tag{2.1}$$

where T is the time interval appearing in the definition of $h_{\mathrm{m.l.}}$, $h(x,y,t)$ is the height of the free ocean surface over the TE. Depending on the choice of T, one can consider different mean levels, i.e., daily, monthly, annual, long-term, etc. [12,26]. The operations of averaging '$\overline{h}$' and calculation of the deviations will be also applied to different functions.

The function $h(x,y,t)$ can be represented in the following form:

$$h = h_{\mathrm{g}}(x,y) + \zeta_{\mathrm{g}}(x,y,t) \tag{2.2}$$

or

$$h = h_{\mathrm{m.l.}}(x,y) + \zeta_{\mathrm{m.l.}}(x,y,t). \tag{2.3}$$

In the first case, $\zeta_{\mathrm{g}}(x,y,t)$ is the deviation of the free ocean surface from the geoid surface (along the direction of Or), which is assumed here not to depend on t. In the second case, $\zeta_{\mathrm{m.l.}}(x,y,t)$ is the deviation of the free ocean surface from $h_{\mathrm{m.l.}}$. Due to the definition of $h_{\mathrm{m.l.}}$, the value $\zeta_{\mathrm{m.l.}}$ always satisfies the equality

$$\int_0^T \zeta_{\mathrm{m.l.}}(x,y,t)\mathrm{d}t = 0 \quad \forall(x,y) \in \Omega_{(1)}. \tag{2.4}$$

The function ζ_{g} is also called the dynamic topography in geodesy and is often denoted by h_{dyn}:

$$\zeta_{\mathrm{g}} \equiv h_{\mathrm{dyn}} = h(x,y,t) - h_{\mathrm{g}}(x,y). \tag{2.5}$$

Note that in oceanology the 'dynamic topography' is defined differently and it is related to the 'dynamical method'.

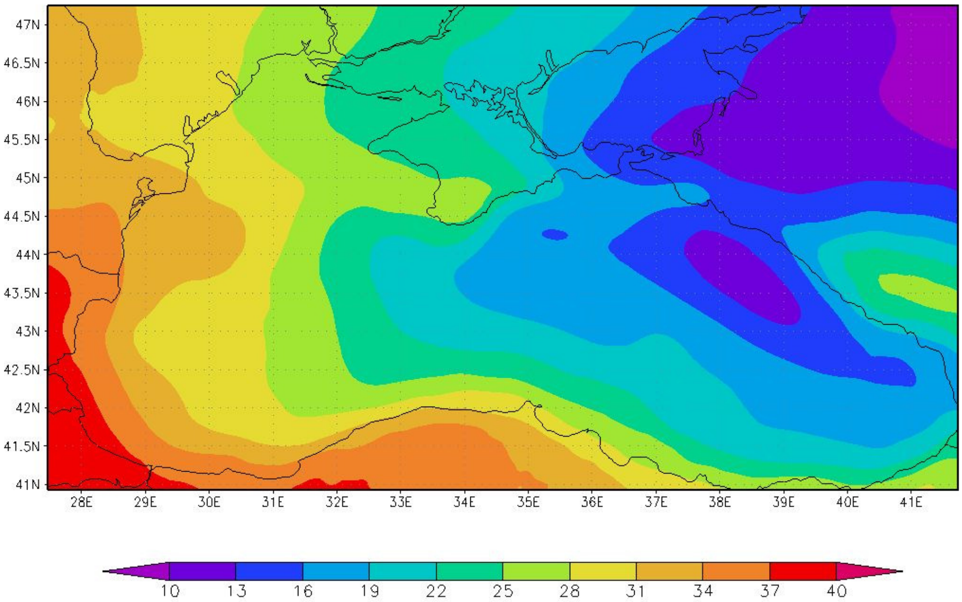

Figure 1. The height of the geoid over the terrestrial ellipsoid in meters.

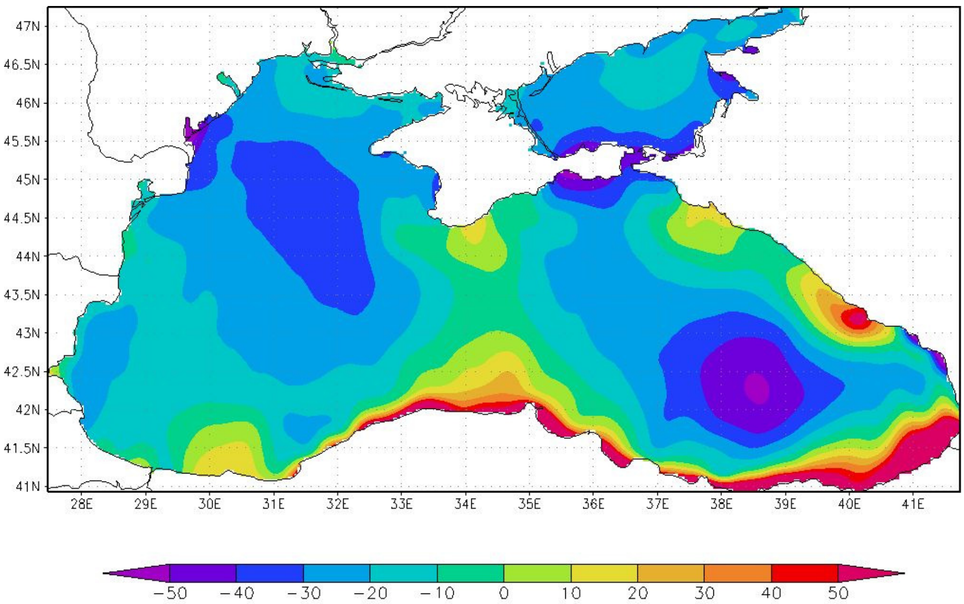

Figure 2. The long-term mean sea level over the geoid (in cm).

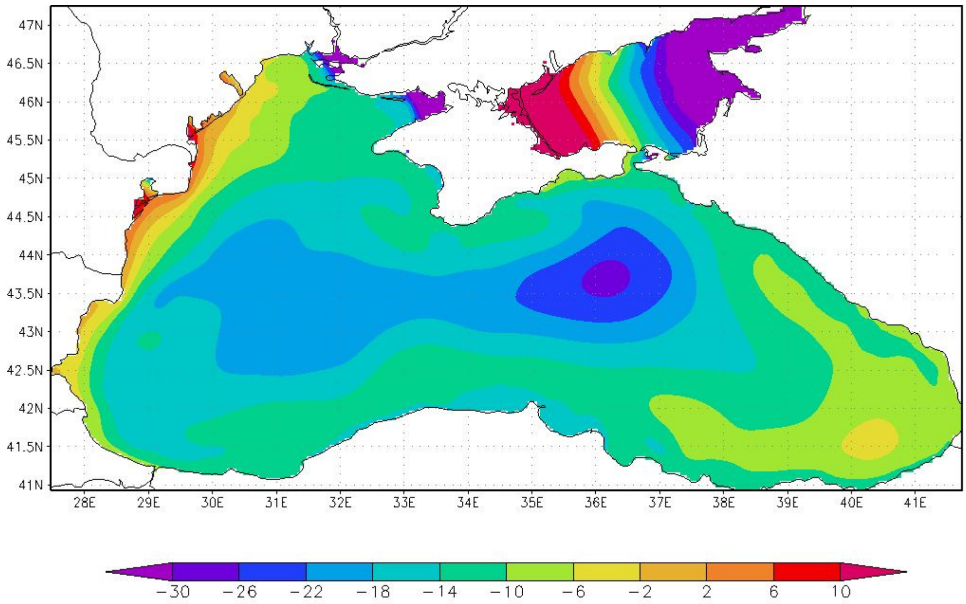

Figure 3. The mean sea level on January 1, 2008, calculated by the model without tidal forces (in cm).

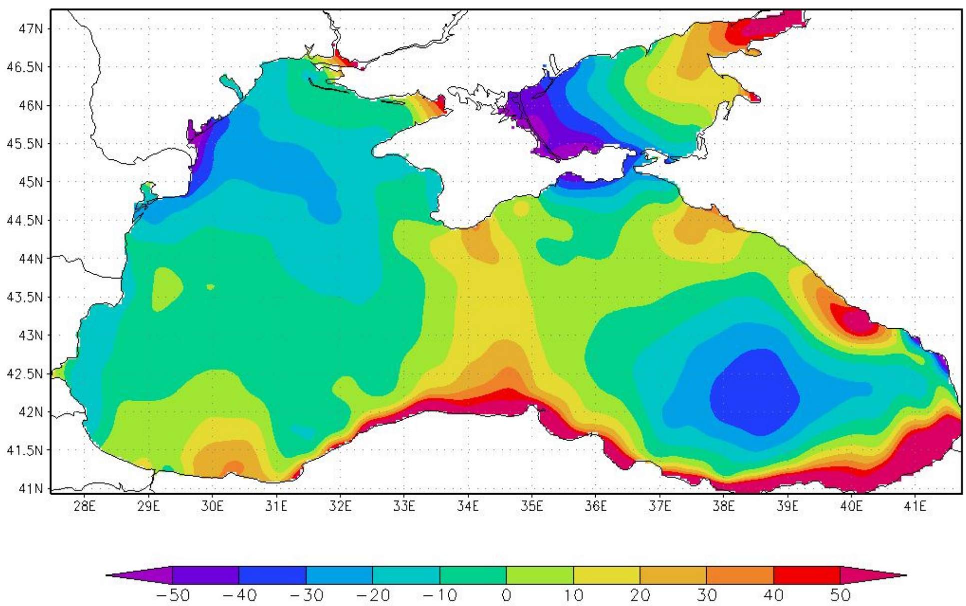

Figure 4. The difference of the actual (Fig. 2) and model (Fig.3) sea levels (in cm).

The functions introduced above (ζ_{g}, $\zeta_{\mathrm{m.l.}}$ and others) are related to the actual physical fields of seas and oceans. However, we also use them in the consideration of the equations of the Black Sea dynamics model and, if this does not lead to misunderstanding, we denote the level by ζ assuming either ζ_{g} or $\zeta_{\mathrm{m.l.}}$.

Thus, the geoid is defined as the equipotential surface of the Earth gravity field (level surface) approximately coinciding with the World Ocean level in its unperturbed state and symbolically extended under the continents [26]. By the definition of an equipotential surface, the geoid surface is everywhere perpendicular to the pluming line. The difference between the actual mean ocean level and the geoid may reach 1 m. The surfaces of the geoid and the mean level in the Black Sea water area are presented in Figs. 1 and 2. (For definiteness sake, in all figures of this paper the axis $0z$ is directed 'downward' as is accepted in many papers in oceanology.)

Figure 2 presents the mean value $\overline{h_{\mathrm{m.l.}}}$. In comparison with Fig.2, in Fig.3 we present the mean long-term (10 years) level calculated by the Black Sea dynamics model not taking into account the tidal forces (see [30]). The difference between the height of the actual sea level over the geoid and the 'model' mean sea level (i.e., calculated with the use of the numerical model) is presented in Fig.4.

Comparing Figs. 2–4, one can make the following conclusion: if we take only the mean long-term height of the Black Sea level as the criterion of the correctness of the description, then the error in the model not taking into account the tide-forming forces (see (2.1)–(2.7)) can reach considerable magnitudes ($50-100\%$).

2.2. Tide-forming forces

2.2.1. The modern approach to the calculation of the tidal potential consists in the use of ephemerides (celestial coordinates) (see [21, 32]). According to [21, 25], the tidal potential caused by each of the tide-forming bodies separately is exactly the following expression:

$$\Omega(A) = \frac{\gamma M}{R}\left(\frac{R}{d} - 1 - \frac{\mathbf{R}\cdot\mathbf{r}}{R^2}\right) \tag{2.6}$$

where A is the point of observation, $\mathbf{R}$ is the radius vector from the center of the Earth to the tide-forming body, $R = |R|$, $\mathbf{r}$ is the radius vector from the center of the Earth to the observation point, d is the distance from the observation point to the tide-forming body, $\gamma = -6.673\cdot 10^{-11}$ m^3/kg$\cdot$s^2 is the gravitational constant, M is the mass of the tide-forming body ($M_{\mathrm{m}} = 7.3477\cdot 10^{22}$kg is the mass of the Moon, $M_S = 1.9891\cdot 10^{30}$ kg is the mass of the Sun). If we assume that $r/R \ll 1$, then $\Omega(A)$ can be expanded into a series over powers of r/R and thus we can reduce the expression to the form

$$\begin{aligned} \Omega(A) &= \sum_n \Omega_n(A) \cong \Omega_2(A) \\ \Omega_n(A) &= \frac{\gamma M}{R}\left(\frac{r}{R}\right)^n P_n(\cos\nu), \quad n = 2, 3, \ldots \end{aligned} \tag{2.7}$$

where $P_n(x)$ are Legendre polynomials. In these calculations it is sufficient to take into account only $\Omega_2(A)$, because even for the closest to the Earth celestial body, i.e., the Moon, we have the relation $r/R \approx 0.017$.

The expression for the tidal potential presented here contains the zenithal angle ν. In order to remove this local value, it can be expressed through the equatorial coordinates (see [21, 32]), the hour angle t, the declination β, and the geographical coordinates (the latitude denoted here by φ, $\varphi \in [-\pi/2, \pi/2]$, and the longitude λ, $\lambda \in [0, 2\pi]$) using the relation $\cos\nu = \sin\varphi \sin\beta + \cos\varphi \cos\beta \cos(t+\lambda)$. Taking into account this relation, we get the following form of the formula for $\Omega_2(A)$:

$$\begin{aligned}\Omega_2(A) = \frac{3}{4}\frac{\gamma M r^2}{R^3}\left(\frac{1}{2} - \frac{3}{2}\sin^2\varphi\right)\left(\frac{2}{3} - 2\sin^2\beta\right) \\ + \frac{3}{4}\frac{\gamma M r^2}{R^3}\left[\sin 2\varphi \sin 2\beta \cos(t+\lambda) + \cos^2\varphi \cos^2\beta \cos 2(t+\lambda)\right]\end{aligned} \tag{2.8}$$

The total tidal potential at the given point A is the sum of the potentials induced at this point by the Moon and the Sun: $\Omega_2(A) = \Omega_2^{\mathrm{M}}(A) + \Omega_2^{\mathrm{S}}(A)$. The hour angle t for the Moon and the Sun is calculated depending on the right ascension (α) and the Greenwich time (Universal Time, UT) by the corresponding algorithms used in the theory of celestial coordinates.

2.2.2. In the harmonic representation of $\Omega \cong \Omega_2$ each component is represented as a sum of harmonics

$$\sum_j C_j \cos(\sigma_j t + s\lambda + q_j). \tag{2.9}$$

Each of these harmonics is characterized by its amplitude C_j and the argument $\sigma_j t + s\lambda + q_j$ related by a linear dependence to the Greenwich Mean Time t and the east longitude λ, $s = 0, 1, 2$. The frequencies of the the harmonics $\{\sigma_j\}$ are determined by the well-known technique (see [12]) and are assumed to be given.

Note some difficulties related to the practical applicability of ‘harmonic expansions’ of the tidal potential $\Omega_2(A)$ determined by (2.9).

The first difficulty is traditional for the theory of series and is related to a slow convergence of series (2.9). For some applications one can restrict oneself to the total of 4–11 harmonics in (2.9). However, for calculations with a sufficient accuracy it is recommended to use 60 and more harmonics (especially for ‘shallow water areas’).

In spite of the seeming simplicity of representations (2.9), they contain sets of the angles $\{q_j\}$, i.e., the ‘initial phases’. These values are calculated according to special handbooks depending on the time interval, where it is necessary to perform the calculation of $\Omega_2(A)$; in this case $\{q_j\}$ must take into account a series of astronomic factors. All these create difficulties and discomfort.

In addition, expansions (2.9) do not take into account the elasticity of the Earth (i.e., of the oceanic bottom), whose influence is commonly accepted. In order to take this into account, each harmonic is supplied with its coefficient γ_j:

$$\gamma_j \equiv 1 + k_j - h_j, \qquad j = 0, 1, 2, \dots \tag{2.10}$$

where $\{k_j\}$ and $\{h_j\}$ are Love numbers calculated separately and the error of their calculation may be essential for 'long-period harmonics'. As the result, instead of (2.9), we should apply the expansion of the form

$$\sum_j \gamma_j C_j \cos(\sigma_j t + s\lambda + q_j) \tag{2.11}$$

and instead of $\Omega_2(A)$, take the potential $\widetilde{\Omega_2(A)} = \gamma_0 \Omega_2(A)$. However, for most of the coefficients γ_j we have

$$\gamma_j \cong 0.7 \tag{2.12}$$

which helps us in this situation (below we introduce the notation $\gamma_0 = 0.7$).

It should be noted that studies aimed to represent $\Omega_2(A)$ in the form of other faster convergent expansions over special functions differing from (2.9) are in progress now.

Based on what has been said above, we can suppose that in some cases it is expedient to apply mixed approximations of $\Omega_2(A)$ using the representation of this potential via 'astronomical parameters' and expressions of type (2.9).

Finally, note that even after the specification of $\Omega_2(A)$ the problem of the description of 'self-attraction' and other effects remains valid. Concerning this problem, we refer the reader to [21], because we do not have the possibility to discuss it here.

2.2.3. The height of the 'static tide' ζ^+ is often more convenient in use than the tidal potential. This value is defined as

$$\zeta^+ \equiv \frac{\Omega_2}{g} + C(t) \tag{2.13}$$

where $C(t)$ is a constant whose value is determined from the condition that the total volume of tidal deformations in the World Ocean turns to zero, $\int_S \zeta^+ \mathrm{d}S = 0$, or from balance relations for each considered basin.

Having some representation of the tide-forming potential (by formula (2.8), or (3.9)), one can calculate the tidal forces used at the right-hand sides of the dynamics equations:

$$\mathbf{f} \equiv g \nabla \zeta^+ \equiv \nabla \Omega_2. \tag{2.14}$$

If we take into account the coefficients $\{\gamma_j\}$, then $\mathbf{f}$ takes the form

$$\mathbf{f} \cong g \nabla \left(\sum_j \gamma_j \zeta_j^+ \right) \cong \nabla(\gamma_0 \Omega_2 + \gamma_0 g C(t)) \tag{2.15}$$

where $\{\zeta_j^+\}$ are expansions of $\{\zeta^+\}$ constructed on the base of (2.9). Below in representation (2.15) we especially distinguish the 'constant' harmonic $\gamma_0\zeta_0^+$ corresponding to the solar S_0 and lunar M_0 harmonics not dependent on time t and dependent on the latitude φ only.

If the method of harmonic representation of the tidal potential is used for obtaining the expression for Ω_2, then the tidal forces are also represented as sums of several harmonics (often 4–8, rarely 11–16). The form of tide-forming forces under the use of 4 principal harmonics (subject to the factors $\gamma_j \cong 0.7$) was presented, e.g., in [8].

2.2.4. Apply the following transformations of the source function of the mathematical model. Consider the function

$$\mathbf{F} \equiv -\frac{1}{\rho_0}\nabla P + \mathbf{f} \tag{2.16}$$

called the 'total source function' in the motion equations of the mathematical model considered here. Here we have

$$\mathbf{f} \equiv g\nabla G, \qquad G \equiv G(x,y,t) \tag{2.17}$$

and P is the pressure function determined (for $f_P \equiv 0$) from

$$\frac{\partial P}{\partial z} = g\rho, \quad P = P_a, \quad z = -h \tag{2.18}$$

where P_a is the atmospheric pressure, h is the height of the free ocean surface.

Introduce the function $\widetilde{h_0}$ defined as

$$\widetilde{h_0} = h_{\mathrm{m.l.}} - h_{\mathrm{g}}. \tag{2.19}$$

Performing the change $z' = z + h_{\mathrm{g}}$ and denoting z' again by z, we shift the origin of the axis OZ onto the surface of the geoid. For the sake of brevity, we do not present this change here, but note that we actually get a new perturbed system of ocean dynamics, which formally coincides with mathematical model (2.1). After the change of coordinates $z' = z + h_{\mathrm{g}}$ *we denote the height of the free ocean surface again by $h(x,y,t)$ (previously, h was measured from TE, and now it is measured from the surface of the geoid).* Represent h in the form

$$h = \widetilde{h_0}(x,y) - \zeta(x,y,t) \tag{2.20}$$

where $\widetilde{h_0}$ is the reference surface for the level function ζ. If we neglect the physically meaningful definition (2.19), then $\widetilde{h_0}$ can be interpreted both as some constant perturbation of the geoid surface and as a reference surface, and ζ as the deviation of the ocean surface from the surface $z = \widetilde{h_0}$. If, for example, we assume $\widetilde{h_0} \equiv 0$

in (2.20), then we get $\zeta \equiv \zeta_{\mathrm{g}}$ is the deviation of the free ocean surface from the geoid surface. If we take $\widetilde{h_0}$ defined according to (2.19), then $\zeta \equiv \zeta_{\mathrm{m.l.}}$.

Based on the following approximate expression for $P(z)$:

$$P(z) = P_a + \rho_0 g(\widetilde{h_0} - \zeta + z) + \int_0^z g\rho_1 \mathrm{d}z' \tag{2.21}$$

we get

$$\mathbf{F} = g\nabla\zeta - \frac{\nabla P_a}{\rho_0} - g\nabla\widetilde{h_0} - \nabla\int_0^z \frac{\rho_1 g}{\rho_0}\mathrm{d}z' + \mathbf{f} \tag{2.22}$$

which is the form of $\mathbf{F}$ actually presented in (2.1) (for $\widetilde{h_0} \equiv 0$).

Now specify the form of the function G generating $\mathbf{f}$. Assume

$$G = \gamma_0\frac{\Omega_2}{g} + \frac{\Omega}{g} \tag{2.23}$$

where Ω_2 is the potential of tide-forming forces (see (2.1)), $\zeta^+ = \Omega_2/g + C(t)$ is the static tide, and $\Omega = \Omega(x,y)$ is a function not dependent on t (in this section we consider this case only).

We assume that Ω_{p} is the potential of forces perturbing the geoid surface (as well as the potential $\gamma_0 g\zeta_0^+$, entering $\gamma_0\Omega_2/g$ and corresponding to the harmonics M_0, S_0 constant in t) and also the mean level surface. As the result of these perturbations, the geoid surface turns to the surface $h_{\mathrm{m.l.}}$, which is the mean level of the sea ('mean level') calculated for the time period $(0,T)$ for sufficiently large T. Due to the small difference between $h_{\mathrm{m.l.}}$ and h_{g}, one may suppose $\Omega_{\mathrm{p}}/g \ll h_{\mathrm{g}} \cong h_{\mathrm{m.l.}}$.

Suppose Ω_{p} is the potential of all potential forces (even those we do not know explicitly) perturbing the geoid surface. Excluding the influence of $g\gamma_0\zeta_0^+$ from these forces (this potential is already included into Ω_2), we assume

$$\Omega_{\mathrm{p}} = g(\widetilde{h_0} - \gamma_0\zeta_0^+), G = g[\gamma_0\zeta^+ + h_0] = g\left[\gamma_0\frac{\Omega_2}{g} + h_0\right] \tag{2.24}$$

where we set $h_0 = \widetilde{h_0} - \gamma_0\zeta_0^+$. Then

$$\mathbf{F} = g\nabla(\zeta + (1-\beta)h_0) - \frac{1}{\rho_0}\nabla P_a + \nabla g\left[\gamma_0\frac{\Omega_2}{g} - \beta\gamma_0\zeta_0^+\right] - g\int_0^z \frac{\rho_1}{\rho_0}\mathrm{d}z'. \tag{2.25}$$

Finally, let

$$C_\zeta(t) \equiv \frac{\int_\Omega \zeta \mathrm{d}\Omega}{\mathrm{mes}(\Omega)}, \quad \zeta = \zeta' + C_\zeta(t), \quad \int_\Omega \zeta' \mathrm{d}\Omega = 0.$$

Since $\nabla C_\zeta = 0$, then redenoting ζ' by ζ, we get the following expression of form (2.25) for **F** and the equation for ζ :

$$\frac{\partial \zeta}{\partial t} + \mathbf{div}\left(\int_0^H \Theta(z)\mathbf{u}\,\mathrm{d}z'\right) = \widetilde{f}_3 \tag{2.26}$$

where

$$\widetilde{f}_3 \equiv f_3 - \frac{\partial C_\zeta}{\partial t}, \qquad \int_\Omega \zeta\,\mathrm{d}\Omega = 0$$

i.e., we have removed the incorrectness related to the non-uniqueness of the function ζ satisfying the equations of the mathematical model.

Remark 2.1. If the problem is considered with the additional condition '$\zeta = \zeta_{\mathrm{obs}}$ in Ω', then, writing down $\zeta_{\mathrm{obs}} \equiv \zeta'_{\mathrm{obs}} + C_{\zeta,\mathrm{obs}}$, where $C_{\zeta,\mathrm{obs}} = \int_\Omega \zeta_{\mathrm{obs}}\mathrm{d}\Omega/\mathrm{mes}(\Omega)$, we get $C_\zeta = C_{\zeta,\mathrm{obs}}$ and $\zeta = \zeta_{\mathrm{obs}}$ in (2.26).

Now we can consider two cases of the specification of **F**.

Case 1: $\beta \equiv 0$. Here $\zeta \equiv \zeta_{\mathrm{g}}$, the potential Ω_2 appears in its 'complete form', and we assume $h_0 = \widetilde{h}_0 - \gamma_0\zeta_0^+ = (h_{cp} - h_{\mathrm{g}}) - \gamma_0\zeta_0^+$ (or the function h_0 must be determined taking into account some other perturbation forces).

Case 2: $\beta \equiv 1$. Here $\zeta \equiv \zeta_{cp}$ and the harmonic '$\gamma_0\zeta_0^+$' is excluded from the potential Ω_2.

Note that in Case 2 we have no need in introduction of some additional 'potential force' perturbing the geoid surface, and we can assume that its influence has been already taken into account by the introduction of $h_{\mathrm{m.l.}}$. If we recall that satellite altimetry only measures the functions $\zeta_{m.,\ \mathrm{obs}} \equiv (\zeta_{m.})_{\mathrm{obs}}$, then, probably, Case 2, where $\beta \equiv 1$ and $\zeta \equiv \zeta_{cp}$, is preferable in practical calculations. However, it is necessary to be aware of the fact that the mathematical models used are inadequate to some extent. Therefore, the inclusion of a function of type h_0 into **F** is possible even in the case $\beta = 1$. Declaring them to be auxiliary unknowns, solving the corresponding inverse problem, and describing these functions with the use of observation data, we can achieve a greater adequacy of the model and improve the description quality concerning the physical processes. We consider one of such inverse problems in the next section.

For definiteness sake, in this paper we consider only Case 1 ($\beta = 0$) with respect to numerical experiments. i.e., the measurement of the level function is performed from the geoid surface.

3. Improvement of the total source function and solution of an inverse problem

In this section we show how one can improve functions of type **F** by solving some inverse problems and using observation data.

3.1. Formulation of the inverse problem

Consider the system of equations for $\mathbf{u} \equiv (u,v), \zeta$ from (2.1) written in the following form:

$$\begin{aligned}
&\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \mathbf{u} - g\nabla\zeta + A_u\mathbf{u} + (A_k)^2\mathbf{u} \\
&\qquad = \nabla g \left[\gamma_0 \frac{\Omega_2}{g} - \beta\gamma_0\zeta_0^+ \right] + \nabla g\Psi - \frac{1}{\rho_0}\nabla P_a - g\int_0^z \frac{\rho_1}{\rho_0}\mathrm{d}z' \quad \text{in } D \quad \forall t \qquad (3.1) \\
&\qquad\qquad \frac{\partial\zeta}{\partial t} - \mathbf{div}\left(\int_0^H \Theta\mathbf{u}\mathrm{d}z' \right) = \widetilde{f}_3 \quad \text{on } \Omega \quad \forall t
\end{aligned}$$

where the additional condition of the form

$$\int_\Omega \zeta\,\mathrm{d}\Omega = 0 \qquad (3.2)$$

is posed on ζ and the motion equation includes the function $\Psi \equiv \Psi_0(x,y) + \Psi_1(x,y,t)$ such that

$$\int_\Omega \Psi_k'\mathrm{d}\Omega = 0, \quad k = 0,1, \qquad \overline{\Psi_1} = \frac{1}{T}\int_0^T \Psi_1\mathrm{d}t = 0. \qquad (3.3)$$

We can consider the function Ψ in (3.1) both for $\beta = 0$ and for $\beta = 1$. We assume that Ψ is an *auxiliary unknown*. In order to close the problem of determination of $\mathbf{u}, \zeta, \Psi$, introduce the auxiliary equation of the form

$$\zeta = -\zeta'_{\mathrm{obs}} \quad \text{in } \Omega \quad \forall t \qquad (3.4)$$

where ζ'_{obs} is obtained from the expansion of the given observation function ζ_{obs} for the ocean level measured either from h_{g}, or from $h_{\mathrm{m.l.}}$:

$$\zeta_{\mathrm{obs}} = \zeta'_{\mathrm{obs}} + C_{\zeta,\mathrm{obs}}(t), \quad C_{\zeta,\mathrm{obs}}(t) = \int_\Omega \zeta_{\mathrm{obs}}\mathrm{d}\Omega/\mathrm{mes}(\Omega)$$

(recall that ζ_{obs} is a function of ‘real’ measurements and for ζ we have performed the change $\zeta \to -\zeta$ after directing the axis $0z$ ‘downward’).

Formulate the following inverse problem: determine $\mathbf{u}, \zeta$, and Ψ so that conditions (3.1)–(3.4) are valid with the corresponding boundary and initial conditions.

Solving this inverse problem and analyzing the values of the function Ψ, we can make conclusions on the adequacy of mathematical model (3.1) where, for simplicity sake, we assume that T, S, and ρ_1 are given.

3.2. The study of the solvability of the approximate inverse problem

Consider the particular case of the inverse problem, the ‘approximate inverse problem’, by assuming

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \equiv \frac{\partial\mathbf{u}}{\partial t}, \quad f \equiv l_0 \text{ is the Coriolis parameter,} \quad \overline{f_3} \equiv \frac{1}{T}\int_0^T \widetilde{f}_3\mathrm{d}t = 0, \quad \overline{\rho_1} \equiv 0$$

and all the coefficients μ, ν are constant in the equations. We also assume that all boundary conditions are homogeneous and their components corresponding to the convection operator are absent. The boundary conditions with respect to t are taken as

$$\mathbf{u}|_{t=0} = \mathbf{u}|_{t=T}, \quad \zeta|_{t=0} = \zeta|_{t=T} \tag{3.5}$$

where $(0,T)$ is a sufficiently large time interval.

Proceed to the consideration of $\Omega_2', (\zeta_0^+)', P_a'$ instead of Ω_2, ζ_0^+, P_a in (3.1) and apply averaging with respect to t to these equations. Then, taking into account (3.5), we get

$$\begin{aligned} &\begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \overline{\mathbf{u}} - g\nabla\overline{\zeta} + A_u\overline{\mathbf{u}} + (A_k)^2\overline{\mathbf{u}} \\ &\quad = \nabla g \left[\gamma_0 \frac{\overline{\Omega_2'}}{g} - \beta\gamma_0 (\overline{\zeta}_0^+)' + \Psi_0 \right] - \frac{1}{\rho_0}\nabla\overline{P_a'} \quad \text{in } D \\ &\mathbf{div}\left(\int_0^H \Theta\overline{\mathbf{u}}\,\mathrm{d}z' \right) = 0 \quad \text{on } \Omega \end{aligned} \tag{3.6}$$

Move $g\nabla\overline{\zeta}$ to the right-hand side of the equation. Then it is easy to see that system (3.6) has the unique solution $\overline{\mathbf{u}}, \overline{\zeta}$, where

$$\overline{\mathbf{u}} \equiv 0, \quad \overline{\zeta} = \frac{\overline{P_a'}}{\rho_0 g} - \left[\gamma_0 \frac{\overline{\Omega_2'}}{g} - \beta\gamma_0 (\overline{\zeta}_0^+)' + \Psi_0 \right]. \tag{3.7}$$

Now from (3.4) we have

$$\overline{\zeta} = -\overline{\zeta_{\mathrm{obs}}'}. \tag{3.8}$$

From (3.7), (3.8) we get one of the required functions Ψ_0:

$$\Psi_0(x,y) = \frac{\overline{P_a'}}{\rho_0 g} + \overline{\zeta_{\mathrm{obs}}'} - \gamma_0 \left[\frac{\overline{\Omega_2'}}{g} - (\overline{\zeta}_0^+)' \right]. \tag{3.9}$$

Calculating $\Psi_0(x,y)$ from (3.8), one can suppose that the inclusion of Ψ_0 into the motion equation may result in a more accurate description of level functions mean with respect to t.

We assume that the function Ψ_0 has been already determined. Now prove the unique solvability of the problem relative to $\mathbf{u}, \zeta, \Psi_1$ considering it with same conditions (3.5).

Suppose two solutions to the problem exist. Then we have the following system of equations for the difference of these solutions:

$$\begin{aligned} &\frac{\partial\mathbf{u}}{\partial t} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \mathbf{u} - g\nabla\zeta + A_u\mathbf{u} + (A_k)^2\mathbf{u} = \nabla g\Psi_1 \quad \text{in } D \quad \forall t \\ &\frac{\partial\zeta}{\partial t} + \mathbf{div}\left(\int_0^H \Theta\mathbf{u}\,\mathrm{d}z' \right) = 0 \quad \text{on } \Omega \quad \forall t \\ &\zeta = 0 \quad \text{in } \Omega \quad \forall t \end{aligned} \tag{3.10}$$

where the latter condition is a corollary from (3.4). Taking into account (3.3), relation (3.10) immediately implies $\mathbf{u} \equiv 0, \Psi_1 \equiv 0$. Thus, the problem of determination of $\mathbf{u}, \zeta, \Psi_1$ considered here can have only a unique solution, i.e., it is uniquely solvable.

3.3. The iterative solution algorithm for the inverse problem

The following iterative method can be such solution algorithm: if $\Psi_1^{(k)}$ and $\Psi^{(k)} \equiv \Psi_0 + \Psi_1^{(k)}$ have been calculated, then the determination of $\Psi_1^{(k+1)}$ (and simultaneously $\mathbf{u}^{(k+1)}, \zeta^{(k+1)}$) is performed by the successive solution of the following problems:

$$\begin{gathered}\frac{\partial \mathbf{u}^{(k)}}{\partial t} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \mathbf{u}^{(k)} - g\nabla\zeta^{(k)} + A_u\mathbf{u}^{(k)} + (A_k)^2\mathbf{u}^{(k)} \\ = \nabla g\left[\gamma_0\frac{\Omega_2}{g} - \beta\gamma_0\zeta_0^+\right] + \nabla g\Psi^{(k)} - \frac{1}{\rho_0}\nabla P_a - g\int_0^z \frac{\rho_1}{\rho_0}\mathrm{d}z' \quad \text{in } D \quad \forall t \\ \frac{\partial \zeta^{(k)}}{\partial t} - \mathbf{div}\left(\int_0^H \Theta\mathbf{u}^{(k)}\mathrm{d}z'\right) = \widetilde{f}_3 \quad \text{on } \Omega\ \forall t\end{gathered} \tag{3.11}$$

$$\begin{gathered}-\frac{\partial \mathbf{q}^{(k)}}{\partial t} + \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix} \mathbf{q}^{(k)} + g\nabla q_3^{(k)} + A_u\mathbf{u}^{(k)} + (A_k)^2\mathbf{u}^{(k)} = 0 \\ -\frac{\partial q_3^{(k)}}{\partial t} + \mathbf{div}\left(\int_0^H \Theta\mathbf{q}^{(k)}\mathrm{d}z'\right) = \zeta + \zeta'_{\mathrm{obs}}\end{gathered} \tag{3.12}$$

$$\Psi^{(k+1)} = \Psi^{(k)} - \tau_k(\alpha\Psi^{(k)} + g\,\mathbf{div}\,\mathbf{q}^{(k)}). \tag{3.13}$$

The numerical solution of problems (3.11) – (3.13) can be performed, e.g., by difference methods.

Algorithm (3.11) – (3.13) can be naturally written for system (4.1) as well.

4. Numerical experiments estimating the effect of tide-forming forces in the mathematical model

In order to study the influence of tidal forces on the dynamics of the Black Sea (first of all, we are interested in their influence on the sea level!) we have performed a series of numerical experiments where the right-hand sides have been modified by tide-forming forces in the form taken from (4.1), (4.2) for $\Psi \cong \Psi_0$. We also considered the problem of the reference level $\widetilde{h}_0$ (see transformation (2.20)) and have performed the corresponding experiments where some value constant in t was taken as $\widetilde{h}_0$.

The calculation domain was $\Omega : \lambda \in [27.475°, 41.775°], \varphi \in [40.93°, 47.29°]$, the grid was $0.05° \times 0.04°$ with 40 levels for the vertical variable (see [30]).

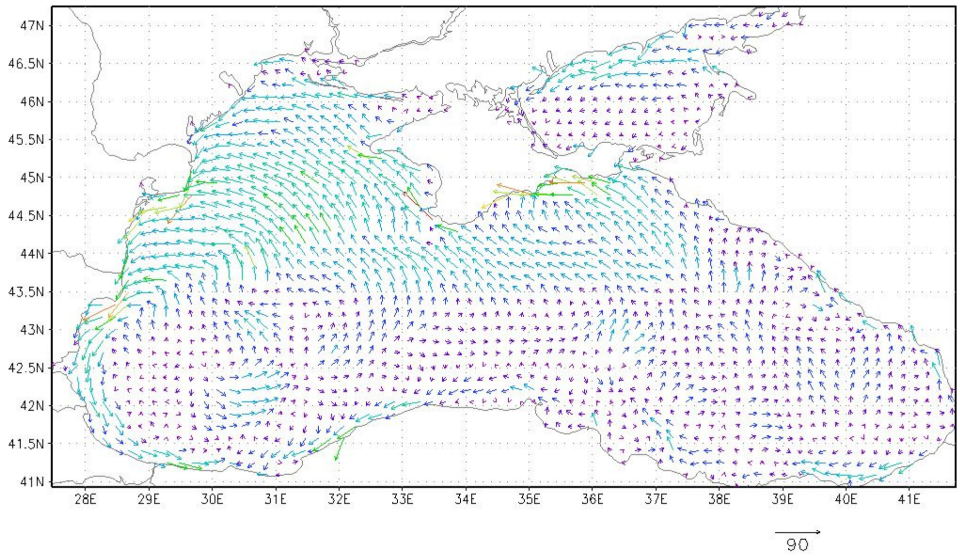

Figure 5. Vector function of the flow velocity at 00 o'clock, January 2, 2008 calculated by the model without tidal forces (in cm/s).

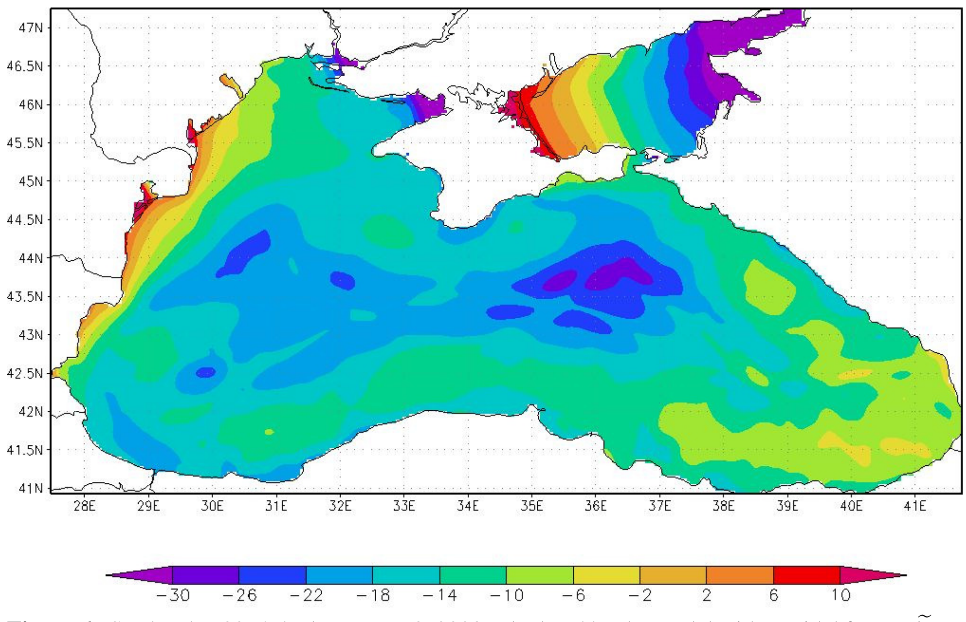

Figure 6. Sea level at 00 o'clock, January 2, 2008 calculated by the model without tidal forces, $\widetilde{h_0} = 0$ (in cm).

All calculations were performed for one day, from January 1 to 2, 2008 with the time step of 5 minutes. The model works in CGS units. Therefore, all variables presented further in figures are measured in centimeters, seconds, and cm/s.

As was shown in [8], one should use the expression for the total tidal potential and the auxiliary potential Ψ_0 in these experiments, which improves the results of calculations compared with observation data.

Before we consider the results of the experiments, note that the observation data of the mean sea level for January 1, 2008 are presented in Fig. 3, the ocean level at 00 o'clock, January 2, 2008 is shown in Fig. 6, and the value C_{obs} equals 14.69 cm for the mean level of the Black Sea.

4.1. Experiment 1, no tidal forces

In the model of the general sea circulation from [30], the level $z = 0$ is called the unperturbed ocean surface (we also assume this fact) and hence the output data of the model represent the deviation from the equilibrium state. A deviation from the equilibrium state in the nature is the deviation from the geoid level. The results of the calculations by the model without the tidal forces for $\widetilde{h_0} = 0$ (i.e., the measurements are performed from the geoid level) are shown in Figs. 3–6. The calculation period covers January 1–2, 2008.

The results of the experiments presented here allow us to conclude that the calculations by the model without tide-forming forces for $\widetilde{h_0} = 0$ qualitatively describe the observed sea level, however, the error may be quite significant.

4.2. Experiment 2, the total potential

In this experiment we performed the calculation with the use of total potential representation (2.8), January 1–2, 2008. The value $\widetilde{h_0}$ equals zero and the constant part of the potential is included. In this experiment the function ζ is the magnitude of the sea level deviation from the equilibrium state, i.e., from the geoid. The tide-forming forces are calculated with the use of (2.15), the coefficient is $\gamma_0 = 0.7$.

The results of the experiments are shown in Figs. 7–10.

Comparing the calculation results shown in Figs. 7 and 8 with those presented in Figs. 2–4, we conclude that for $\Psi_0 = 0$ the value $\overline{\zeta}$ differs from the observation data and $\|\overline{\zeta} - \overline{\zeta'_{\mathrm{obs}}}\|/\|\overline{\zeta_{\mathrm{obs}}}\| \simeq 1.06$. Thus, the inclusion of all tide-forming forces does not compensate the errors of the mathematical model, because their influence is not great, see Figs. 8 and 9, which is confirmed by the known facts (see [14, 15]). (We also note that the effect of inclusion of tide-forming forces is more significant in the case of the consideration of the whole World Ocean, see [8].) Therefore, one can suppose that in order to obtain more adequate calculations of the level function, it is necessary to take into account the curvature of the geoid surface as the reference surface in some form in the mathematical model. The same is true for the influence of various forces perturbing this surface (see the next section).

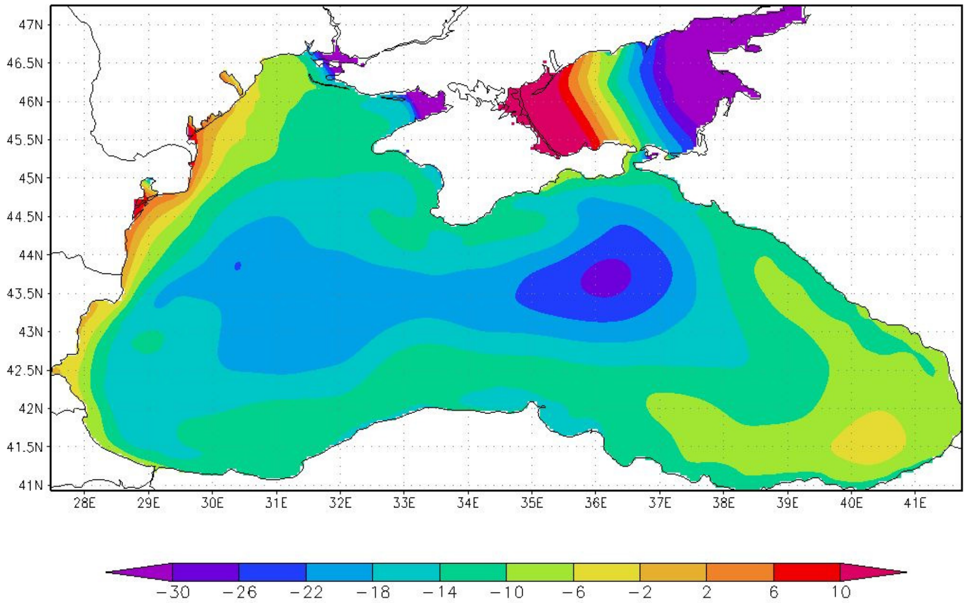

Figure 7. Mean sea level calculated by the model with the total potential (in cm).

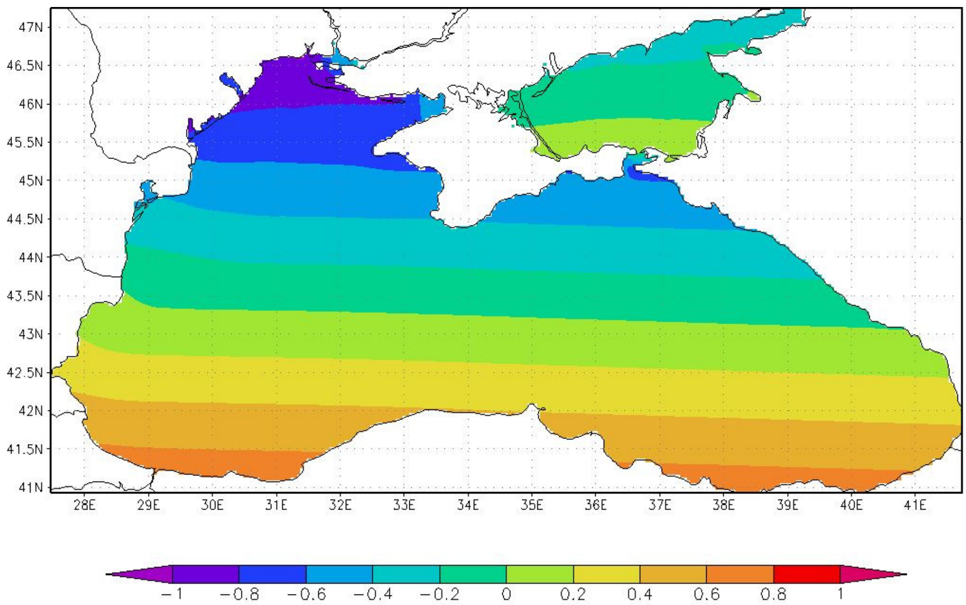

Figure 8. Deviation of the mean sea level calculated by the model with the total potential from the mean level obtained in Experiment 1 (without tide-forming forces, Fig. 7), (in cm).

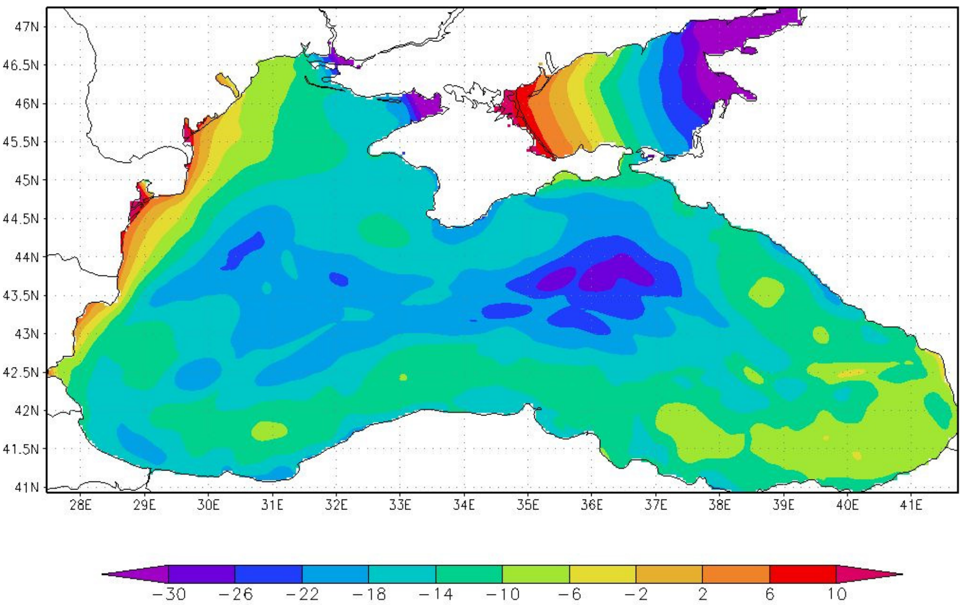

Figure 9. Sea level at 00 o'clock, January 2, 2008 calculated by the model with the total potential (in cm).

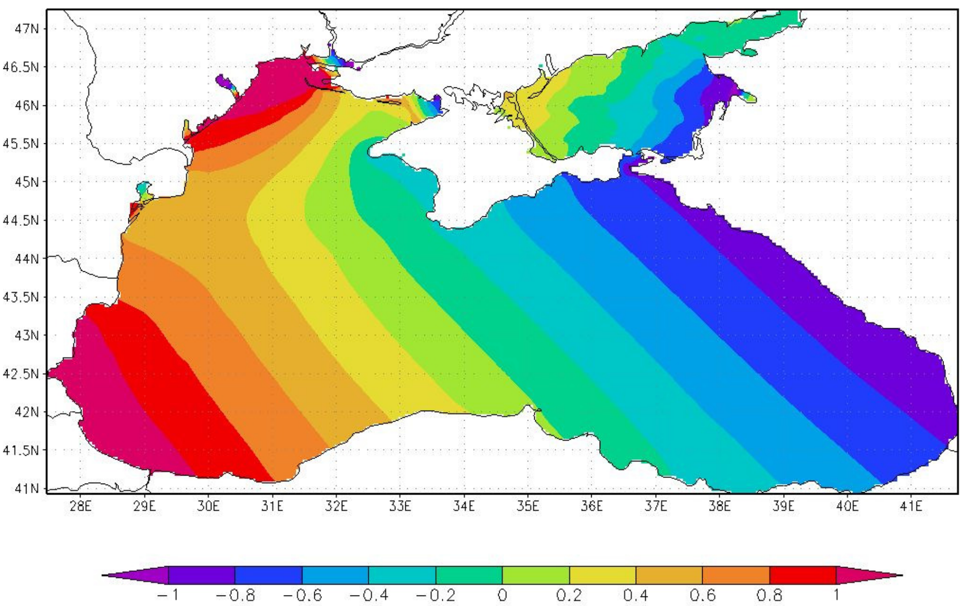

Figure 10. Deviation of the sea level at 00 o'clock, January 2, 2008 calculated by the model with the total potential from the level obtained in Experiment 1 (without tide-forming forces, Fig 6), (in cm).

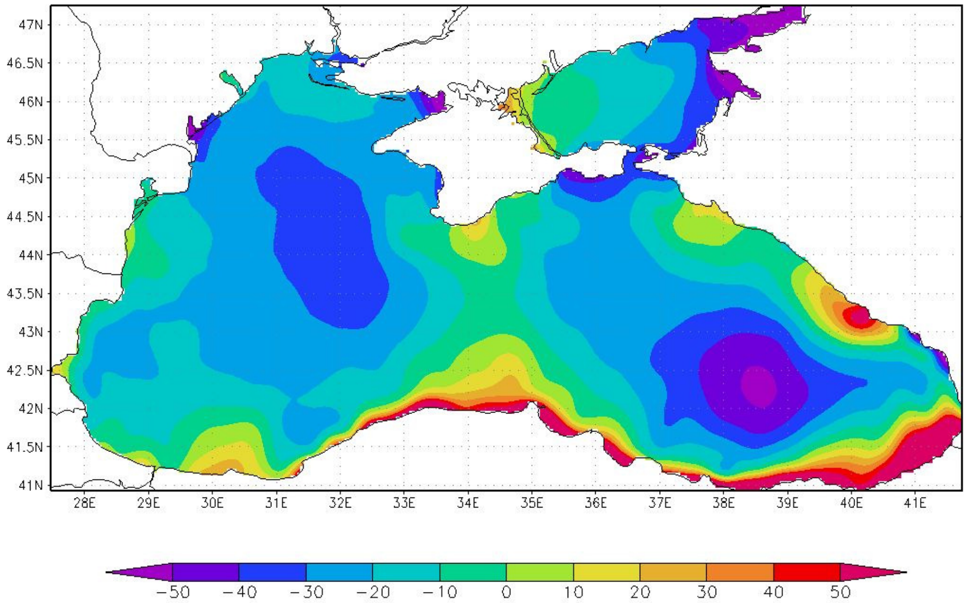

Figure 11. The value $\overline{\zeta}$ for $\Psi_0 = \overline{\zeta_\tau} + \overline{\zeta'_{\text{obs}}}$, (in cm) (cf. Fig. 2).

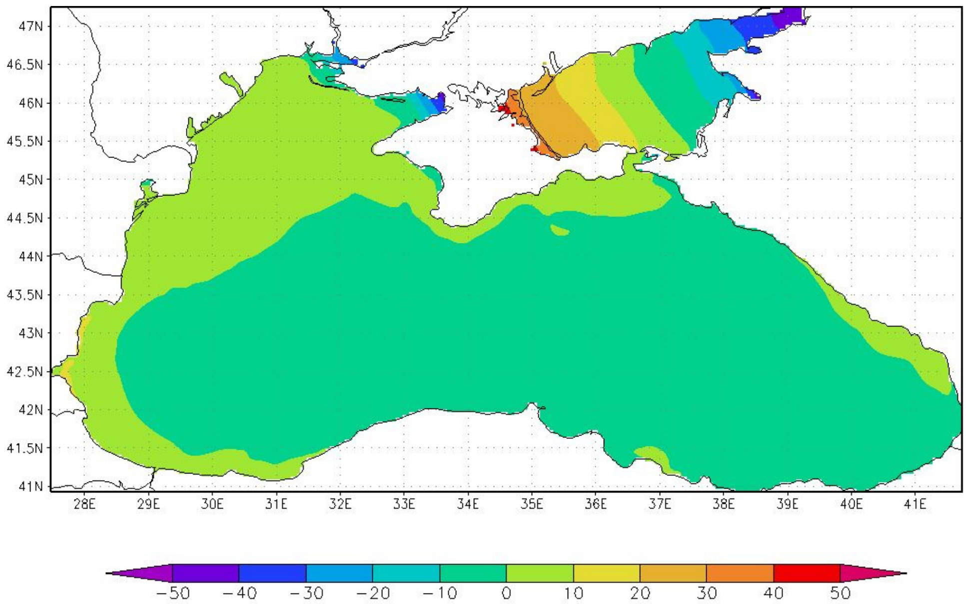

Figure 12. The difference $\overline{\zeta'_{\text{obs}}} - \overline{\zeta}$, Figs. 11 and 2 (in cm). (cf. Fig.4).

4.3. Numerical experiments for the estimation of the influence of auxiliary potential

Additional experiments were performed relative to the water area of the Black Sea to estimate the effect of 'self-attracting forces' and different forces (including even unknown ones!) influencing the deformation of the geoid surface (and hence the mean level surface).

In these experiments the model included the tide-forming forces and auxiliary forces from the potential Ψ_0, whose calculation was based on observation data (see the previous section). Some of the results of these experiments are presented in Figs. 11 and 12.

If we use the expression $\Psi_0 = \overline{\zeta_\tau} + \overline{\zeta'_{\mathrm{obs}}}$, the calculation results are noticeably closer to the observation data. The error $(\|\overline{\zeta} - \overline{\zeta'_{\mathrm{obs}}}\|)/\|\overline{\zeta_{\mathrm{obs}}}\|$ in the common water area of the Black Sea and the Sea of Azov is of order 0.27. If we calculate this error only for the open parts of the Black Sea, it gets the value $\approx$0.04–0.06 (which is comparable with the results from [8] obtained in the consideration of the whole World Ocean). The principal errors appearing here relate to the shelf zone of the Black Sea and especially to the basin of the Sea of Azov. Probably, this is primarily related to the inaccuracy of the direct 'mathematical' model (see [30]) used in this paper in numerical experiments.

5. Conclusion

The experiments performed here show that the use of total tidal potential (2.8) in the mathematical model of the dynamics of the Black Sea and the Sea of Azov is quite realizable, but requires additional data (positions of celestial bodies). However, this allows us to perform calculations for any dates and to take all components of tidal forces, including long-period ones. Note also that tidal waves are clearly expressed in the calculations, which opens new possibilities for applying harmonic analysis to tidal currents and waves. This may also play an important role in the study of resonance phenomena of tidal waves and free oscillations in the Black Sea and the Sea of Azov, where the wave amplitudes may essentially grow (see [14, 15]).

The experiments performed here confirm the necessity to take into account the 'self-attraction' forces and the forces causing the distortion of the geoid and the mean level surfaces. The improvement of the accuracy of the calculations of the ocean level function is observed even for an approximate calculation of those forces and solution of the corresponding inverse problem. If we compare the results of calculations of the Black Sea dynamics with the actually observed physical fields, it is necessary to take into account the form of the mean sea level more precisely in mathematical models, because this is the main component of the level function in this basin. It its turn, this requires a correct inclusion of the geoid surface or the mean level surface representations in the mathematical model.

Based on numerical investigation of the Black Sea dynamics taking into account tide-forming forces presented above, one can make a series of conclusions concerning the adequacy of the used mathematical models: (1) If the closeness of the ocean level functions is taken as the criterion, the mathematical model without tide-forming forces should be assumed inadequate for the description of the physical process. (2) If we use the mathematical model taking into account the tide-forming forces and the curvature of the geoid surface (or the mean level surface) and consider the level functions in the open part of the sea, the model can be assumed approximately adequate. If we consider the level functions in the neighbourhood of the shore lines and shelf zones, on 'shallow water' areas, the model is inadequate and its correction is required. (3) As was shown by numerical experiments, the influence of the inclusion of tide-forming forces on the temperature and salinity fields is insignificant (at least in the calculations for small periods, which is in accordance with the conclusions from [8]). (4) Probably, in order to get more complete effects of the inclusion of tide-forming forces, one should use numerical models of a higher grid resolution and also take into account the river runoff, describe shallow water parts of basins, etc. Moreover, it is expedient to apply assimilation procedures for the observation data obtained at the stations positioned in the Black Sea and the Sea of Azov, because altimetry data may have considerable errors in the shelf zones and in the basin of the Sea of Azov.

It is worth noting that the consideration of the curvature of the geoid (or mean level) surface as the reference surface in the mathematical model also influences the accuracy of calculations of the vertical flow velocity component. However, the discussion of this issue requires separate analysis and lies beyond the scope of this paper.

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