# On Vidal's trivalent explanations for defective conditional in mathematics

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**ABSTRACT**
The paper deals with a problem posed by Mathieu Vidal to provide a formal representation for defective conditional in mathematics (Vidal, 2014). The key feature of defective conditional is that its truth-value is indeterminate if its antecedent is false. In particular, we are interested in two explanations given by Vidal with the use of trivalent logics. By analyzing a simple argument from plane geometry, where defective conditional is in use, he gives two trivalent formal explanations for it. For both explanations, Vidal rigorously shows that (most well-known) trivalent logics cannot adequately represent defective conditional. Preserving Vidal's criteria of defective conditional *ad max*, we indicate some arguable points in his explanations and present an alternative explanation containing the original conjunction and disjunction in order to show that there are trivalent logics that might be an adequate formal explanation for defective conditional.

**KEYWORDS**
defective conditional; defective implication; conditional; implication; trivalent logic; three-valued logic; many-valued logic; non-classical logic.

## 1. Introduction

In his paper (Vidal, 2014), Vidal analyses a simple argument from mathematics (to be precise, from plane geometry) that is valid, according to classical logic, on one hand, and is obviously counter-intuitive or unacceptable to mathematical practice, on the other hand.[1] Let us quote this argument to make reading self-contained and refer the reader for more details to (Vidal, 2014, p. 169):

"**(0)** Let q be a quadrilateral.
**(1)** For every quadrilateral, if it is a rhombus and a rectangle, it is a square.
**(∴)** ————————————————————————
**(2)** If q is a rhombus, then q is a square or if q is a rectangle, then q is a square."

This argument is valid from the standpoint of classical logic and yet is "highly counter-intuitive" because "in Euclidean geometry, if a quadrilateral is a rhombus and

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[1]Note that the correspondence between proofs in mathematical practice and the ways they are represented by formal proofs in logic has been fruitfully discussing in the literature. See the status praesens in (Hamami, 2018).

a rectangle, then it is a square. However, the possession of only one of the properties 'rhombus' and 'rectangle' does not ensure that of the property 'square', as stated by (2)" (Vidal, 2014, p. 169). The scheme $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ is responsible, in Vidal's view, for drawing such an unwanted conclusion.[2]

In the paper we focus on the part of Vidal's work containing two explanations that deals with *trivalent* logics only. We stress the fact that the purpose of this paper isn't to find some gaps in Vidal's approach. Rather, based on indicating some arguable points in his presentation, its purpose is to provide an alternative explanation which leads to a positive conclusion, contrary to Vidal's negative verdict which is that most well-known trivalent logics can't adequately represent defective conditional. At least, through his paper he has never claimed a formal representation can't be given with the use of some trivalent logic. The existence of such an alternative explanation, again, isn't a gap in his paper. Such a possibility is believed to lie in the nature of defective conditional itself. To put it specifically, this notion is vague, and, hence, allows several equitable explanations. See the list of references in Vidal's paper for the details.[3]

We, however, are inspired by Vidal's explanations and want to stick to them and their notation *ad max.* It's not the case that he explicates every criterion, and sometimes the criterion is given implicitly. So, we will support our guesses in such cases with quotations and discuss alternative, parallel interpretations.

The paper is organised as follows. In section 2, we discuss Vidal's criteria for the logics which can solve his problem. Besides, in section 2.1 and 2.2, respectively, we discuss his first and second explanations. Section 3 is devoted to our own explanation of Vidal's problem. Section 4 deals with concluding remarks.

## 2. The criteria of both Vidal's explanations

We start out with the criteria that both explanations share and then discuss the criteria that each explanation shares separately. Possibly, due to the vague character of defective conditional, Vidal doesn't give a strict definition of it. The general argumentation form of his two formal explanations for defective conditional (Proposition 1 and Proposition 2, respectively) is *ad absurdum.* Suppose that there exist some trivalent logics that can formally represent defective reasoning. Hence, defective connectives as well as the relation of logical consequence in these logics must have some specific properties. Then Vidal gives two formal proofs that $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ is valid in these logics. Therefore, the logics in question aren't defective. In this paper, we don't want to give any definition of defective logic, too, and we analyse the crucial criteria of two explanations given by Vidal with indicating arguable and discussable points.

The 1st criterion says that the inference scheme $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ must be invalid. As illustrated above with the argument from plane geometry, this inference scheme leads to a mathematically counter-intuitive conclusion. We indicate that it's not clear whether this inference scheme is unique with respect to characterizing defective conditional or there are other inference schemes that lead to the above-

[2] We indicate that by "the possession of *only one* of the properties" Vidal interprets inclusive disjunction in a clear-cut *exclusive* sense. Exclusive disjunction denoted here by $\veebar$ is never under discussion in Vidal's paper. It's likely to be explainable by the fact that the inference scheme $(A \wedge B) \rightarrow C \models (A \rightarrow C) \veebar (B \rightarrow C)$ is invalid even classically.

[3] We may add that in theoretical computer science the term 'defective logic' often refers to a hardware error, for example, 'a defective logic chip' etc. (Menon and Sinha, 2014; Pan and Breuer, 2007).

mentioned or some other mathematically counter-intuitive conclusions, too. Quite obviously, $(A \wedge B) \to C \models (A \to C) \vee (B \to C)$ isn't unique. For example, contraposition inferences are sometimes thought to be suspicious (Jacquette, 2000). We, however, find this topic being beyond this paper's scope. On the other hand, we indicate that this inference scheme might be derivable in some proof system with a certain basic set of inference schemes. Does it mean that, by transitivity, any basic inference scheme occurring in a formal proof of $(A \wedge B) \to C \models (A \to C) \vee (B \to C)$ must also be mathematically counter-intuitive just on the basis of the fact that it leads to the inference scheme that was originally shown to lead to the unacceptable conclusion? We suggest it's quite obviously, too, that the positive answer to this question would make mathematically counter-intuitive almost all inference schemes. Again, we repeat the mantra that this methodological question goes beyond this paper's scope.

The 2nd criterion says that defective conditional is the one, where its truth-value is indeterminate if its antecedent is false (Vidal, 2014, p. 169). Denoted by "I", the indeterminate value is a truth-value gap (Vidal, 2014, p. 173). Such an interpretation of the third value (in contradistinction to the interpretation as, say, a truth-value glut) clearly presupposes that "I" isn't a designated value.[4]

The 3rd criterion deals with defective disjunction. Vidal gives two formulations of this criterion. The first one says that "we can eliminate theories where the result of a disjunction is indeterminate when it contains at least one indeterminate disjunct" (Vidal, 2014, p. 173). According to the second one, "we argued that a mathematician should consider the disjunction to be false if the first disjunct is false and the other one is indeterminate." (Vidal, 2014, p. 173). We indicate the formulations under consideration are not equivalent, namely, the first one is stronger. To prove it, suppose one of the disjuncts is true and the other is indeterminate. According to the first formulation, disjunction mustn't be indeterminate while the second formulation doesn't deal with this case. With regard to the cases, when one of the disjuncts is indeterminate and the other one is false, disjunction is false, under the second formulation, and mustn't be indeterminate, under the first formulation. Hence, both formulations deal with these cases.

The 4th criterion blocks the following interdefinability of defective connectives: $(A \to B) \leftrightarrow (\neg A \vee B)$ and $(A \to B) \leftrightarrow \neg(A \wedge \neg B)$ because both defective disjunction and conjunction are commutative while defective conditional is not. Without blocking, one loses the specifics of defective conditional, where *only one* of its components has a defective aspect while commutativity of defective disjunction/conjunction implies that *both* of its components has a defective aspect (Vidal, 2014, p. 172).

The 5th criterion manifestly underlies the previous one and says both conjunction and disjunction must be commutative (Vidal, 2014, p. 172).[5]

The 6th criterion implicitly follows from the 4th one. $(A \to B) \leftrightarrow (\neg A \vee B)$ and $(A \to B) \leftrightarrow \neg(A \wedge \neg B)$ hold iff their negation is Łukasiewicz-style negation, i.e. it satisfies the classical negation clauses (negation of T is F, and vice versa) added with

[4]Note that we don't mean here to unduly restrict the constraint of the first explanation that the set of designated values is an upper set of the set of values.

[5]We provide another argument in favour of commutativity of $\wedge$ and $\vee$. The inference scheme $(A \wedge B) \to C \models (A \to C) \vee (B \to C)$ is equivalent to $(B \wedge A) \to C \models (A \to C) \vee (B \to C)$. It is also equivalent to $(A \wedge B) \to C \models (B \to C) \vee (A \to C)$ and the other inference schemes obtainable via commutativity. We stress the fact that we neither mean the validity of $(A \wedge B) \to C \models (A \to C) \vee (B \to C)$ can't be treated in non-commutative logics nor we say that there are no logics, where the equivalencies in question are invalid. We just mean if one considers highly counter-intuitive the fact that the conclusion $(A \to C) \vee (B \to C)$ follows from the premise $(A \wedge B) \to C$, then she/he is also supposed to consider highly counter-intuitive the fact that, say, the conclusion $(B \to C) \vee (A \to C)$ follows from the premise $(B \wedge A) \to C$.

the clause that negation of I remains I.[6]

### *2.1. Vidal's first explanation*

The first feature of this explanation is that there exists the linear order T > I > F between the truth-values. It allows Vidal to define a truth-value of disjunction as the maximum among the truth-values of its disjuncts and a truth-value of conjunction as the minimum among the truth-values of its conjuncts, as usual. In fact, in the case of disjunction Vidal generalizes the traditional definition with the maximum and defines a truth-value of disjunction is to be *greater* or equal to the maximum among the truth-values of its disjuncts. We indicate that this generalisation allows, among other ones, such a strange *inclusive* disjunction that takes T when both of its disjuncts take F. To the best of our knowledge, there is no such disjunction in the literature. To be sure, a question what is it to be disjunction remains open in the literature, where some conceptual backgrounds could be found in supporting unconventional kinds of disjunction (e.g. non-commutative one (Abrusci and Ruet, 1999; Fitting, 1994)). However, let us remind the reader that the concern is mathematical practice, and we do doubt about the context containing mathematical arguments (at least, in plane geometry from the textbook), where *inclusive* disjunction satisfies this clause.[7]

There is another thing about defective disjunction worth to be discussed here. Defined with the maximum and the minimum over this order, defective disjunction is incompatible with both formulations occurring in the 3rd criterion. Let's recall the reader that the first one says that "we can eliminate theories where the result of a disjunction is indeterminate when it contains at least one indeterminate disjunct" (Vidal, 2014, p. 173). According to the second one, "we argued that a mathematician should consider the disjunction to be false if the first disjunct is false and the other one is indeterminate." (Vidal, 2014, p. 173). If a truth-value of disjunction to be greater or equal to the maximum among the truth-values of its disjuncts, then it's easy to see that the first formulation fails in the case, when one of the disjuncts is false, another one is indeterminate and disjunction itself is indeterminate. Moreover, the second formulation says it straight that disjunction must be false in this case.

The second feature of this explanation is that $D$ is an upper set of $\langle V, > \rangle$, where $D$ is a set of (designated) truth-values and $>$ is the linear order in the first feature. Hence, $A \models B$ if, for all possible interpretations, the truth-value of $A$ is less or equal to the truth-value of $B$ (Vidal, 2014, p. 174).

As a result, Vidal proves Proposition 1 to show that, for any trivalent logic $L$ so defined, the inference scheme $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ is $L$-valid. Therefore, $L$ can't be an adequate formal representation of defective conditional. We confine ourselves to the formulation of Proposition 1 only, and refer the reader for the proof of it to (Vidal, 2014). $f_c$ is the truth-function denoted by the connective $c$ belonging to the set $C$ of all connectives, $\rightarrow$ is defective conditional, and $[S]$ is the truth-value of a sentence $S$.

**Proposition 1.** (Vidal, 2014, p. 174). *Let $L$ be a many-valued logic $\langle V, D, f_c : c \in C \rangle$*

[6] This criterion excludes so called cyclic (Post-style) negation (Post, 1921) as well as *dual* cyclic negation (Petrukhin, 2018a). Moreover, it excludes Heyting's negation (Heyting, 1930) which transforms I to F as well as Heyting's dual negation which transforms I to T (Brunner and Carnielli, 2008).

[7] On the other hand, it's so mathematically, if one proves the desired result as a corollary of more general one. We thank the anonymous referee for attracting our attention to the fact that the advantage of such a general definition is that it concerns more potential definitions of disjunction, and among them the ones following the 3rd criterion and that with this more universal conception of disjunction, the proof applies to more systems.

*such that:*

(1) *There exists a linear order* $>$ *between the truth-values in* $V$.
(2) *If* $[A] \leq [B]$ *for all possible interpretations, then* $A \models_L B$.
(3) $[A \vee B] \geq max([A],[B])$; $[A \wedge B] = min([A],[B])$; $[A \rightarrow B] = f([A],[B])$.

*Then* $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ *is valid in* $L$.

### *2.2. Vidal's second explanation*

The second explanation starts with clear-cut changes in the definitions of both conjunction and disjunction occurring in the first explanation. Vidal uses the truth-tables for conjunction and disjunction that appear originally in (Ebbinghaus, 1969; Sobociński, 1952). Combining them and the original truth-table for defective conditional together with the logical consequence relation from the first explanation, Vidal seems to obtain the desired defective logic, where $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ is invalid. We indicate that the definition of disjunction satisfies the weak version of the 3rd criterion.[8]

However, Vidal says that this logic can't be a defective one because in the countermodel for $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$, the value of the premise $(A \wedge B) \rightarrow C$ is I and the value of the conclusion $(A \rightarrow C) \vee (B \rightarrow C)$ is F: "If we consider defective sentences to be irrelevant, a semantic consequence whose premises are defective is not conclusive" (Vidal, 2014, p. 174). Hence, the relation of logical consequence $A \models B$ is restricted to the case that $B$ is T, whenever $A$ is T.[9] With this restriction, $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ is valid. Moreover, Vidal imposes two other linear orders that are possible between T, I, F, viz, the order T $>_1$ F $>_1$ I to define disjunction as the maximum and the order F $>_2$ T $>_2$ I to define *conjunction as the maximum*. Note that the idea to impose various orders on truth-values within one logic or, to put it differently, to relativise an order over truth-values with respect to connectives is widely used in logics of generalized truth values (Grigoriev, 2016; Shramko, Dunn and Takenaka, 2001; Zaitsev, 2009). Hence, it'd be desirable to have some plausible background to justify it (for example, a justification of interconnections between defectiveness and the orders in question).

As the second result, Vidal proves Proposition 2 to show that, for any trivalent logic $L$ so defined, the inference scheme $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ is $L$-valid. Therefore, $L$ can't be an adequate formal representation of defective conditional. We, again, confine ourselves to the formulation of Proposition 2 only, and refer the reader for the proof of it to (Vidal, 2014). The notation is the same as in Proposition 1 above.

**Proposition 2.** (Vidal, 2014, p. 175) *Let* $L$ *be a trivalent logic* $\langle V, D, f_c : c \in C\rangle$ *with* $V = \{\mathrm{T}, \mathrm{I}, \mathrm{F}\}$ *such that:*

(1) *We have two linear orders:* T $>_1$ F $>_1$ I *and* F $>_2$ T $>_2$ I.
(2) *If, for all possible interpretations,* $[B] = \mathrm{T}$ *if* $[A] = \mathrm{T}$, *then* $A \models_L B$.
(3) $[A \vee B] \geq max_1([A],[B])$; $[A \wedge B] = max_2([A],[B])$; $[A \rightarrow B] = f([A],[B])$.

*Then* $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ *is valid in* $L$.

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[8] We recall the reader the definition of disjunction from the first explanation is shown to satisfy neither the weak, nor the strong versions of the 3rd criterion.

[9] Note, however, footnote 4.

## 3. Our explanation

In this section, we present our explanation of Vidal's problem. First of all, following Vidal's first explanation *ad max*, we present Proposition 3. Then we show that the combination of Vidal's both explanations gives some fruitful results (Proposition 4). Next we consider the relations of logical consequence which have not been viewed by Vidal, but either naturally arise from his second explanation (Propositions 5 and 6) or are common for many-valued logics (Proposition 7). Moreover, in Proposition 7, we deal with the relation of logical consequence which was used in Vidal's second explanation. Besides, in the case of Propositions 6 and 7 we consider the connectives which have not been studied by Vidal. Furthermore, we present Proposition 8 which can be viewed as one more alternative approach to Vidal's problem.

We propose the following truth tables for the main connectives. Some of them are introduced in the literature for different purposes while $\wedge_D$ and $\vee_D$ are original.

| $A$ | $\neg_K$ |
|---|---|
| T | F |
| I | I |
| F | T |

| $\vee_E$ | T | I | F |
|---|---|---|---|
| T | T | T | T |
| I | T | I | F |
| F | T | F | F |

| $\vee_D$ | T | I | F |
|---|---|---|---|
| T | T | I | T |
| I | I | I | F |
| F | T | F | F |

| $\vee_K$ | T | I | F |
|---|---|---|---|
| T | T | T | T |
| I | T | I | I |
| F | T | I | F |

| $\wedge_E$ | T | I | F |
|---|---|---|---|
| T | T | T | F |
| I | T | I | F |
| F | F | F | F |

| $\wedge_D$ | T | I | F |
|---|---|---|---|
| T | T | T | F |
| I | T | I | I |
| F | F | I | F |

| $\wedge_K$ | T | I | F |
|---|---|---|---|
| T | T | I | F |
| I | I | I | F |
| F | F | F | F |

| $\wedge_W$ | T | I | F |
|---|---|---|---|
| T | T | I | F |
| I | I | I | I |
| F | F | I | F |

The connectives $\neg_K$, $\vee_K$, and $\wedge_K$ are $\mathbf{K_3}$'s ones, where $\mathbf{K_3}$ is Kleene's strong logic (Kleene, 1938)[10] which is the $\{\neg, \vee, \wedge\}$-fragment of Łukasiewicz's logic $\mathbf{Ł_3}$ (Łukasiewicz, 1920). The connectives $\neg_K$, $\vee_E$, and $\wedge_E$ are $\mathbf{E}$'s ones, where $\mathbf{E}$ is Finn and Grigolia's nonsense logic (Finn and Grigolia, 1993) which is the $\{\neg, \vee, \wedge\}$-fragment of Ebbinghaus' logic $\mathbf{E_3}$ (Ebbinghaus, 1969). Note that $\vee_E$ and $\wedge_E$ were first studied in (Sobociński, 1952). However, the entailment relation of Sobociński's logic $\mathbf{S_3}$ differs from the one of $\mathbf{E_3}$. Moreover, $\neg_K$, $\vee_E$, and $\wedge_W$ are connectives of Hałkowska's (Hałkowska, 1989) nonsense logic $\mathbf{Z}$.[11] Furthermore, $\neg_K$, $\vee_W$, and $\wedge_W$, where $A \vee_W B = \neg_K(\neg_K A \wedge_W \neg_K B)$, are connectives of Kleene's weak logic $\mathbf{K_3^w}$ (Kleene, 1938)[12] which is a fragment of Bochvar's nonsense logic $\mathbf{B_3}$ (Bochvar, 1938). Besides, the notion of logical consequence in $L \in \{\mathbf{K_3}, \mathbf{Ł_3}, \mathbf{E}, \mathbf{E_3}, \mathbf{Z}, \mathbf{K_3^w}, \mathbf{B_3}\}$ is defined in the following way: $\Gamma \models_L A$ iff it holds that if, for each $G \in \Gamma$, $[G] = \mathrm{T}$, then $[A] = \mathrm{T}$, for all possible interpretations. To the best of our knowledge, $\vee_D$ and $\wedge_D$ have not been mentioned in the literature before. Let us call them *defective disjunction* and *defective conjunction*, respectively. Note that $A \vee_D B = \neg_K(\neg_K A \wedge_D \neg_K B)$ and $A \wedge_D B = \neg_K(\neg_K A \vee_D \neg_K B)$.

Moreover, let us define the following classes $\mathfrak{DC}$ and $\mathfrak{DC}_R$ ($R$ stands for *restricted*) of defective conditionals such that $\mapsto \in \mathfrak{DC}$ and $\mapsto_R \in \mathfrak{DC}_R$, if $\mapsto$ and $\mapsto_R$ are defined

[10](Petrukhin and Shangin, 2018) is devoted to automated proof searching for all $\mathbf{K_3}$'s truth-functional binary extensions.

[11](Petrukhin, 2018b) is devoted to the presentation of the natural deduction systems for $\mathbf{E}$ and $\mathbf{Z}$.

[12]In (Petrukhin, 2017), a natural deduction system for $\mathbf{K_3^w}$ is presented.

as follows, where $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \{\mathrm{T}, \mathrm{I}, \mathrm{F}\}$:[13]

| $\mapsto$ | T | I | F |
|---|---|---|---|
| T | T | $a_1$ | F |
| I | $a_2$ | $a_3$ | $a_4$ |
| F | I | I | I |

| $\mapsto_R$ | T | I | F |
|---|---|---|---|
| T | T | $a_5$ | F |
| I | $a_6$ | $a_7$ | T |
| F | I | I | I |

The classes $\mathfrak{DC}$ and $\mathfrak{DC}_R$ have 81 and 27 elements, respectively.

Let us, following Vidal's *first* explanation *ad max*, present some logics which solve his problem. We generalise the notion of logical consequence on the case of sets, not just formulae. Besides, we don't define disjunction, in contrast to Vidal's Proposition 1, as $\vee_K$, but as $\vee_E$. Actually, we stress, again, the fact that disjunction in Vidal's Proposition 1 does not satisfy his 3rd criteria.

**Proposition 3.** *Let $L$ be a trivalent logic $\langle V, D, f_c \colon c \in C\rangle$ such that*

(1) $\mathrm{T} > \mathrm{I} > \mathrm{F}$.
(2) $\Gamma \models_L A$ *iff it holds that, for each $G \in \Gamma$, $[G] \leq [A]$, for all possible interpretations. If $\Gamma$ is empty, then $\models_L A$ iff it holds that, for each formula $F$, $[F] \leq [A]$, for all possible interpretations.*[14]
(3) $[A \vee B] = [A \vee_E B]$, $[A \wedge B] = [A \wedge_K B] = min([A], [B])$, $[A \to B] = [A \mapsto B]$, *where* $\mapsto \in \mathfrak{DC}$, $[A \leftrightarrow B] = [(A \to B) \wedge (B \to A)]$, $[\neg A] = [\neg_K A]$.

*Then it holds that*:

(*a*) $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$ (*1st criteria*).
(*b*) $\to$ *is defective* (*2nd criteria*).
(*c*) $\vee$ *satisfies the 3rd criteria in its weak version.*[15]
(*d*) $\not\models_L (\neg p \vee q) \leftrightarrow (p \to q)$ *and* $\not\models_L \neg(p \wedge \neg q) \leftrightarrow (p \to q)$ (*4th criteria*).
(*e*) *Both* $\vee$ *and* $\wedge$ *are commutative* (*5th criteria*).
(*f*) $\neg$ *satisfies the 6th criteria.*[16]

***Proof.*** (*a*) Consider an interpretation such that $[p] = \mathrm{T}$, $[q] = [r] = \mathrm{F}$. Then $[p \wedge q] = \mathrm{F}$. Since $\to$ is defective conditional, so $[(p \wedge q) \to r] = \mathrm{I}$. Moreover, since $[p] = \mathrm{T}$ and $[r] = \mathrm{F}$, so $[p \to r] = \mathrm{F}$. Besides, since $[q] = \mathrm{F}$ and $\to$ is defective conditional, so $[q \to r] = \mathrm{I}$. Therefore, $[(p \to r) \vee (q \to r)] = \mathrm{F}$. Thus, $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$.[17]

(*b*) Immediately follows from the definitions of the class $\mathfrak{DC}$ and defective conditional.

(*c*) Immediately follows from the definition of $\vee_E$.

[13] Note that Vidal allows $\to$ to be any binary connective. We require $\to$ to be a defective implication, i.e. $[A \to B] = \mathrm{I}$ if $[A] = \mathrm{F}$. Besides, we require $\to$ to be classical conditional when $[A] = [B] = \mathrm{T}$ and when both $[A] = \mathrm{T}$ and $[B] = \mathrm{F}$. As a result, we obtain the class $\mathfrak{DC}$. In the case of $\mathfrak{DC}_R$, we have one more condition: if $[A] = \mathrm{I}$ and $[B] = \mathrm{F}$, then $[A \to B] = \mathrm{T}$. We need this restriction for the technical reasons only in the case of Propositions 5-7 below.

[14] Note that since T is the maximum value, if $[A] = \mathrm{T}$, then, for an arbitrary formula $\varphi$, it holds that $[\varphi] \leq [A]$, for all possible interpretations.

[15] "We argued that a mathematician should consider the disjunction to be false if the first disjunct is false and the other one is indeterminate." (Vidal, 2014, p. 173).

[16] We thank the anonymous referee for attracting our attention to the fact that $\neg p \vee \neg q \models_L \neg(p \wedge q)$ and $\neg p \wedge \neg q \models_L \neg(p \vee q)$ while $\neg(p \wedge q) \not\models_L \neg p \vee \neg q$ and $\neg(p \vee q) \not\models_L \neg p \wedge \neg q$.

[17] Note that in this proof, we have not looked over all the logics for which this Proposition holds, because we have not dealt with the cases when $[A \to B] = a_i$, where $1 \leqslant i \leqslant 4$ and $a_i \in \{\mathrm{T}, \mathrm{I}, \mathrm{F}\}$ (see the definition of the class $\mathfrak{DC}$). However, we have found a valuation such that $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$, for each logic $L$ in question.

($d$) Consider an interpretation such that $[p] = \mathrm{T}$ and $[q] = \mathrm{F}$. Then $[p \to q] = \mathrm{F}$, $[\neg p \vee q] = \mathrm{F}$, and $[\neg(p \wedge \neg q)] = \mathrm{F}$. Then $[(p \to q) \to (\neg p \vee q)] = \mathrm{I}$, $[(\neg p \vee q) \to (p \to q)] = \mathrm{I}$, $[(p \to q) \to \neg(p \wedge \neg q)] = \mathrm{I}$, and $[\neg(p \wedge \neg q) \to (p \to q)] = \mathrm{I}$. Therefore, $[((\neg p \vee q) \to (p \to q)) \wedge ((p \to q) \to (\neg p \vee q))] = \mathrm{I}$. Thus, $[(\neg p \vee q) \leftrightarrow (p \to q)] = \mathrm{I}$. Similarly, $[\neg(p \wedge \neg q) \leftrightarrow (p \to q)] = \mathrm{I}$. Besides, $[\neg(p \to q)] = \mathrm{T}$. Hence, $[\neg(p \to q)] \nleq [(\neg p \vee q) \leftrightarrow (p \to q)]$ and $[\neg(p \to q)] \nleq [\neg(p \wedge \neg q) \leftrightarrow (p \to q)]$. Thus, it's *not* the case that, for each formula $F$ and all possible interpretations, $[F] \leq [(\neg p \vee q) \leftrightarrow (p \to q)]$ and $[F] \leq [\neg(p \wedge \neg q) \leftrightarrow (p \to q)]$. Thus, $\not\models_L (\neg p \vee q) \leftrightarrow (p \to q)$ and $\not\models_L \neg(p \wedge \neg q) \leftrightarrow (p \to q)$.

($e$) By a routine proof, $A \vee B \models_L B \vee A$ and $A \wedge B \models_L B \wedge A$.[18]

($f$) Immediately follows from $[\neg_K A]$'s definition. □

Nothing prevents us from combining two Vidal's approaches. We take all the conditions from Vidal's second approach except the definition of logical consequence which we take from Vidal's first approach.

**Proposition 4.** *Let $L$ be a trivalent logic $\langle V, D, f_c \colon c \in C\rangle$ such that*

(1) $\mathrm{T} > \mathrm{I} > \mathrm{F}$.
(2) $\Gamma \models_L A$ *iff it holds that, for each $G \in \Gamma$, $[G] \leq [A]$, for all possible interpretations. If $\Gamma$ is empty, then $\models_L A$ iff it holds that, for each formula $F$, $[F] \leq [A]$, for all possible interpretations.*
(3) $[A \vee B] = [A \vee_E B]$, $[A \wedge B] = [A \wedge_E B]$, $[A \to B] = [A \mapsto B]$, *where* $\mapsto \in \mathfrak{DC}$, $[A \leftrightarrow B] = [(A \to B) \wedge (B \to A)]$, $[\neg A] = [\neg_K A]$.

*Then it holds that*:

($a$) $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$ (*1st criteria*).
($b$) $\to$ *is defective* (*2nd criteria*).
($c$) $\vee$ *satisfies the 3rd criteria in its weak version.*
($d$) $\not\models_L (\neg p \vee q) \leftrightarrow (p \to q)$ *and* $\not\models_L \neg(p \wedge \neg q) \leftrightarrow (p \to q)$ (*4th criteria*).
($e$) *Both $\vee$ and $\wedge$ are commutative* (*5th criteria*).
($f$) $\neg$ *satisfies the 6th criteria.*

***Proof.*** ($a$) Consider an interpretation such that $[p] = [r] = \mathrm{F}$ and $[q] = \mathrm{T}$.
The other cases are proved similarly to the previous propositions. □

Now we try to follow Vidal's second approach *ad max*, but we define the notion of logical consequence via $\leq_1$ and $\leq_2$. We start with $\leq_2$, because in this case we need to restrict the class $\mathfrak{DC}$ only. In the case of $\leq_1$, we will need more changes to be made.

**Proposition 5.** *Let $L$ be a trivalent logic $\langle V, D, f_c \colon c \in C\rangle$ such that*

(1) $\mathrm{F} >_2 \mathrm{T} >_2 \mathrm{I}$.
(2) $\Gamma \models_L A$ *iff it holds that, for each $G \in \Gamma$, $[G] \leq_2 [A]$, for all possible interpretations. If $\Gamma$ is empty, then $\models_L A$ iff it holds that, for each formula $F$, $[F] \leq_2 [A]$, for all possible interpretations.*[19]

[18]We use the computer program *MaTest* (González, 2012) to check this assertion.

[19]Such a definition of the semantic consequence seems to be an unusual one. For instance, the interpretation, where $[G] = \mathrm{T}$ and $[A] = \mathrm{F}$, is not a counter-model to $G \models A$. On the other hand, $>_2$ itself is an unusual order of truth values, since $\mathrm{F} >_2 \mathrm{T}$. But $>_2$ is treated in (Vidal, 2014). Moreover, the definition of semantic consequence via an order of truth values is one of standard approaches to semantic consequence in many-valued logic.

(3) $[A \vee B] = [A \vee_E B]$, $[A \wedge B] = [A \wedge_E B] = max_2([A], [B])$, $[A \to B] = [A \mapsto_R B]$, *where* $\mapsto_R \in \mathfrak{DC}_R$, $[A \leftrightarrow B] = [(A \to B) \wedge (B \to A)]$, $[\neg A] = [\neg_K A]$.

*Then it holds that*:

(a) $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$ *(1st criteria).*
(b) $\to$ *is defective (2nd criteria).*
(c) $\vee$ *satisfies the 3rd criteria in its weak version.*
(d) $\not\models_L (\neg p \vee q) \leftrightarrow (p \to q)$ *and* $\not\models_L \neg(p \wedge \neg q) \leftrightarrow (p \to q)$ *(4th criteria).*
(e) *Both* $\vee$ *and* $\wedge$ *are commutative (5th criteria).*
(f) $\neg$ *satisfies the 6th criteria.*

***Proof.*** (a) Consider an interpretation such that $[p] = \mathrm{T}$, $[q] = \mathrm{I}$, $[r] = \mathrm{F}$.
The other cases are proved similarly to the previous propositions. □

Now let's turn to the abovementioned $\leq_1$. In contrast to the case of $\leq_2$, we need to change the definitions of disjunction and conjunction also.

**Proposition 6.** *Let* $L$ *be a trivalent logic* $\langle V, D, f_c \colon c \in C\rangle$ *such that*

(1) $\mathrm{T} >_1 \mathrm{F} >_1 \mathrm{I}$.
(2) $\Gamma \models_L A$ *iff it holds that, for each* $G \in \Gamma$, $[G] \leq_1 [A]$, *for all possible interpretations. If* $\Gamma$ *is empty, then* $\models_L A$ *iff it holds that, for each formula* $F$, $[F] \leq_1 [A]$, *for all possible interpretations.*
(3) $[A \vee B] = [A \vee_D B]$, $[A \wedge B] = [A \wedge_D B]$, $[A \to B] = [A \mapsto_R B]$, *where* $\mapsto_R \in \mathfrak{DC}_R$, $[A \leftrightarrow B] = [(A \to B) \wedge (B \to A)]$, $[\neg A] = [\neg_K A]$.[20]

*Then it holds that*:

(a) $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$ *(1st criteria).*
(b) $\to$ *is defective (2nd criteria).*
(c) $\vee$ *satisfies the 3rd criteria in its weak version.*
(d) $\not\models_L (\neg p \vee q) \leftrightarrow (p \to q)$ *and* $\not\models_L \neg(p \wedge \neg q) \leftrightarrow (p \to q)$ *(4th criteria).*
(e) *Both* $\vee$ *and* $\wedge$ *are commutative (5th criteria).*
(f) $\neg$ *satisfies the 6th criteria.*

***Proof.*** (a) Consider an interpretation such that $[p] = \mathrm{I}$, $[q] = [r] = \mathrm{F}$.
The other cases are proved similarly to the previous propositions. □

Now we consider one of the most popular definitions of logical consequence in many-valued logic, when I is not a designated value, i.e. we define the notion of logical consequence via the preservation of the only designated value T. Recall that this definition is used in the abovementioned logics $\mathbf{K_3}$, $\mathbf{Ł_3}$, $\mathbf{E}$, $\mathbf{E_3}$, $\mathbf{Z}$, $\mathbf{K_3^w}$, and $\mathbf{B_3}$. Moreover, this definition of logical consequence is used in Vidal's second approach.

**Proposition 7.** *Let* $L$ *be a trivalent logic* $\langle V, D, f_c \colon c \in C\rangle$ *such that*

(1) $\Gamma \models_L A$ *iff it holds that if, for each* $G \in \Gamma$, $[G] = \mathrm{T}$, *then* $[A] = \mathrm{T}$, *for all possible interpretations. If* $\Gamma$ *is empty, then* $\models_L A$ *iff* $[A] = \mathrm{T}$, *for all possible interpretations.*

[20] In this Proposition, we use conjunction that, as far as we know, has not been mentioned before in the literature. However, we can replace it with the more familiar one, i.e. $[A \wedge B] = [A \wedge_W B]$, and, still, keep the same results as well as their proofs.

(2) $[A \vee B] = [A \vee_D B]$, $[A \wedge B] = [A \wedge_D B]$, $[A \to B] = [A \mapsto_R B]$, *where* $\mapsto_R \in \mathfrak{DC}_R$, $[A \leftrightarrow B] = [(A \to B) \wedge (B \to A)]$, $[\neg A] = [\neg_K A]$.[21]

*Then it holds that*:

(*a*) $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$ (*1st criteria*).
(*b*) $\to$ *is defective* (*2nd criteria*).
(*c*) $\vee$ *satisfies the 3rd criteria in its weak version.*
(*d*) $\not\models_L (\neg p \vee q) \leftrightarrow (p \to q)$ *and* $\not\models_L \neg(p \wedge \neg q) \leftrightarrow (p \to q)$ (*4th criteria*).
(*e*) *Both* $\vee$ *and* $\wedge$ *are commutative* (*5th criteria*).
(*f*) $\neg$ *satisfies the 6th criteria.*

***Proof.*** (*a*) Consider an interpretation such that $[p] = \mathrm{I}$, $[q] = [r] = \mathrm{F}$.
The other cases are proved similarly to the previous propositions. □

Now let us present one more explanation of Vidal's problem. We follow Vidal's *second* explanation *ad max*, except the definition of the logical consequence and the 5th criteria (commutativity of conjunction and disjunction).

**Proposition 8.** *Let* $L$ *be a trivalent logic* $\langle V, D, f_c \colon c \in C\rangle$ *such that*

(1) *There are two linear orders:* $\mathrm{T} >_1 \mathrm{F} >_1 \mathrm{I}$ *and* $\mathrm{F} >_2 \mathrm{T} >_2 \mathrm{I}$.
(2) $\Gamma \models_L A$ *iff it holds that, for each* $G \in \Gamma$, $[G] = [A] = \mathrm{T}$, *for all possible interpretations. If* $\Gamma$ *is empty, then* $\models_L A$ *iff* $[A] = \mathrm{T}$, *for all possible interpretations.*
(3) $[A \vee B] = [A \vee_E B] = max_1([A], [B])$, $[A \wedge B] = [A \wedge_E B] = max_2([A], [B])$, $[A \to B] = [A \mapsto B]$, *where* $\mapsto \in \mathfrak{DC}$, $[A \leftrightarrow B] = [(A \to B) \wedge (B \to A)]$, $[\neg A] = [\neg_K A]$.[22]

*Then it holds that*:

(*a*) $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$ (*1st criteria*).
(*b*) $\to$ *is defective* (*2nd criteria*).
(*c*) $\vee$ *satisfies the 3rd criteria in its weak version.*
(*d*) $\not\models_L (\neg p \vee q) \leftrightarrow (p \to q)$ *and* $\not\models_L \neg(p \wedge \neg q) \leftrightarrow (p \to q)$ (*4th criteria*).
(*e*) *Both* $\vee$ *and* $\wedge$ *are **not** commutative* (*5th criteria*).
(*f*) $\neg$ *satisfies the 6th criteria.*

***Proof.*** (*a*) Consider an interpretation such that $[p] = [q] = [r] = \mathrm{F}$. Then $[p \wedge q] = \mathrm{F}$, $[p \to r] = \mathrm{I}$, and $[q \to r] = \mathrm{I}$. Then $[(p \wedge q) \to r] = \mathrm{I}$ and $[(p \to r) \vee (q \to r)] = \mathrm{I}$. Since $[(p \wedge q) \to r] \neq \mathrm{T}$ and $[(p \to r) \vee (q \to r)] \neq \mathrm{T}$, so $(p \wedge q) \to r \not\models_L (p \to r) \vee (q \to r)$.

(*d*) Consider an interpretation such that $[p] = \mathrm{T}$ and $[q] = \mathrm{F}$. Then $[(\neg p \vee q) \leftrightarrow (p \to q)] = \mathrm{I}$ and $[\neg(p \wedge \neg q) \leftrightarrow (p \to q)] = \mathrm{I}$ (see the proof of Proposition 3). Thus, $\not\models_L (\neg p \vee q) \leftrightarrow (p \to q)$ and $\not\models_L \neg(p \wedge \neg q) \leftrightarrow (p \to q)$.

(*e*) Consider an interpretation such that $[p] = [q] = \mathrm{F}$. Then $[p \wedge q] = [p \vee q] = \mathrm{F}$. Thus, $A \vee B \not\models_L B \vee A$ and $A \wedge B \not\models_L B \wedge A$.

Proofs of (*b*), (*c*), and (*f*) are the same as in the case of Proposition 3. □

[21] As in the case of the previous Proposition we can, still, keep all the results, if we define conjunction in the following way: $[A \wedge B] = [A \wedge_W B]$.
[22] Let us define conjunction as in Proposition 3, i.e. $[A \wedge B] = [A \wedge_K B]$. However, in this case the same results and proof hold.

## 4. Conclusion

In this paper, we proposed several classes of logics which solve Vidal's problem and can be viewed as suitable candidates for defective reasoning. A further research can be devoted to searching for alternative many-valued explanations of defective logic. Besides, the combination of defective and relevant logics promises fruitful results. Vidal (2014) writes that in some relevant logics with intensional implication $(A \wedge B) \rightarrow C \models (A \rightarrow C) \vee (B \rightarrow C)$ isn't valid. Moreover, he suggests to consider four-valued relevant logics in order to find a solution to his problem.

We would like to mention one more issue. There are two folklore types of paradoxes of classical conditional and logical consequence which motivated the development of relevant logic. The first type of paradoxes consists of certain formulae, for example, $A \rightarrow (B \rightarrow A)$, $(A \wedge \neg A) \rightarrow B$, $B \rightarrow (A \vee \neg A)$ etc. The second type consists of formulae of the type $p \rightarrow q$, where $p$ and $q$ don't have any common content or causal relationship.[23] For example, in classical logic, the statement "If $2+2=5$, then Moscow is a big city" is true, although, intuitively, it is a nonsense. Relevant logic usually deals with paradoxes of the first type and has some problems with the second type. The well-known requirement that in formulae of the type $A \rightarrow B$ formulae $A$ and $B$ should have at least one common propositional variable, even excludes from consideration formulae of the type $p \rightarrow q$. However, it's not always the case that $p \rightarrow q$ is a paradoxical sentence. For example, the sentence "If I know the Pythagorean theorem, then I know how to calculate the hypotenuse via the other two sides" has the form $p \rightarrow q$, but it's not a paradoxical one. In the case of defective logic, the sentence "If $2+2=5$, then Moscow is a big city" is neither true nor false, since it's antecedent is false. Moreover, the sentence "If I know the Pythagorean theorem, then I know how to calculate the hypotenuse via the other two sides" is true. It seems that defective logic can solve some problems which are in "possession" of relevant logic, but relevant logic fails to solve them. The question is whether defective relevant logic will manage to solve the paradoxes of the first and the second types?

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[23] Formulae of the second type are mentioned in the beginning of logical textbooks to illustrate the fact that the classical truth-table definition for conditional doesn't fully capture its intuitive usage (see, for example, (Mendelson, 1997, p. 12)).

of the classical extended functional calculus. *History and Philosophy of Logic*, *2*(1-2), 87–112.[24]

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[24]The English translation mistakenly dates Bochvar's original paper with the year 1937.