Algebra, Geometry and Analysis of Commuting Ordinary Differential Operators

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1 Introduction

There are two classical problems related to integrable systems, appeared and studied already in the works of I. Schur, J. Burchnall, T. Chaundy in the beginning of 20th century: how to construct explicitly a pair of commuting differential operators and how to classify all commutative subalgebras of differential operators. Both problems have broad connections with many branches of modern mathematics, first of all with integrable systems, since explicit examples of commuting operators provide explicit solutions of many non-linear partial differential equations. The theory of commuting differential operators is far to be complete, but it is well developed for commuting *ordinary differential operators*.

This course involves an explanation of basic ideas and constructions from the theory of commuting ordinary differential operators as well as an overview of related open problems from algebra, algebraic geometry and complex analysis.

We meet ordinary differential operators every time when we want to solve a *linear* differential equation:

$$(a_n\partial^n + \ldots + a_0)\psi = 0,$$

where a_i, ψ are (usually) smooth functions, and even *non-linear* equations.

Consider a ring $R = C^{\infty}(\mathbb{R})$ of smooth (or analytic) functions on the line (or on a open neighbourhood of zero), denote $\partial := \partial/\partial x$. For any function $f \in R$ denote by $\hat{f}$ the *operator* of multiplication on f in R: $\hat{f}(g) := f \cdot g$. Then the Leibniz rule

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

is equivalent to the equality of *operators*: $\partial\hat{f} = \widehat{f'} + \hat{f}\partial$. Later we will omit the sign $\hat{}$ to simplify the notation.

Example 1.1. Consider two operators:

$$L = \partial^2 + \hat{u}, \quad P = 4\partial^3 + 6\hat{u}\partial + 3\widehat{u'}.$$

Then $[P, L] = 6\widehat{uu'} + \widehat{u'''}$ (check it! Hint: in order to check it, apply $P \circ L$ and $L \circ P$ to a test function $\varphi \in R$. Then the following equation must hold: $(P \circ L - L \circ P)\varphi = (u''' + 6uu')\varphi$). Now if we take $u = u(x,t)$, where t is a new variable, and set $\frac{\partial}{\partial t}(\partial) = 0$, we obtain a famous non-linear equation of mathematical physics, the Korteweg de Vries equation: $u_t = 6uu_x + u_{xxx}$. Namely, the equation

$$\frac{\partial L}{\partial t} = [P, L]$$

is equivalent to it.

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First explicit examples appeared already in 1903 in the work of Wallenberg [30]:

Example 1.2. Let $\Lambda \simeq \mathbb{Z}^2 \subset \mathbb{C}$ be a lattice and

$$\wp(x) = \sum_{\lambda \in \Lambda \setminus \{0,0\}} \left(\frac{1}{(x+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

be the corresponding Weierstrass function (a meromorphic function on a torus, we'll return to special functions later). Wallenberg observed that the ordinary differential operators L, P from previous example with $u(x) = -2\wp(x+\alpha)$, $\alpha \in \mathbb{C}$, or with $u(x) = -2/(x+\alpha)^2$ or with $u(x) = -2/\sin^2(x+\alpha)$ (degenerations of $\wp(x)$), commute.

I. Schur in 1905 and Burchnall, Chaundy in 1920-th got more examples of operators of relatively prime orders (Burchnall and Chaundy even classified such pairs).

In 1968 Dixmier discovered another interesting example [10]: for any $\lambda \in \mathbb{C}$ put $Q = \partial^2 + x^3 + \lambda$ and consider operators

$$L = Q^2 + 2x, \quad P = 2Q^3 + 3(Qx + xQ).$$

Then L and P commute and satisfy the relation $Q^2 = P^3 - \lambda$.

Two problems mentioned in the beginning appear to be connected with many problems from different branches of mathematics (search for them in internet), e.g.:

- Complex analysis (the Schottky problem, solved)
- Non-linear partial differential equations (find new exact solutions)
- Deformation quantisation
- Algebra (the Dixmier or Jacobian or Poisson conjectures, highly non-trivial and still open)

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2 Lectures guide and List of notations

The lectures consist of Theorems, Propositions, Lemmas, Remarks, Exercises and Comments. The comments are not necessary for the first reading, but they contains useful information for curious or advanced readers. We tried to keep the exposition of our lectures as self-contained as it is possible. The material is based upon various texts from references.

Calligraphic letters denote generic algebras and rings. Usual capital letters denote commutative algebras and rings. Almost always the letter D (combined together with various indices) is reserved to denote rings of ordinary differential operators. Recall the following commonly used definitions.

Definition 2.1. Let K be a field. An *algebra* over K or K-algebra is a vector space over K equipped with a bilinear product $\cdot$. A K-linear map $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ is a *homomorphism* of algebras if $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \ \ \forall x, y \in \mathcal{A}_1$.

A *Lie algebra* over K is an algebra with a product usually denoted by $[,]$ that satisfies the axioms of alternativity, i.e. $[y, y] = 0 \ \ \forall y$, and the Jacoby identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

A *ring* over K is an associative K-algebra with the multiplicative unity 1. In particular, the ring over K contains a copy of K. A K-linear map $\varphi : \mathcal{R}_1 \to \mathcal{R}_2$ is a *homomorphism* of rings if it is a homomorphism of algebras and $\varphi(1) = 1$.

A left (right) $\mathcal{R}$-module M, where $\mathcal{R}$ is a ring, is an additive group with the left (right) action of $\mathcal{R}$: $\mathcal{R}\times M \to M$ ($M\times\mathcal{R}\to M$) with usual axioms. Homomorphisms of $\mathcal{R}$-modules are defined as linear maps compatible with the action of the ring.

A left (right) ideal I in $\mathcal{R}$ is an abelian subgroup such that $ry\in I$ ($yr\in I$) $\forall r\in\mathcal{R}$, $\forall y\in I$. A (two-sided) ideal is a left and right ideal.

A ring $\mathcal{R}$ is an *integral domain* is it contains no non-zero zero divisors, i.e. $xy\neq 0$ for all non-zero $x,y\in\mathcal{R}$.

If M is a $\mathcal{R}$-module (left or right), then $1\in\mathcal{R}$ acts trivially on M. An R-module M is of finite type if it is generated by finitely many elements, that is, if there exist $a_1,\dots,a_n\in M$ such that any $y\in M$ can be written as $y=r_1a_1+\ldots+r_na_n$ for some $r_i\in\mathcal{R}$. If a module A over a ring R also has a ring structure (compatible with that of R in the sense that the map $R\to A$ given by $r\mapsto r\cdot 1_A$ is a ring homomorphism), then A is called an R-algebra. An R-algebra A is of finite type (or finitely generated) if there exist $a_1,\dots,a_n\in A$ such that any $y\in A$ can be written as a polynomial in $a_1,\dots,a_n$ with coefficients in R.

An ideal I is called *principal* if it is generated by one element: $I=(y)$.

Definition 2.2. Let $\mathcal{R}$ be a ring. By a *discrete valuation* on $\mathcal{R}$ we will understand a function v on $\mathcal{R}$ with values in $\mathbb{Z}\cup\infty$ ($\mathbb{Z}\cup\infty$ form a monoid with the operation $y+\infty=\infty+y=\infty$ for all $y\in\mathbb{Z}\cup\infty$) subject to the conditions:

1. $v(y)\in\mathbb{Z}\cup\infty$ and v assumes at least two values,
2. $v(xy)=v(x)+v(y)$,
3. $v(x+y)\geq\min\{v(x),v(y)\}$

The set

$$\ker v=\{y\in\mathcal{R}|v(y)=\infty\}$$

is easily verified to be an ideal of $\mathcal{R}$, which is proper by (3). If $\ker v=0$, v is said to be *proper*; e.g. on a field every valuation is proper, because 0 is the only proper ideal. If $\mathcal{R}$ is a ring over K we will consider discrete K-valuations, i.e. discrete valuations trivial on K: $v(K)=0$. In our lectures we'll meet only proper discrete K-valuations. In general it follows easily from the conditions above that $v(1)=v(-1)=0$ and that $v(-y)=v(y)$ $\forall y\in\mathcal{R}$.

If we have a discrete valuation v, we can define a metric on $\mathcal{R}/\ker v$ by choosing a real constant c between 0 and 1 and defining

$$d(x,y)=c^{v(x-y)}.$$

It is easily verified, using the conditions above, that the usual axioms of a metric hold, and moreover $d(x+a,y+a)=d(x,y)$ (check it). Thus, if v is proper, $\mathcal{R}$ becomes a topological ring with a Hausdorff topology. As with every metric space, one can form the completion of R, which plays an important role in commutative ring theory.

Exercise 2.1. 1) Prove that in fact $d(x,z)\leq\max\{d(x,y),d(y,z)\}$, i.e. every triangle is isosceles. In terms of the original valuation this states that if $v(x+y)>\min\{v(x),v(y)\}$, then $v(x)=v(y)$.

2) An easiest example of a complete discrete valuated ring is the ring $K[[z]]$ of formal power series with a proper valuation defined as $v(u)=n$, if $u=\sum_{i=n}^{\infty}c_iz^i$. Recall the multiplication of two series:

$$(\sum_{i=0}^{\infty}a_iz^i)(\sum_{j=0}^{\infty}b_jz^j)=\sum_{k=0}^{\infty}(\sum_{i+j=k}a_ib_j)z^k$$

Show that $K[[z]]$ is complete. In particular, if $u\in K[[z]]$ is such that $u=c-\tilde{u}$ with $0\neq c\in K$ and $\tilde{u}(0)=0$, then the inverse element $u^{-1}=c^{-1}(1+\sum_{i=1}^{\infty}c^{-i}\tilde{u}^i)$ is well defined.

Comment 2.1. For further reading about the theory of valuations for non-commutative rings see e.g. books [25], [8].

List of notations.

- $\mathrm{Der}_K(\mathcal{A})$ denotes the space of K-derivation of the algebra $\mathcal{A}$
- $C_n^i = \frac{n(n-1)\dots(n-i+1)}{i!}$
- For any ring $\mathcal{R}$ we denote by $\mathcal{R}[[z]] = \{\sum_{i=0}^{\infty} u_i z^i, u_i \in \mathcal{R}\}$ the *ring of formal power series* with usual multiplication (i.e. z is a formal variable which commutes with all elements from $\mathcal{R}$)
- $\mathcal{R}((z)) = \{\sum_{i=N\in\mathbb{Z}}^{\infty} u_i z^i, u_i \in \mathcal{R}\}$ denotes the *ring of formal Laurent series*
- $\mathrm{End}(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}, \mathcal{A})$ denotes the space of all endomorphisms of an algebra (ring) $\mathcal{A}$
- $\mathrm{Aut}(\mathcal{A})$ denotes the group of automorphisms, i.e. invertible endomorphisms
- $\mathcal{R}^*$ denotes the group of units
- $D(\mathcal{R}) = \mathcal{R}[\partial]$ denotes the ring of ordinary differential operators with coefficients in $\mathcal{R}$
- $E(\mathcal{R}) = \mathcal{R}((\partial^{-1}))$ denotes the ring of pseudo-differential operators with coefficients in $\mathcal{R}$
- Spectral curve: see definition 4.6
- Spectral module: see definition 4.7
- Rank of a ring, $\mathrm{rk}\, B$: see definition 4.8
- Rank of the spectral module (sheaf): see definition 5.1
- Spectral sheaf: see definition 5.3
- Projective spectral data: see section 6.2

3 Basic algebraic properties of the ring of ordinary differential operators

3.1 Basic definitions

Let K be an algebraically closed field of characteristic zero. Almost always we can assume that $K = \mathbb{C}$, but many algebraic results we will use in our lectures hold also for arbitrary K.

First recall the most important for us basic definitions.

Definition 3.1. Let $\mathcal{A}$ be an algebra over K. A K-*derivation* of $\mathcal{A}$ is a K-linear map $\partial : \mathcal{A} \to \mathcal{A}$ such that the Leibniz rule hold:

$$\partial(a \cdot b) = \partial(a) \cdot b + a \cdot \partial(b), \text{ for any } a, b \in \mathcal{A},$$

where $\cdot$ means the multiplication in the algebra $\mathcal{A}$. For shortness we'll write $a^{(k)}$ instead of $\partial^k(a)$, and we'll omit $\cdot$ in formulas with multiplication of elements. The collection of all K-derivations of $\mathcal{A}$ is denoted by $\mathrm{Der}_K(\mathcal{A})$.

Remark 3.1. If $\mathcal{A}$ has a unit 1, then $\partial(1) = \partial(1^2) = 2\partial(1)$, so that $\partial(1) = 0$. Thus by K-linearity, $\partial(k) = 0$ for all $k \in K$.

Comment 3.1. $\mathrm{Der}_K(\mathcal{A})$ is a Lie algebra with Lie bracket defined by the commutator:

$$[\partial_1, \partial_2] = \partial_1 \circ \partial_2 - \partial_2 \circ \partial_1$$

(here $\circ$ means the composition; check it!).

If we have a K-algebra $\mathcal{A}$ with a K-derivation ∂, we can consider formal symbols of the form $\sum_{i=0}^{n} u_i \partial^i$ and think of these symbols as acting on elements of $\mathcal{A}$ by multiplication and differentiation: $(u\partial)(f) = u \cdot \partial(f)$. Thus we obtain a big space of K-linear operators acting on $\mathcal{A}$. The Leibniz rule can be considered as an equality of operators:

$$\partial f = f' + f\partial,$$

what motivates the following definition

Definition 3.2. Let $\mathcal{R}$ be a ring over K and let ∂ be a K-derivation. We define the ring of *ordinary differential operators with coefficients in* $\mathcal{R}$ as the set

$$D(\mathcal{R}) := \mathcal{R}[\partial] = \{\sum_{i=0}^{n} u_i \partial^i, \; u_i \in \mathcal{R}\}$$

(which is obviously a linear space over K) with the composition rule

$$\partial^n u = \sum_{i=0}^{n} C_n^i u^{(i)} \partial^{n-i},$$

where $C_n^i = \frac{n(n-1)\dots(n-i+1)}{i!}$, and $u^{(0)} = u$.

Exercise 3.1. Extending the composition rule by linearity we can write down its general form: if $P = \sum_{k=0}^{n} a_k \partial^k$, $Q = \sum_{l=0}^{m} b_l \partial^l$, then

$$PQ = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{0 \le i \le k} C_k^i a_k b_l^{(i)} \partial^{k+l-i}. \tag{1}$$

Proposition 3.1. *The space* $\mathcal{R}[\partial]$ *with the composition rule* (1) *is a ring over* K.

Proof. The distributivity of the multiplication can be easily checked directly. Obviously, the multiplicative identity of $\mathcal{R}$ is the multiplicative identity of $\mathcal{R}[\partial]$. So, we need to check only the associativity of the multiplication. We use the following trick (cf. [20, Ch.III,§11]). Let's extend the derivation ∂ on $\mathcal{R}[\partial]$ by setting $\partial(\partial) = 0$. Introduce a new derivation δ on $\mathcal{R}[\partial]$ by setting $\delta(a\partial^n) = na\partial^{n-1}$ (check that ∂, δ are derivations). Then for any $P, Q \in \mathcal{R}[\partial]$ we have

$$PQ = \sum_{k=0}^{\infty} \frac{1}{k!} \delta^k(P) * \partial^k(Q), \tag{2}$$

where $*$ means the multiplication of series from the *ring of formal power series* $\mathcal{R}[[z]]$, where z is just replaced by ∂, and $*$ has the effect of bringing all elements from $\mathcal{R}$ to the left and powers of ∂ to the right, e.g. $(a\partial^n) * (b\partial^m) = (ab)\partial^{n+m}$ (in fact, the sum is finite, since P, Q are polynomials in ∂). Indeed, note that it is enough to check this equality for monomials of the form $f\partial^n$, $g\partial^m$. We have

$$f\partial^n g\partial^m = \sum_{k=0}^{n} C_n^k f g^{(k)} \partial^{n-k+m} = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{n!}{(n-k)!} f \partial^{n-k} \right) * g^{(k)} \partial^m = \sum_{k=0}^{\infty} \frac{1}{k!} \delta^k(f\partial^n) * \partial^k(g\partial^m).$$

Then,

$$\delta(P*Q)=\delta(P)*Q+P*\delta(Q), \quad \partial(P*Q)=\partial(P)*Q+P*\partial(Q)$$

and $\delta\circ\partial=\partial\circ\delta$. Therefore,

$$\delta^k(P*Q)=\sum_{i=0}^{k}C_k^i\delta^i(P)*\delta^{k-i}(Q), \quad \partial^l(P*Q)=\sum_{i=0}^{l}C_l^i\partial^i(P)*\partial^{l-i}(Q)$$

and

$$(PQ)T=\sum_{k,l\geq 0}\frac{1}{l!k!}\delta^l(\delta^k(P)*\partial^k(Q))*\partial^l(T)=\sum_{k,l,p\geq 0}\frac{1}{l!k!}C_l^p\delta^{p+k}(P)*\delta^{l-p}\partial^k(Q)*\partial^l(T)$$

and

$$P(QT)=\sum_{l',k'\geq 0}\frac{1}{k'!l'!}\delta^{k'}(P)*\partial^{k'}(\delta^{l'}(Q)*\partial^{l'}(T))=\sum_{l',k',p'\geq 0}\frac{1}{k'!l'!}C_{k'}^{p'}\delta^{k'}(P)*\partial^{k'-p'}\delta^{l'}(Q)*\partial^{p'+l'}(T).$$

Replacing $p'+l'$ by l, $k'-p'$ by k and p' by p in the second formula, we obtain the first one, since

$$\frac{1}{k'!l'!}C_{k'}^{p'}=\frac{1}{l'!p'!(k'-p')!}=\frac{1}{(l-p)!p!k!}=\frac{1}{l!k!}C_l^p$$

for $p'\leq k'$ and $p\leq l$. Thus, the multiplication is associative and $\mathcal{R}[\partial]$ is a ring. □

On each ring of ordinary differential operators (ring of ODO's for short) there is a natural order function, which defines a discrete valuation and the corresponding metric topology on this ring.

Definition 3.3. For any non-zero operator $P=\sum_{i=0}^n u_i\partial^i$ of the ring $\mathcal{R}[\partial]$ we define its *order* to be

$$\mathrm{ord}(P)=n=\max\{k| \quad u_k\neq 0\}.$$

The non-zero coefficient $u_n\in\mathcal{R}$ is called *the highest coefficient* $HT(P)$ of the operator P, and the term $u_n\partial^n$ is called *the highest symbol* $\sigma(P)$ of the operator P.

The operator P is called *monic* if $HT(P)=1$. It is called *normalized* if it has the form

$$P=\partial^n+u_{n-2}\partial^{n-2}+\ldots+u_0.$$

From the composition rule (1) immediately follows

Lemma 3.1. *Let $\mathcal{R}[\partial]$ be a ring of ODOs. For any non-zero elements $P,Q\in\mathcal{R}[\partial]$ we have*

- $\mathrm{ord}(PQ)\leq\mathrm{ord}(P)+\mathrm{ord}(Q)$, *and the equality holds iff* $HT(P)HT(Q)\neq 0$;
- $HT(PQ)=HT(P)HT(Q)$, *provided* $HT(P)HT(Q)\neq 0$.

Exercise 3.2. Let $\mathcal{R}$ be an integral domain. Show that $\mathcal{R}[\partial]$ is an integral domain and that $-\mathrm{ord}$ is a proper discrete valuation.

3.2 Basic algebraic properties of the ring $D = K[[x]][\partial]$

Ordinary differential operators appears naturally every time when we study *linear differential equations* $P(\psi) = 0$. In this case coefficients of P and the function ψ are usually assumed to be smooth or analytic in some open neighborhood of 0. Since analytic functions admit the Taylor series expansion in appropriate neighborhoods, it is reasonable to study rings of ODO's with coefficients in the *commutative* ring $R = K[[x]]$ or in its field of fractions $\mathrm{Quot}(R) = K((x))$, with the derivation $\partial = \partial/\partial x$. Let's collect basic algebraic facts about these rings.

Theorem 3.1. *Let $D = K[[x]][\partial]$ be the ring of ODOs with coefficients in the commutative ring $R = K[[x]]$; let $\tilde{D} = K((x))[\partial]$ be the ring of ODOs with coefficients in the field of fractions $\mathrm{Quot}(R)$. Then we have:*

1. *The rings D, $\tilde{D}$ are integral domains; their units (i.e. invertible elements) are the units of R or $\mathrm{Quot}(R)$ correspondingly (the units R^* are just elements $w \in R$ with $w(0) \neq 0$).*

2. *The rings D, $\tilde{D}$ are simple, i.e. there are no non-zero two-sided ideals.*

3. *The ring $\tilde{D}$ is a domain with a Euclidean algorithm: that is, given operators M and L with (say) $\mathrm{ord}\, M \geq \mathrm{ord}\, L$, there are operators Q_i, R_i such that $M = Q_1 L + R_1$, $\mathrm{ord}\, R_1 < \mathrm{ord}\, L$, $L = Q_2 R_1 + R_2$, $\mathrm{ord}\, R_2 < \mathrm{ord}\, R_1$, and so on. If $R_i \neq 0$, $R_{i+1} = 0$, then R_i is called the right GCD of L and M (the left GCD is analogously defined).*[1]

   In particular, all left and right ideals in $\tilde{D}$ are principal (i.e. generated by one element).

4. *Let φ be a non-zero ring endomorphism of D. Then this is an automorphism, i.e. $\mathrm{End}(D)\backslash\{0\} = \mathrm{Aut}(D)$. More precisely, there exist $u \in K[[x]]$ satisfying $u(0) = 0$ and $u'(0) \neq 0$, and $v \in K[[x]]$ such that*

$$\begin{cases} x \overset{\varphi}{\mapsto} u \\ \partial \overset{\varphi}{\mapsto} \frac{1}{u'}\partial + v. \end{cases} \tag{3}$$

5. *Let $P = u_n\partial^n + u_{n-1}\partial^{n-1} + \cdots + u_0 \in D$, where $u_n(0) \neq 0$. Then there exists $\varphi \in \mathrm{Aut}(D)$ such that $\varphi(P)$ is normalized. Moreover, if $Q \in D$ is a normalized differential operator of positive order and ψ is an inner automorphism of D (i.e. is of the form $\psi : y \mapsto w^{-1}yw$, $w \in R^*$) such that $\psi(Q) = Q$, then $\psi = \mathrm{id}$.*

Remark 3.2. Let $w \in K[[x]]$ be a unit. Then for the inner automorphism $\mathsf{Ad}_w : D \to D,\ P \mapsto w^{-1}Pw$, we have:

$$\begin{cases} x \mapsto x \\ \partial \mapsto \partial + \frac{w'}{w}. \end{cases}$$

Note that for any $K[[x]] \ni v = \sum_{i=0}^{\infty} \beta_i x^i = \beta_0 + \tilde{v}$, the formal power series $w := \exp(v) = e^{\beta_0}\exp(\tilde{v})$ is a unit in $K[[x]]$. Therefore, any automorphism $\varphi \in \mathrm{Aut}(D)$ satisfying $\varphi(x) = x$ is inner, see (3)

Proof. 1) follows from exercise 3.2. The description of units then follows from lemma 3.1. Indeed, the order of a unit must be zero, i.e. any unit is an element of R or $\mathrm{Quot}(R)$. The description of units in R follows from the observation that elements $u \in R$ with $u(0) = 0$ form the

[1] Thus, $M = M'R_i$, $L = L'R_i$ for some operators M', L'. Note that the GCD is well defined up to left (right) multiplication by a unit in the ring $\tilde{D}$.

(unique) maximal ideal in R, i.e. such elements are not invertible. Other elements are invertible by exercise 2.1.

2) Let I be a non-zero proper ideal. First note that I must contain an element of order zero. Indeed, if $P\in I$ and $\mathrm{ord}(P)>0$, consider the element $[P,x]=Px-xP\in I$. By lemma 3.1 $\mathrm{ord}(Px-xP)<\mathrm{ord}(P)$. From (1) or (2) it follows that $[P,x]=\delta(P)$, thus if $\mathrm{ord}(P)>0$, then $[P,x]\neq 0$. Repeating this procedure, we obtain an element of order zero in I, i.e. I contains an element u from R. Taking commutators of u with ∂, we obtain that $u^{(k)}\in I$ for all k by the same arguments. Thus I contains a unit, a contradiction.

3) The proof is straightforward: since $K((x))$ is a field, we can always find an element $u\in K((x))$ such that $HT(M)=uHT(L)$. Therefore, $\mathrm{ord}(M-u\partial^{\mathrm{ord}(M)-\mathrm{ord}(L)}L)<\mathrm{ord}(M)$. Repeating this procedure, we obtain Q_1 and R_1, then Q_2 and R_2 and so on.

4) Let $u:=\varphi(x)\in D$. First note that $\mathrm{ord}(u)=0$. For, if $\mathrm{ord}(u)>0$, the image of any infinite series from $K[[x]]$ will not belong to D. By the same reason $u(0)=0$. Let $P:=\varphi(\partial)=a_n\partial^n+a_{n-1}\partial^{n-1}+\cdots+a_0\in D$ for some $n\in\mathbb{N}$, where $a_n\neq 0$. Clearly, $[P,u]=nu'a_n\partial^{n-1}+\text{l.o.t}$, hence $[\partial,x]=1=[P,u]$ if and only if $n=1$ and $a_1=\dfrac{1}{u'}$.

5) By assumption, a_n is a unit in $K[[x]]$. Therefore, there exists $a\in K[[x]]$ such that $a^n=u_n$. It implies that $P=\big(a\partial\big)^n+\text{l.o.t}$. Hence, there exists a change of variables as in (3) transforming P into an operator of the form $\widetilde{P}:=\partial^n+c_{n-1}\partial^{n-1}+\cdots+c_0$. Applying now to $\widetilde{P}$ an automorphism (3) with $u=x$ and $v=-\dfrac{c_{n-1}}{n}$, we get a normalized operator Q. This proves the first statement. The proof of the second statement is straightforward. □

Remark 3.3. The ring D contains another well known ring, called the *first Weyl algebra*: $A_1=K[x][\partial]$. Amazingly the fourth property from theorem 3.1 is still unknown for it. This problem is called the *Dixmier conjecture*: is it true that $\mathrm{End}(A_1)\backslash\{0\}=\mathrm{Aut}(A_1)$? It was the first problem (among six) posed by J. Dixmier in [10].

The same conjecture exists for many variables: the n-th Weyl algebra is defined as $A_n=K[x_1,\dots,x_n][\partial_1,\dots\partial_n]$, where $\partial_i=\partial/\partial x_i$. The conjecture says that every non-zero endomorphism of A_n is an automorphism. The Dixmier conjecture is equivalent to other famous open conjectures: the Jacobian conjecture and the Poisson conjecture. This relation can serve as an illustration of unity in mathematics.

The simple and attractive problem we want to study in our lectures dates back to works of Wallenberg ([30]), Schur ([26]) and Burchnall-Chaundy ([5], [6], [7], [2]). The problem asks to find and classify all non-trivial commutative subrings of $D=K[[x]][\partial]$ in the sense that we are looking for subrings not isomorphic to $K[P]$. Originally this problem was considered for ODOs with analytic coefficients. In this case for each operator P there is a shift of variables $x\mapsto x+\varepsilon$, $\partial\mapsto\partial$ making the highest coefficient of P not vanishing at zero (note that such a shift is not an endomorphism of D, but is an endomorphism of some smaller rings, e.g. of the first Weyl algebra). Therefore, due to theorem 3.1, item 5) the problem reduces to the classification of commutative subrings containing monic operators (we'll return to this point in section 3).

Definition 3.4. A differential operator $P=u_n\partial^n+u_{n-1}\partial^{n-1}+\cdots+u_0\in D$ of positive order n is called *formally elliptic* if $u_n\in K^*$.

Exercise 3.3. Let B be a commutative subring of D containing a formally elliptic element P. Show that *all* elements of B are formally elliptic.

According to theorem 3.1, item (5), we can transform P into a normalized formally elliptic differential operator. Therefore, in the sequel all commutative subrings of D are assumed

- to contain an elliptic operator of positive order (i.e. being *elliptic*)
- to be *normalized*, meaning that all elements of B of minimal positive order are normalized.

The last assumption eliminates redundant degrees of freedom in the problem of classification of commutative subalgebras of differential operators: if $B \subset D$ is a normalized elliptic subalgebra and φ is an inner automorphism of D such that $\varphi(B) = B$, then $\varphi = \mathrm{id}$.

To study basic algebraic properties of *commutative subrings* of differential operators, we need to explain the Schur theory of pseudo-differential operators.

3.3 Schur's theory

Definition 3.5. Let $\mathcal{R}$ be a ring over K and let ∂ be a K-derivation. We define the ring of *pseudo-differential operators with coefficients in* $\mathcal{R}$ as the set

$$E(\mathcal{R}) := \mathcal{R}((\partial^{-1})) = \{ \sum_{i=-\infty}^{N} u_i \partial^i,\ u_i \in \mathcal{R},\ N \in \mathbb{Z} \}$$

(which is obviously a linear space over K) with the composition rule

$$\partial^n u = \sum_{i=0}^{\infty} C_n^i u^{(i)} \partial^{n-i}$$

for all $n \in \mathbb{Z}$.

Exercise 3.4. Extending the composition rule by linearity we can write down its general form: if $P = \sum_{k=-\infty}^{n} a_k \partial^k$, $Q = \sum_{l=-\infty}^{m} b_l \partial^l$, then

$$PQ = \sum_{k=-\infty}^{n} \sum_{l=-\infty}^{m} \sum_{i=0}^{\infty} C_k^i a_k b_l^{(i)} \partial^{k+l-i}. \tag{4}$$

Note that for each $n \in \mathbb{Z}$ the number of terms with $k + l - i = n$ is finite, so the sum is well defined.

Proposition 3.2. *The space* $\mathcal{R}((\partial^{-1}))$ *with the composition rule* (4) *is a ring over* K*.*

Exercise 3.5. Check that the proof of proposition 3.1 works also for this proposition.

Obviously, $D(\mathcal{R}) \subset E(\mathcal{R})$ and the order function can be extended to the ring $E(\mathcal{R})$ just in the same way (see definition 3.3). I particular, the function $(-\operatorname{ord})$ is a proper discrete valuation on $E(\mathcal{R})$ if $\mathcal{R}$ is an integral domain. The notions of *monic* and *normalized* pseudo-differential operators are defined in the same way.

Lemma 3.2. *Let* $\mathcal{R}$ *be an integral domain. Then* $E(\mathcal{R})$ *is a complete ring with respect to the valuation topology defined by the valuation* $v = -\operatorname{ord}$*.*

Proof. Let $\{P_n \in E(\mathcal{R})\}$, $n \in \mathbb{N}$ be a Cauchy sequence. By definition of the Cauchy sequence for each N there exists $n(N) \in \mathbb{N}$ such that for all $m, k > n(N)$ $v(P_m - P_k) > N$. Consider the sequence $\{-\operatorname{ord}(P_n)\}$. Then we have two possibilities: either it stabilizes (i.e. there exists $n_0 \in \mathbb{N}$ such that $-\operatorname{ord}(P_n) = const$ for all $n \geq n_0$), or not.

In the second case we claim that the limit of the sequence $\{P_n\}$ is zero. Indeed, if 0 is not the limit, there exists N such that for all $n \in \mathbb{N}$ there is $m(n) > n$ such that $v(P_{m(n)}) < N$. But then for all $k > n(N)$ we must have $v(P_{m(n(N))}) = v(P_k)$, i.e. the sequence $\{-\operatorname{ord}(P_n)\}$ stabilizes, a contradiction.

If the sequence $\{-\operatorname{ord}(P_n)\}$ stabilizes, then we can built the limit recursively, by finding the sequence of its coefficients. The order of the limit must be, obviously, equal to $-n_0$. For $N = n_0$ we must have $v(P_m - P_k) > n_0$ for all $k, m > n(N)$ and therefore the operators P_m

and P_k must have equal highest coefficients. Thus we take this highest coefficient as the first coefficient of the limit. Taking $N = n_0+1$ we obtain by analogous arguments that the operators P_m and P_k have equal coefficients at ∂^{-n_0} and at ∂^{-n_0-1} for all $m, k > n(N)$. The coefficient at ∂^{-n_0-1} is the second coefficient of the limit. Continuing this line of reasoning we'll find all coefficients of the limit operator (we leave to the reader to check that it is indeed the limit). □

Theorem 3.2. *The following results are true.*

1. *The spaces* $E(\mathcal{R})^{\le i} = \{P \in E(\mathcal{R}) | \operatorname{ord}(P) \le i\}$ *define a structure of filtered ring on* $E(\mathcal{R})$*:* $E(\mathcal{R})^{\le i} E(\mathcal{R})^{\le j} \subset E(\mathcal{R})^{\le i+j}$*.*

2. $E(\mathcal{R})$ *is a graded Lie algebra with respect to the commutator bracket, besides* $[E(\mathcal{R})^{\le i}, E(\mathcal{R})^{\le j}] \subset E(\mathcal{R})^{\le i+j-1}$*.*

3. *There is a decomposition of the vector space* $E(\mathcal{R})$ *into direct sum of subalgebras* $E(\mathcal{R}) = E(\mathcal{R})^{\le -1} \oplus D(\mathcal{R})$*. The projections of an operator* P *onto these subrings are denoted by* P_- *and* P_+ *correspondingly.*

4. *For any monic operator* $P = \partial^d + a_{d-1}\partial^{d-1} + \dots$ *there exists the inverse operator* $P^{-1} = \partial^{-d} + b_{-d-1}\partial^{-d-1} + \dots$*.*

5. *For any monic operator* $P = \partial^d + a_{d-1}\partial^{d-1} + \dots$ *there exists a unique monic* d*-th root, i.e. a monic operator* $P^{1/d} = \partial + u_0 + u_{-1}\partial^{-1} + \dots$ *such that* $(P^{1/d})^d = P$*.*

6. *Assume that the derivation* $\partial : \mathcal{R} \to \mathcal{R}$ *is surjective. Assume also that the equation* $dLog(y) = c$ *has a solution in* $\mathcal{R}$ *for any* $c \in \mathcal{R}$ *(these properties hold e.g. for* $\mathcal{R} = K[[x]]$ *).*

   Then for every first order operator $L = \partial + u_0 + u_{-1}\partial^{-1} + \dots$ *there exists an invertible zero-th order operator*
   $$S = s_0 + s_1\partial^{-1} + s_2\partial^{-2} + \dots$$
   (the Schur operator) such that $S^{-1}LS = \partial$*. If* $\bar{S}$ *is another operator such that* $\bar{S}^{-1}L\bar{S} = \partial$*, then there is an invertible zero-th order operator* S_c *with constant coefficients such that* $\bar{S} = S \cdot S_c$*.*

Proof. 1) follows from the multiplication law 4. 2) follows from 1). 3) is obvious. 4) If we rewrite the operator P as $P = (1 + a_{d-1}\partial^{-1} + \dots)\partial^d = (1 - P_0)\partial^d$ (note that $\operatorname{ord}(P_0) < 0$), then $P^{-1} = \partial^{-d}(1 + P_0 + P_0^2 + \dots)$.

5) We will find the operator $Y = P^{1/d}$ as the limit of a Cauchy sequence. Set $Y_1 = \partial$. Then $Y_1^d = P + O(d-1)$ (here we denote by $O(k)$ elements from $E(\mathcal{R})^{\le k}$). Now let's construct the sequence by induction. Let Y_k be such that $Y_k^d = P + O(d-k)$ and $Y_k - Y_{k-1} = O(-k+2)$. Consider the operator $Y_{k+1} = Y_k + b\partial^{-k+1}$ (here b is unknown coefficient). Then

$$Y_{k+1}^d = Y_k^d + dbY_k^{d-1}\partial^{-k+1} + O(d-k-1) = Y_k^d + db\partial^{d-k} + O(d-k-1).$$

On the other hand, $Y_k^d = P + a\partial^{d-k} + O(d-k-1)$ for some $a \in \mathcal{R}$. Thus, setting $b = -a/d$, we define Y_{k+1}, after that we proceed by induction. Clearly, $\{Y_k\}$ is a Cauchy sequence. So, there is the limit Y by lemma 3.2, and $Y^d = P$.

6) As in 5) we will find the operator S as the limit of a Cauchy sequence, which can be found by induction. Set $S_0 = w$, where w is a solution of the equation $dLog(w) = u_0$ (cf. remark 3.2). Then $S_0^{-1}LS_0$ is a normalized operator. Assume we have found the operator S_k, $k > 0$ such that $S_k^{-1}LS_k = \partial + a\partial^{-k-1} + O(-k-2)$. It is enough to find $\bar{S}_{k+1} = 1 + s_{k+1}\partial^{-k-1}$ such that

$$\bar{S}_{k+1}^{-1}(\partial + a\partial^{-k-1} + O(-k-2))\bar{S}_{k+1} = \partial + O(-k-2).$$

Then the $(k+1)$-th operator from the Cauchy sequence is $S_{k+1} = S_k \bar{S}_{k+1}$. It is easy to check that this sequence is indeed a Cauchy sequence.

Now direct calculations show that

$$\bar{S}_{k+1}^{-1}(\partial + a\partial^{-k-1} + O(-k-2))\bar{S}_{k+1} = \partial + s'_{k+1}\partial^{-k-1} + a\partial^{-k-1} + O(-k-2).$$

Since ∂ gives a surjective map, the equation $s'_{k+1} = -a$ has a solution, and we are done.

If $\bar{S}$ is another operator with these properties, then $S^{-1}\bar{S}\partial = \partial S^{-1}\bar{S} = S^{-1}\bar{S}\partial + \partial(S^{-1}\bar{S})$. Hence, $\partial(S^{-1}\bar{S}) = 0$, i.e. the operator $S_c = S^{-1}\bar{S}$ is a zero-th order operator with constant coefficients. □

Corollary 3.1. *Let $P \in K[[x]][\partial]$ be a monic operator. Denote by B_P the set of operators commuting with P. Then B_P is a commutative ring over K.*

Moreover, there is an embedding $B_P \hookrightarrow K[[z]]$ of the ring B_P into the ring of formal power series $K((z))$.

Proof. Let S be a Schur operator for the operator P from theorem 3.2, item (6). Then $S^{-1}B_P S$ is a set of pseudo-differential operators commuting with ∂. As we have seen above, each such operator has constant coefficients. But all operators with constant coefficients commute. Thus they form a commutative subring. At last, $S^{-1}B_P S \subset K((\partial^{-1})) \simeq K((z))$. □

As we will see, commutative subrings of ODOs can be classified in terms of algebro-geometric *spectral data*, which in particular consist of an algebraic curve and a spectral sheaf (or spectral bundle). These objects are very well known in *algebraic geometry*.

4 Basic facts about commutative subrings of ODOs and constructions from Commutative Algebra and Algebraic Geometry

In this section we collect basic facts about commutative subrings of ODOs and basic constructions from Commutative Algebra and Algebraic Geometry which will be used in further lectures. Main references for this section are the books [1], [15], [12, Ch.1], and short course [28]. We will focus mostly on "algebraic" side of Algebraic geometry. We recommend to read also lectures by Professor Wei-Ping Li, where the algebraic geometry is considered from the "complex geometry" side. Since we will be interested mostly in algebraic curves, students who will read this lecture can try to "test" the results on them during the reading. All rings in this chapter are *commutative* and K denotes a field of characteristic zero.

Definition 4.1. The elements $r_1, \ldots, r_n$ of a ring R over K are *algebraically independent* (over K) if the only irreducible polynomial $f(x_1, \ldots, x_n)$ with coefficients in K such that $f(r_1, \ldots, r_n) = 0$, is the zero polynomial (here we assume that f depends on all variables $x_1, \ldots, x_n$).

Notation 4.1. Let's denote the filtration on D induced by the filtration $E(K[[x]])^{\leq n}$ by

$$D^{(n)} := D \cap E(K[[x]])^{\leq n}$$

and for any subring $B \subset D$

$$B^{(n)} := B \cap D^{(n)}.$$

Proposition 4.1. *Let $B \subset D$ be a commutative subring over K (not necessarily elliptic or normal), containing an operator of positive order. Then B is finitely generated over K.*

Proof. We'll need the following claim.
Claim. Let $N_B = \{\operatorname{ord} P| \quad P \in B\} \subset \mathbb{N} \cup \{0\}$. Then there exists a finite subset $F_B \subset \mathbb{N}$ such that $N_B = r(\mathbb{N} \backslash F_B) \cup \{0\}$, where $r = GCD\{\operatorname{ord} P| \quad P \in B\}$.

Proof. Since r is a GCD there exist operators $P, Q \in B$ such that $\operatorname{ord}(P)i + \operatorname{ord}(Q)j = r$ for some $i, j \in \mathbb{Z}$ (prove this!). Since $r > 0$, i or $j > 0$. Without loss of generality let $i > 0$. Since $r|\operatorname{ord}(P)$ and $r|\operatorname{ord}(Q)$ we must have $j \leq 0$. Note that if $j = 0$ then $\operatorname{ord}(P) = r$, so $B = K[P]$ and we are done.

So let $j < 0$, $\alpha = \operatorname{ord} P$, $\beta = \operatorname{ord} Q$, $\alpha = \alpha' r$, $\beta = \beta' r$. Obviously, $N_B \subset r\mathbb{N} \cup \{0\}$. Now it suffices to show that $N_B \supset rn$ for any $n \gg 0$.

We claim that $rn \in N_B$ for any $n \geq -j\alpha'\beta'$. Indeed, let $n > -j\alpha'\beta'$. Applying the Euclidean algorithm to $n + j\alpha'\beta'$, we find unique numbers $m \geq 0$, $0 \leq l < \alpha'$ such that $n = -j\alpha'\beta' + m\alpha' + l$. Therefore,

$$rn = -rj\alpha'\beta' + m\alpha + l(i\alpha + j\beta) = (m + il)\alpha - (\alpha' - l)j\beta = \operatorname{ord}(P^{m+il}Q^{-(\alpha'-l)j}) \in N_B$$

Now $F_B = \{n \in \mathbb{N}| \quad rn \notin N_B\}$, therefore F_B is finite. □

Let's prove that B is finitely generated. Let's denote by $\tilde{F}_B = \{\operatorname{ord}(P)| \quad P \in B^{(-rj\alpha'\beta'-r)}\}$. Then $\tilde{F}_B \cup \{nr| \quad n \geq -j\alpha'\beta'\} = N_B$. Let $s = \sharp\tilde{F}_B$. Choose operators $T_1, \ldots, T_s \in B$ such that $\{\operatorname{ord}(T_1), \ldots \operatorname{ord}(T_s)\} = \tilde{F}_B$. Then we claim that $B = K[P, Q, T_1, \ldots, T_s]$.

Indeed, let $L \in B$, $\operatorname{ord}(L) = t \in N_B$. Then there is an operator $L' \in K[P, Q, T_1, \ldots, T_s]$ such that $\operatorname{ord} L' = t$. Let $L = a\partial^t + l.o.t.$, $L' = b\partial^t + l.o.t.$. Then

$$0 = [L, L'] = tab'\partial^{2s-1} - tba'\partial^{2s-1} + l.o.t.$$

Hence $ab' = ba'$, where from $b = ac$, $c \in K$. Then $\operatorname{ord}(L - c^{-1}L') < t$ and $(L - c^{-1}L') \in B$. Repeating the same arguments with this new operator and so on, we will come to an operator of order < 0, i.e. to the zero operator. Hence $L \in K[P, Q, T_1, \ldots, T_s]$, and we are done. □

Exercise 4.1 (*)**.** Let $P, Q \in D$ and P be a monic operator. Let $F \in K[X, Y]$, $F(X, Y) = \sum_{i+j \leq N} c_{ij} X^i Y^j$ be a polynomial. Assume that $F(P, Q) = \sum c_{ij} P^i Q^j = 0$. Is it true that $[P, Q] = 0$?

Comment 4.1. This exercise is connected with the following interesting conjecture of Y. Berest, cf. [16].

The group of automorphisms of the first Weyl algebra A_1 acts on the set of solutions of the equation $F(X, Y) = 0$, i.e. if $X, Y \in A_1$ satisfy the equation and $\varphi \in Aut(A_1)$, then $\varphi(X), \varphi(Y)$ also satisfy the equation. The group $Aut(A_1)$ is generated by the following automorphisms

$$\varphi_1(x) = \alpha x + \beta \partial_x, \quad \varphi_1(\partial_x) = \gamma x + \delta \partial_x, \quad \alpha, \beta, \gamma, \delta \in K, \quad \alpha\delta - \beta\gamma = 1,$$

$$\varphi_2(x) = x + P_1(\partial_x), \quad \varphi_2(\partial_x) = \partial_x,$$

$$\varphi_3(x) = x, \quad \varphi_2(\partial_x) = \partial_x + P_2(x),$$

where P_1, P_2 are arbitrary polynomials (see [10]). So, $Aut(A_1)$ consists of tame automorphisms. A natural and important problem is to describe the orbit space of the group action of $Aut(A_1)$ in the set of solutions. If one describes the orbit space it gives a chance to compare $End(A_1)$ and $Aut(A_1)$ ($End(A_1)$ consists of endomorphisms $\varphi : A_1 \to A_1$, i.e. $[\varphi(\partial_x), \varphi(x)] = 1$). Berest has proposed the following interesting conjecture:

If the Riemann surface corresponding to the equation $F = 0$ with generic $c_{ij} \in \mathbb{C}$ has genus $g = 1$ then the orbit space is infinite, and if $g > 1$ then there are only finite number of orbits.

One can prove that if there are finite number of orbits for some equation F then $End(A_1) \backslash \{0\} = Aut(A_1)$.

Definition 4.2. Let $A \subset R$ be an extension of integral domains. An element $x \in R$ is called *integral* over A if there exists a monic polynomial $f \in A[T]$ such that $f(x) = 0$.

Let $\tilde{K}/k$ be a field extension. An element $x \in \tilde{K}$ is *algebraic* over k if it is integral over k. $\tilde{K}/k$ is an algebraic extension if any $x \in \tilde{K}$ is algebraic over k.

Recall the following facts from algebra:

Theorem 4.1. *[15, Ch. V §6] The ring $k[T_1, \dots T_n]$ is a unique factorisation domain (UFD for short), i.e. any polynomial $f \in k[T_1, \dots T_n]$ has a unique decomposition (up to multiplication on a unit)*

$$f = f_1^{k_1} \dots f_q^{k_q},$$

where f_i are irreducible polynomials (i.e. they are not divisible by any other non-constant polynomials).

Theorem 4.2. *[15, Ch. V, §10] Two polynomials $F, G \in A[T]$, where A is an integral domain, $F = a_0 + \dots + a_n t^n$, $G = b_0 + \dots + b_m t^m$ have a common zero (in some extension of* $\mathrm{Quot}(A)$ *) if and only if their resultant*

$$Res(F,G) = \det \begin{pmatrix} a_0 & a_1 & \dots & a_n & 0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & a_{n-1} & a_n & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & a_0 & \dots & \dots & \dots & a_n \\ b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & b_0 & \dots & \dots & \dots & \dots & b_m \end{pmatrix}$$

is equal to zero. Moreover, $Res(F,G) = 0$ if and only if $r_1 F$, $r_2 G$ have a common divisor of positive degree for some non-zero $r_1, r_2 \in A$.

To prove the second algebraic property of a commutative ring of ODOs we need some preliminary results. Consider first one special case: assume a (or c) is algebraic over K.

Lemma 4.1. *Let R be an integral domain over K. If $a \in R$ is algebraic over K and $a, b \in R$ are algebraically dependent over K then b is algebraic over K.*

Proof. First note that a is algebraic over K if and only if there exists an irreducible *homogeneous* polynomial $F(T_1, T_2)$ such that $F(1, a) = 0$ (just take the homogenisation of the corresponding irreducible monic polynomial). Let $G(T_2, T_3)$ be an irreducible polynomial such that $G(a, b) = 0$. Let $n = \deg F$, $m = \deg G$, $F = a_0 T_1^n + \dots + a_n T_2^n$, $G = b_0 + \dots + b_m T_2^m$, where $a_i \in K$, $b_i \in K[T_3]$. Then the polynomial $H(T_1, T_3) := Res(F(T_2), G(T_2)) \in K[T_1, T_3]$ is not zero. For, if $H = 0$, then F and G must have a common divisor of positive degree in $K[T_1, T_2, T_3]$ by theorem 4.2 (since $K[T_1, T_2, T_3]$ is a UFD), a contradiction.

Note that H depend on T_1 non-trivially. For, the determinant of the Sylvester matrix contains the unique monomial of degree $n \cdot m$ with respect to T_1, namely, $(a_0 T_1^n)^m \cdot b_m^n$ (all other monomials have smaller degree, and $a_0, b_m \neq 0$ since F, G are irreducible). Also note that $H(1, T_3) \neq 0$. For, if it is zero, then again $F(1, T_2)$ and $G(T_2, T_3)$ must have a common divisor of positive degree in $K[T_2, T_3]$ by theorem 4.2, a contradiction with irreducibility of G.

Note that $H(1, b) = Res(F(1, T_2), G(T_2, b) = 0$ in the ring $K(b) \subset R$, since $a \in R$ is a common zero. Hence, any irreducible factor of $H(T_1, T_3)$ that vanishes at $(1, b)$ depend either on T_1, T_3 or on T_3. In both cases we are done: b is algebraic over K. □

Corollary 4.1. *Let R be an integral domain over K. If $a, b \in R$ are algebraic over K, then a and b are algebraically dependent over K. Moreover, $a \cdot b$ is algebraic over K.*

Proof. Let $F(T_1,T_2)$, $G(T_2,T_3)$ be irreducible homogeneous polynomials such that $F(a,1) = G(1,b) = 0$. Then $H(T_1,T_3)$ from the proof of previous lemma is a *homogeneous* polynomial. Indeed, the determinant of the Sylvester matrix is a sum of monomials of the form $\pm a_{1,j_1}\cdots a_{m,j_m}\cdot b_{1,k_1}\cdots b_{n,k_n}$, where a_{i,j_i} denotes the element at i-th row and j_i-th column and b_{i,k_i} denotes the element at $(i+m)$-th row and k_i-th column, and $\{j_1,\dots,j_m,k_1,\dots,k_n\} = \{1,\dots n+m\}$. Note that $a_{i,j_i} = c_{i,j_i}T_1^{n-(j_i-i)}$ for some $c_{i,j_i}\in K$ and $b_{i,k_i} = d_{i,k_i}T_1^{m-(k_i-i)}$ for some $d_{i,k_i}\in K$. So,

$$\deg(a_{1,j_1}\cdots a_{m,j_m}\cdot b_{1,k_1}\cdots b_{n,k_n}) = 2mn - \frac{(m+n)(m+n+1)}{2} + \frac{m(m+1)}{2} + \frac{n(n+1)}{2} = mn.$$

Since any factor of a homogeneous polynomial H is again homogeneous, there exists an irreducible homogeneous polynomial $\tilde{H}$ in two variables which vanishes at (a,b).

Let $\tilde{H} = \sum_{i=0}^{n} c_{i,n-i}T_1^iT_2^{n-i}$. Consider the polynomial $H_n(T_1,T_2) = \sum_{i=0}^{n} c_{i,n-i}T_1^iT_2^{2(n-i)}$. Then $H_n(ab,b) = b^n\tilde{H}(a,b) = 0$. Note that H_n has no irreducible factors depending on T_2 and vanishing at b, since otherwise $\tilde{H}(T_1,b) = 0$, a contradiction. Therefore, there is an irreducible factor vanishing at (ab,b) and depending on T_1. Thus, ab is algebraic over K by lemma 4.1. □

Lemma 4.2. *Let R be an integral domain over K. If $a,b\in R$ are algebraically dependent over K and $b,c\in R$ are algebraically dependent over K, then $a,c\in R$ are algebraically dependent over K.*

Proof. Let $F(a,b)=0$, $G(b,c)=0$, where $F(T_1,T_2)$, $G(T_2,T_3)$ are irreducible polynomials.

As in proofs above consider F as a polynomial in the variable T_2 with coefficients in $K[T_1]\subset K[T_1,T_3]$ and G as a polynomial in T_2 with coefficients in $K[T_3]\subset K[T_1,T_3]$. Then the polynomial $H(T_1,T_3) := Res(F(T_2),G(T_2))\in K[T_1,T_3]$ is not zero. For, if $H=0$, then F and G have a common divisor in $K[T_1,T_2,T_3]$ by theorem 4.2, a contradiction.

Again we have $H(a,c) = Res(F(a,T_2),G(T_2,c)) = 0$ in the ring $K[a,c]$, since $b\in R\supset K[a,c]$ is a common zero. Any irreducible factor of H that vanishes at (a,c) depends either on two or one variable. If it depends on two variables, we are done. If it depends on one variable, then a (or c) is algebraic over K. Therefore, b is algebraic over K by lemma 4.1, and therefore c (or a) is algebraic over K by the same reason. Hence, a,c are algebraically dependent by corollary 4.1. □

Corollary 4.2. *If any two non-algebraic over K elements in R are algebraically dependent, then any two non-algebraic over K elements in* $\mathrm{Quot}(R)$ *are algebraically dependent.*

Proof. Take any element $\frac{1}{a}\in\mathrm{Quot}(R)$, where a is not algebraic. Then $(\frac{1}{a})a = 1$, so $\frac{1}{a}$, a are algebraically dependent over K and therefore $\frac{1}{a}$ and b are algebraically dependent for any non-algebraic over K element $b\in R$ by lemma 4.2. In particular, $\frac{1}{a}$ is not algebraic by lemma 4.1.

Now take any element $\frac{b}{a}\in\mathrm{Quot}(R)$ which is not algebraic over K. If a is algebraic over K, then $1/a$ is algebraic over K by lemma 4.1 and therefore b (and $1/b$) can not be algebraic over K by corollary 4.1. Let $F(T)$ be an irreducible polynomial such that $F(1/a)=0$. Then the polynomial $\tilde{F}(T_1,T_2) = F(T_1T_2)$ vanishes at $(b/a,1/b)$. Clearly, $\tilde{F}$ has no irreducible factors depending on one variable and vanishing at $(b/a,1/b)$ (because $b/a,1/b$ are not algebraic over K). Thus, $b/a,1/b$ are algebraically dependent and therefore b/a is algebraically dependent with any non-algebraic element from R by lemma 4.2. The same arguments work if b is algebraic over K.

If a,b are not algebraic over K, let $F(T_1,T_2) = \sum_{i,j\geq 0} c_{ij}T_1^iT_2^j$ be an irreducible polynomial such that $F(1/a,b)=0$. Then there exists some $N\in\mathbb{Z}$ such that $j-i-N\geq 0$ for all j,i

from the finite sum, so that

$$0=\sum_{i,j\geq 0}c_{ij}\frac{1}{a^i}b^j=b^N(\sum_{i,j\geq 0}c_{ij}\frac{b^i}{a^i}b^{j-i-N}).$$

Therefore, the polynomial $\tilde{F}(T_1,T_2)=\sum_{i,j\geq 0}c_{i,j}T_1^iT_2^{j-i-N}$, where sum is taken over the same set if indices, (note it is not identically zero) vanishes at $(b/a,b)$. Since $b/a,b$ are not algebraic over K , $\tilde{F}$ has no irreducible factors vanishing at $(b/a,b)$ and depending only on one variable. Thus, $b/a,b$ are algebraically dependent.

Now by lemma 4.2 any two non-algebraic over K elements in $\mathrm{Quot}(R)$ are algebraically dependent. □

Corollary 4.3. *If non-algebraic over K elements $a,b\in R$ are algebraically dependent, then any two non-algebraic over K elements from $K[a,b]$ are algebraically dependent.*

Proof. Obviously, any non-algebraic element in $K[a]$ is algebraically dependent with a (we can write down the polynomial explicitly). Therefore, any two non-algebraic elements are algebraically dependent in $K[a]$. Analogously any two non-algebraic elements are algebraically dependent in $K[b]$. So, by 4.2, 4.2 any two non-algebraic elements from $K(a)$ and $K(b)$ are algebraically dependent.

Now take any non-algebraic over K element $f\in K[a,b]$, say $f=\sum_{i=0}^n f_ib^i$, $f_i\in K(a)$. We claim that f,b are algebraically dependent. Use induction on n : if $n=0$, then we already know that f is algebraically dependent with any element from $K[b]$. In general situation

$$f=f_n(b^n+\frac{f_{n-1}}{f_n}b^{n-1}+\ldots+\frac{f_0}{f_n})=:f_n(b^n+f').$$

If (b^n+f') is algebraic over K , then by the arguments from the proof of corollary 4.2 f,f_n are algebraically dependent, whence f,b are algebraically dependent by lemma 4.2. So, we can assume (b^n+f') is not algebraic over K . If f' is algebraic over K , then we can use the same trick: let $F(T)$ be an irreducible polynomial such that $F(f')=0$. Then the polynomial $F(T_2-T_1^n)$ vanishes at $(b^n+f'),b$, and this leads to algebraic dependence of $(b^n+f'),b$. If f' is not algebraic over K , by induction f',b are algebraically dependent. Therefore, $(b^n+f'),b$ are algebraically dependent: if $F(T_1,T_2)$ is an irreducible polynomial with $F(f',b)=0$, then $F(T_1-T_2^n,T_2)$ is a polynomial vanishing at $(b^n+f'),b$, and this leads to algebraic dependence of $(b^n+f'),b$.

If f_n is algebraic, the arguments above say that f,b are algebraically dependent. If not, by lemma 4.2 $(b^n+f'),f_n$ are algebraically dependent, and we can use again the trick from previous corollary to show that f,f_n are algebraically dependent, whence f,b are algebraically dependent and we are done. □

Now let's prove the second property of a commutative ring of ODOs.

Proposition 4.2. *Let $B\subset D$ be a commutative subring as in proposition 4.1. Then any two non-algebraic over K elements are algebraically dependent.*

Proof. We use notations from the proof of proposition 4.1.

Take any $P\in B$ and choose $Q\in B$ such that $GCD(\mathrm{ord}\,P,\mathrm{ord}\,Q)=r$ (note: such Q exists, because for any $n\gg 0$ $nr\in N_B$, see proposition 4.1). Consider now the ring $\tilde{B}=K[P,Q]\subset B$ and repeat arguments from proposition 4.1. Then $N_{\tilde{B}}\supset rn$, $n\geq -j\alpha'\beta'$. Again note that $j<0$, for if $j=0$ then $Q\in K[P]$ (or $P\in K[Q]$), therefore P,Q are algebraically dependent.

Let $\{u_1,\ldots,u_q\}$ be a K -linear basis for $\tilde{B}^{(-rj\alpha'\beta'-r)}$ and let $\varphi_n=P^{m+il}Q^{-(\alpha'-l)j}$ for $n\geq -j\alpha'\beta'$. Then $\{u_1,\ldots,u_q\}\cup\{\varphi_n\}_{n\geq -j\alpha'\beta'}$ form a K -basis for $\tilde{B}$. Since any φ_n is not the power of P, for $N \gg 0$ P^N is a linear combination of $u_1, \dots, u_q, \varphi_{n_1}, \dots, \varphi_{n_k}$ (and u_i include restricted powers of P, so that N can be chosen bigger than they together). Thus we get a non-trivial polynomial relation (with the highest coefficient one in variable P):

$$P^N = a_{k_1}P^{k_1} + \dots + a_0, \quad k_i < N\,,\ a_{k_i} \in K[Q]\,,$$

and P, Q are algebraically dependent. Note that for any $P_1, P_2 \in B$ there exists $Q \in B$ such that $GCD(\operatorname{ord} P_1, \operatorname{ord} Q) = r$ and $GCD(\operatorname{ord} P_2, \operatorname{ord} Q) = r$. Then P_1, Q and P_2, Q are algebraically dependent and by Lemma 4.2 P_1, P_2 are algebraically dependent. □

The proposition 4.2 can be reformulated using the notion of a *transcendence basis.*

Definition 4.3. Let $\tilde{K}/k$ be a field extension. A *transcendence basis* of $\tilde{K}/k$ is a set T of algebraically independent elements over k (here we mean that any *finite number* of elements from T are algebraically independent) such that $\tilde{K}$ is algebraic over $k(T)$.

On a set σ of algebraically independent subsets there is a partial order: $T_1 \le T_2$ if $T_1 \subseteq T_2$. For any chain $T_1 \subset T_2 \subset \dots$ there is an upper bound $T = \cup_i T_i$ (i.e. $T \ge T_i$ for any i). Then by Zorn's lemma[2] there are maximal elements in σ. Therefore, the transcendence basis exists. We'll consider only extensions with *finite* transcendence bases.

Theorem 4.3. *Let $\tilde{K} \supset k$ be an extension of fields. Then any two transcendence bases of $\tilde{K}$ over k have equal cardinality.*

If $\tilde{K} = k(\Gamma)$, where Γ is a set of generators, and $T \subset \Gamma$ is a subset of algebraically independent elements, then there exists a transcendence basis β of $\tilde{K}$ over k such that $T \subset \beta \subset \Gamma$.

Proof. Let $\{x_1, \dots x_m\}$ be a transcendence basis, and $\{w_1 \dots, w_n\}$ are algebraically independent elements. It is suffice to prove that $n \le m$, since then by symmetry $m \le n$ and therefore $m = n$.

Let's prove it. There exists a non-zero polynomial $f_1(w_1, x_1, \dots, x_m) = 0$ (since w_1 is algebraic over $k(x_1, \dots, x_m)$). Without loss of generality we can assume that f_1 depends on x_1. Then it means that w_1, x_1 are algebraically dependent over $k(x_2, \dots x_m)$ and x_1 is algebraic over $k(w_1, x_2, \dots, x_m)$. Therefore, $\tilde{K}$ is algebraic over $k(w_1, x_2, \dots, x_m)$. For, for any non-algebraic over $k(x_2, \dots, x_m)$ element $\alpha \in \tilde{K}$ the elements α, x_1 are algebraically dependent over $k(x_2, \dots, x_m)$, hence by lemma 4.2 α, w_1 are algebraically dependent over $k(x_2, \dots, x_m)$ and therefore α is algebraic over $k(w_1, x_2, \dots, x_m)$.

Now we use induction: if $\tilde{K}$ is algebraic over $k(w_1, \dots, w_r, x_{r+1}, \dots, x_m)$ for $r < n$ then there exists a non-zero polynomial $f(w_{r+1}, w_1, \dots, w_r, x_{r+1}, \dots, x_m) = 0$. Without loss of generality (by renumbering the variables) we can assume that f depends on x_{r+1} (if f contains no x_i this would mean that $w_1, \dots, w_{r+1}$ are algebraically dependent, a contradiction). Then x_{r+1} is algebraic over $k(w_1, \dots, w_{r+1}, x_{r+2}, \dots, x_m)$.

Now again for any non-algebraic over $k(w_1, \dots, w_r, x_{r+2}, \dots, x_m)$ element $\alpha \in \tilde{K}$ the elements α, x_{r+1} are algebraically dependent over $k(w_1, \dots, w_r, x_{r+2}, \dots, x_m)$ and x_{r+1}, w_{r+1} are algebraically dependent over
$k(w_1, \dots, w_r, x_{r+2}, \dots, x_m)$. Hence by lemma 4.2 α, w_{r+1} are algebraically dependent over $k(w_1, \dots, w_r, x_{r+2}, \dots, x_m)$ and therefore α is algebraic over $k(w_1, \dots, w_{r+1}, x_{r+2}, \dots, x_m)$. Then, if $n > m$ we deduce that $\tilde{K}$ is algebraic over $k(w_1, \dots, w_m)$, a contradiction, since $w_1, \dots, w_n$ are algebraically independent. □

[2] Zorn's lemma says: Let S be a non-empty partially ordered set (i.e. we are given a relation $x \le y$ on S which is reflexive and transitive and such that $x \le y$ and $y \le x$ together imply $x = y$). A subset T of S is a chain if either $x \le y$ or $y \le x$ for every pair of elements x, y in T. If every chain of T has an upper bound in S (i.e. if there exists $x \in S$ such that $t \le x$ for all $t \in T$) then S has at least one maximal element.

Notation 4.2. We denote by $\mathrm{trdeg}(\tilde{K}/k)$ the cardinality of a transcendence basis.

Corollary 4.4. *Let $B \subset D$ be a commutative subring as in proposition 4.1. Then* $\mathrm{trdeg}(\mathrm{Quot}(B)/K) = 1$ *.*

Proof. Indeed, by corollaries 4.2 and 4.3 any two elements in $\mathrm{Quot}(B)$ are algebraically dependent over K. Obviously, an operator of positive order is transcendental over K. □

Remark 4.1. It is useful to mention another notion of "measure" on the commutative ring (except the transcendence degree), the *Krull dimension.* Here we list important results on the Krull dimension of rings, which we will not prove. The proofs one can find e.g. in [1, Ch.11].

Definition 4.4. Let R be a ring. The *height* of a prime ideal $\wp$ in the ring R is a supremum of the lengths of all chains of prime ideals $\wp_0 \subset \wp_1 \subset \dots \wp_n = \wp$, and is denoted by $ht(\wp)$.

The *Krull dimension* of R is a supremum of the lengths of all chains of prime ideals, and is denoted by $\dim(R)$.

Examples of rings with finite dimension: rings finitely generated over K, local Noetherian rings (see below).

Proposition 4.3. *[1, Ch. 11] Let B be an integral domain finitely generated over K. Then*

1. $\dim(B) = trdeg_K(\mathrm{Quot}(B))$ *;*
2. *for any prime ideal $\wp \subset B$ we have*

$$ht(\wp) + \dim(B/\wp) = \dim B.$$

In order to explain what is the *spectral curve* of a commutative ring of ODOs, we need more facts from Commutative Algebra.

4.0.1 More from CA about Noetherian rings

Let R be a commutative ring. We shall always consider ideals $I \subset R$ different from R itself. Then the quotient ring R/I is also a commutative ring.

Definition 4.5. An ideal $I \subset R$ is called *prime* if the quotient ring R/I has no zero divisors.

An ideal I is *maximal* if it is not contained in another ideal (different from R). Then the ring R/I has no non-zero ideals (otherwise its preimage in R would be an ideal containing I), hence every element $x \in R/I$, $x \neq 0$, is invertible (since the principal ideal (x) must coincide with R/I, and thus contain 1), in other words, R/I is a field. Since a field has no non-trivial ideals, the converse is also true, so that $I \subset R$ is maximal iff R/I is a field. By Zorn's lemma, each ring contains a maximal ideal.

An important finiteness property of rings is encoded in a notion *Noetherian ring*, given in the following proposition-definition.

Proposition 4.4. *A ring R satisfying any of the following equivalent properties is called Noetherian:*

1. *any chain of ideals $I_1 \subset I_2 \subset I_3 \subset \dots$ of R stabilizes (that is, there is an integer m such that $I_m = I_{m+1} = I_{m+2} = \dots$),*
2. *any set of ideals of R contains a maximal element,*
3. *any ideal of R is generated by finitely many elements, that is, is an R-module of finite type.*

Proof. The equivalence of (1) and (2) is completely formal.

(3) $\Rightarrow$ (1): Let $I = \sum I_j$, then I is an ideal which is generated, say, by $x_1, \dots, x_n$ as an R-module. Take m such that I_m contains all the x_i, then the chain stabilizes at I_m.

(2) $\Rightarrow$ (3) is based on a trick called "Noetherian induction" (cf. [12, Ch.2, Exer. 3.16]). Suppose that $I \subset R$ is an ideal which is not of finite type as an R-module. Consider the set of subideals of I which are of finite type as R-modules. This set is not empty: it contains 0. Now it has a maximal element $J \neq I$. Take $x \in I \backslash J$, then the ideal $J + (x) \subset I$ is strictly bigger than J, but is of finite type as an R-module. Contradiction. □

Exercise 4.2. i) Prove equivalence of (1) and (2);

ii) Let R be a Noetherian ring, $I \subset R$ is an ideal. Show that R/I as also a Noetherian ring.

The easiest example of a Noetherian ring is a field. Hilbert's basis theorem produces a lot of examples of Noetherian rings.

Theorem 4.4 (Hilbert's basis theorem)**.** *If R is a Noetherian ring, then so are the polynomial ring $R[z]$ and the formal power series ring $R[[z]]$.*

Proof. Let $I \subset R[z]$ be an ideal. We associate to it a series of ideals in R: $I_0 \subset I_1 \subset I_2 \subset \dots$, where I_j is generated by the leading coefficients of polynomials in I of degree j. Since R is Noetherian, this chain of ideals stabilizes, say, at I_r. Then we have a finite collection of polynomials whose leading coefficients generate $I_0, \dots, I_r$. Then the ideal of $R[z]$ generated by these polynomials is I. The same idea works with $R[[z]]$, if we use the discrete valuation instead of degree function. □

Exercise 4.3. Prove the theorem 4.4 for $R[[z]]$.

Definition 4.6. Now we are able to formulate what is the spectral curve of B. There is the following basic geometric observation: if $B = K[P_1, \dots, P_n]$ is a finitely generated ring, then there is a surjective homomorphism of rings

$$\varphi : K[T_1, \dots, T_n] \to B, \quad T_i \mapsto P_i.$$

Note that $I := \ker \varphi$ is an ideal. By Hilbert's basis theorem I is finitely generated by, say, $f_1, \dots, f_k$. Then $B \simeq K[T_1, \dots, T_n]/(f_1, \dots, f_k)$. Let $K = \mathbb{C}$. The set

$$\begin{cases} f_1(T_1, \dots, T_n) = 0 \\ f_2(T_1, \dots, T_n) = 0 \\ \qquad \vdots \\ f_k(T_1, \dots, T_n) = 0 \end{cases}, \qquad (T_1, \dots, T_n) \in \mathbb{C}^n$$

is a geometric object, an algebraic set (affine variety), called the *affine spectral curve.* We'll denote it by $C_0 = \operatorname{Spec} B$.

Later we will see that the affine curve can be completed by one smooth point. So, if the curve is smooth, then the completion will be a Riemann surface of finite genus.

Example 4.1. Consider the following example of Wallenberg:

$$P = \partial^2 - \frac{2}{(x+1)^2}, \quad Q = 2\partial^3 - \frac{6}{(x+1)^2}\partial + \frac{6}{(x+1)^3}$$

commute. The ring $B = \mathbb{C}[P, Q] \simeq \mathbb{C}[T_1, T_2]/(f)$, where $f = T_2^2 - 4T_1^3$. Thus the spectral curve in this case is a plane curve defined by the equation $f(T_1, T_2) = 0$.

Remark 4.2. First proof of the fact that two commuting operators are algebraically dependent belongs to Burchnall and Chaundy [5]. Their famous lemma was proved by different method, which gives in particular an explicit form of the equation of the algebraical dependence, cf. [22, Lemma 1.11]. Moreover, their method is applicable also in the case of commuting *difference operators*.

Let's define the spectral module:

Definition 4.7. Let $B \subset D$ be a commutative subring. Consider the right D–module $F := D/xD \simeq K[\partial]$, $\overline{a(x)\partial^n} \mapsto a(0)\partial^n$. Clearly, the *right* action of D on $K[\partial]$ is given by the following rules:

$$\begin{cases} p(\partial) \circ \partial &= \partial \cdot p(\partial) \\ p(\partial) \circ x &= p'(\partial). \end{cases} \tag{5}$$

Restricting the action (5) on the subalgebra B, we endow F with the structure of a B–module. Since the algebra B is commutative, we shall view F as a *left* B–module (although having the natural right action in mind). The module F is called the *spectral module*.

Note that if B is elliptic, then F is torsion free, i.e. for any non-zero $f \in F$ and for any non-zero $b \in B$ $fb \neq 0$.

Definition 4.8. Let B be a commutative subring of D. We call the natural number

$$r = \mathrm{rk}(B) = \gcd\left\{\mathsf{ord}(P) \middle| P \in B\right\}$$

the *rank* of B.

To explain the dictionary between geometric and algebraic objects, as well as to explain the construction of completion of the spectral curve we need to recall more facts from CA.

4.0.2 More from CA about integral elements

Proposition 4.5. *Let $A \subset B$ be integral domains. The following conditions are equivalent:*

1. *$x \in B$ is integral over A,*
2. *$A[x]$ is an A-module of finite type,*
3. *There exists an A-module M of finite type such that $A \subset M \subset B$ and $xM \subset M$.*

Proof. The proof of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) is direct. Suppose we know (3). Let $m_1, \ldots, m_n$ be a system of generators of M. Then $xm_i = \sum_{i=1}^n b_{ij} m_j$, where $b_{ij} \in A$.

Recall that in any ring R we can do the following "determinant trick". Let N be a matrix with entries in R. Let $adj(N)$ be the matrix with entries in R given by

$$adj(N)_{ij} = (-1)^{i+j} \det(N(j,i)),$$

where $N(i,j)$ is N with i-th row and j-th column removed. It is an exercise in linear algebra that the product $adj(N) \cdot N$ is the scalar matrix with $\det(N)$ on the diagonal.

We apply this trick to the polynomial ring $R = A[T]$. For N we take the $n \times n$-matrix $Q(T)$ such that $Q(T)_{ij} = T\delta_{ij} - b_{ij}$. Let $f(T) = \det(Q(T)) \in A[T]$ (this is the analogue of the characteristic polynomial of x). We have a matrix identity

$$adj(Q(T)) \cdot Q(T) = diag(f(T)).$$

We consider this as an identity between matrices over the bigger ring $B[T]$. We are free to assign T any value in B. Substitute $T = x \in B$, and apply these matrices to the column vector $(m_1, \ldots, m_n)^T$. Then the left hand side is zero. Hence $f(x)m_i = 0$ for any i. Since the m_i generate M, the whole module M is annihilated by $f(x) \in B$. In particular, $f(x) \cdot 1 = 0$, that is, $f(x) = 0$. Now note that $f(T)$ has coefficients in A and leading coefficient 1. □

Definition 4.9. Let $A \subset B$ be integral domains, then B is integral over A if every its element is integral over A. The set of elements of B which are integral over A is called the *integral closure* of A in B.

Let us prove some basic properties of integral elements.

Proposition 4.6. *1. The integral closure is a ring.*

2. Suppose that B is integral over A, and is of finite type as an A-algebra. Then B is of finite type as an A-module.

3. Suppose that C is integral over B, and B is integral over A, then C is integral over A.

Proof. (1) Let $x, y \in B$ be integral over A. Consider the A-module generated by all the monomials $x^i y^j$, $i, j \geq 0$. All higher powers of x, y can be reduced to finitely many of its powers using monic polynomials whose roots are x, y. So, the module is of finite type, and xy and $x+y$ act on it. Then by proposition 4.5 (3) xy and $x+y$ are integral.

(2) Suppose that B is generated by $b_1, \ldots, b_n$ as an A-algebra, then B is generated by monomials $b_1^{i_1} \ldots b_n^{i_n}$ as an A-module. As in item 1), all higher powers of each of the b_i's can be reduced to finitely many of its powers using a monic polynomial whose root is b_i. There remain finitely many monomials which generate B as an A-module.

(3) Let $x \in C$. Consider the A-subalgebra $F \subset C$ generated by x and the coefficients b_i of a monic polynomial with coefficients in B, whose root is x. Then F is an A-module of finite type, as only finitely many monomials generate it (the b_i are integral, and the higher powers of x can be reduced to lower powers). Now use item 3) of proposition 4.5. □

Definition 4.10. A ring is *integrally closed* or *normal* if it is integrally closed in its field of fractions.

Example 4.2. The rings $K[x]$ and $K[x, y]$ are integrally closed, but $K[x, y]/(y^2 - x^2 - x^3)$ is not.

Theorem 4.5 (Noether's normalization lemma)**.** *Let k be any field, and $I \subset k[T_1, \ldots, T_n]$ be an ideal, $R = k[T_1, \ldots, T_n]/I$. There exist algebraically independent elements $Y_1, \ldots, Y_m \in R$ such that R is integral over $k[Y_1, \ldots, Y_m]$.*

Proof. If $I = 0$ there is nothing to prove. Suppose we have a non-zero polynomial $f \in I$. Let d be a positive integer greater than $\deg(f)$. Let us choose new variables in the following tricky way:

$$T_2' = T_2 - T_1^d, \quad T_3' = T_3 - T_1^{d^2}, \quad T_4' = T_4 - T_1^{d^3}, \quad \ldots, \quad T_n' = T_n - T_1^{d^{n-1}}.$$

Substituting $T_2 = T_2' + T_1^d, \ldots$ into f we rewrite it as a linear combination of powers of T_1 and a polynomial, say, g containing no pure powers of T_1. We observe that the pure powers of T_1 are of the form $i_1 + di_2 + d^2 i_3 + \ldots + d^{n-1} i_n$. Since $d > i_s$, all these integers are different, hence there is no cancellation among the pure powers of T_1. At least one such power enters with a nonzero coefficient. On the other hand, any power of T_1 in g is strictly less than the corresponding pure power. Therefore, we get a polynomial in T_1 with coefficients in $k[T_2', \ldots, T_n']$ and leading coefficient in k. Normalizing this polynomial we conclude that T_1 is integral over $R_1 = k[T_2', \ldots, T_n']/(I \cap k[T_2', \ldots, T_n'])$. Hence R is integral over R_1. We now play the same game with R_1 instead of R, and obtain a subring R_2 over which R_1 is integral. By proposition 4.6, item (3) R is also integral over R_2. We continue like that until we get a zero ideal, which means that the variables are algebraically independent. □

Exercise 4.4. Show that for $B \subset D$ from proposition 4.1 the field $\mathrm{Quot}(B)$ is a finite algebraic extension, i.e. a module of finite type, over $K(Q)$, $Q \in B$, $\mathrm{ord}(Q) > 0$.

4.0.3 Nullstellensatz

Theorem 4.6. *Let k be an algebraically closed field. All maximal ideals of $k[X_1, \dots, X_n]$ are of the form $(X_1 - a_1, \dots, X_n - a_n)$, $a_i \in k$, that is, consist of polynomials vanishing at a point $(a_1, \dots, a_n) \in k^n$.*

Proof. Any polynomial has a Taylor expansion at the point $(a_1, \dots, a_n)$. The canonical map

$$k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]/(X_1 - a_1, \dots, X_n - a_n)$$

sends f to $f(a_1, \dots a_n)$, hence is surjective onto k. It follows that the ideal $(X_1 - a_1, \dots, X_n - a_n)$ is maximal.

Let M be a maximal ideal (recall that $M \neq k[X_1, \dots, X_n]$), then $\tilde{K} := k[X_1, \dots, X_n]/M$ is a field containing k. By Noetherian normalization $\tilde{K}$ is integral over its subring $A = k[Y_1, \dots, Y_m]$. But $\tilde{K}$ is a field, and we now show that then A must also be a field, in which case $k[Y_1, \dots, Y_m] = k$ (no variables at all), and hence $\tilde{K}$ is integral over k. Indeed, let $x \in A$, then it is enough to show that $x^{-1} \in \tilde{K}$ also belongs to A. Since $x^{-1} \in \tilde{K}$ is integral over A it is subject to a polynomial relation $(x^{-1})^n + a_{n-1}(x^{-1})^{n-1} + \dots + a_1 x^{-1} + a_0 = 0$, for some $a_i \in A$. Multiplying this by x^{n-1} we express x^{-1} as a polynomial in x with coefficients in A, hence $x^{-1} \in A$.

The k-algebra of finite type $\tilde{K}$ is integral over k, hence by proposition 4.5, item 2) $\tilde{K}$ is a k-module (= vector space over k) of finite type (= of finite dimension). Since k is algebraically closed, we must have $k = K$. Now let $a_i \in k$ be the image of X_i under the map $k[X_1, \dots, X_n] \to k = k[X_1, \dots, X_n]/M$. Then M contains the maximal ideal $(X_1 - a_1, \dots, X_n - a_n)$, hence coincides with it. □

Remark 4.3. When k is not supposed to be algebraically closed, this proof shows that the quotient by a maximal ideal of $k[X_1, \dots, X_n]$ is a finite extension of k.

Corollary 4.5. *Let k be an algebraically closed field. If the polynomials of an ideal $I \subset k[X_1, \dots, X_n]$ have no common zeros in k^n, then $I = k[X_1, \dots, X_n]$.*

Proof. Assume $I \neq k[X_1, \dots, X_n]$. Hilbert's basis theorem says that $k[X_1, \dots, X_n]$ is Noetherian. Then I is contained in a maximal ideal, since the set of ideals that contain I has a maximal element, by item 2) of proposition 4.4 above. Therefore $I \subset (X_1 - a_1, \dots, X_n - a_n)$, for some $a_i \in k$, since all the maximal ideals are of this form by the previous result. But then all the polynomials of I vanish at the point $(a_1, \dots, a_n)$, which is a contradiction. □

Theorem 4.7 (Nullstellensatz)**.** *Let k be an algebraically closed field. If a polynomial f vanishes at all the zeros of an ideal $I \subset k[X_1, \dots, X_n]$, then $f^m \in I$ for some positive integer m.*

Remark 4.4. Let $I \subset A$ be an ideal in a ring A. The ideal

$$\sqrt{I} = \{ f \in A | \quad f^r \in I \text{ for some integer } r > 0 \}$$

is called the *radical* of the ideal I.

Proof. We know that I is generated by finitely many polynomials, say, $I = (g_1, \dots, g_r)$. Let T be a new variable. Consider the ideal $J \subset k[T, X_1, \dots, X_n]$ generated by $g_1, \dots, g_r$ and $Tf - 1$. We observe that these polynomials have no common zero. The previous corollary implies that $J = k[T, X_1, \dots, X_n]$, in particular, J contains 1. Then there exist polynomials $p, p_1, \dots, p_r$ in variables $T, X_1, \dots, X_n$ such that

$$1 = p(Tf - 1) + p_1 g_1 + \dots + p_r g_r.$$

Note that this is an identity in variables $T, X_1, \dots, X_n$. Thus we can specialize the variables anyway we like. For example, we can set $T = 1/f$. Multiplying both sides by an appropriate power of f we get an identity between polynomials in variables $X_1, \dots, X_n$, which gives that some power of f belongs to $I = (g_1, \dots, g_r)$. □

The Nullstellensatz and Hilbert's basis theorem form a foundation of a "dictionary" between commutative algebra and algebraic geometry. Below we give an overview of basic concepts from affine and projective algebraic geometry. We need these concepts to explain the connection between algebraic objects arising from a commutative ring of differential operators and geometric objects from complex algebraic geometry.

4.1 Affine algebraic geometry

Affine algebraic geometry studies the solutions of systems of polynomial equations with coefficients in k (k is any field). Let $A = k[X_1, \dots, X_n]$ be the polynomial ring in n variables. We can consider elements of A as functions on the *affine space* k^n. Let

$$Z(T) = \{Q \in k^n | \quad f(Q) = 0 \text{ for all } f \in T\}$$

be the set of zeros of a subset $T \subset A$. Instead of a set of polynomials it is better to consider the ideal of the polynomial ring A generated by them. The subsets of k^n consisting of common zeros of the subset of polynomials are called *closed algebraic sets.* They define the *Zariski topology* on k^n.

4.1.1 Zariski topology

Let us prove some easy facts about closed algebraic sets. If $X \subset k^n$ we denote by $I(X) \subset k[X_1, \dots, X_n]$ the ideal consisting of polynomials vanishing at all the points of X. It is a tautology that $X \subset Z(I(X))$ and $J \subset I(Z(J))$. If X is a closed algebraic set, then $X = Z(I(X))$ (if $X = Z(J)$, then $I(Z(J)) \supset J$, hence $Z(I(Z(J))) \subset Z(J)$).

It is clear that the function $J \to Z(J)$ reverses inclusions; associates the empty set to the whole ring, and the whole affine space k^n to the zero ideal; sends the sum of (any number of) ideals to the intersection of corresponding closed sets; and sends the intersection $I_1 \cap I_2$ to $Z(I_1) \cup Z(I_2)$.

Because of these properties we can think of closed algebraic sets as the closed sets for some topology on k^n (any intersections and finite unions are again closed, as are the empty set and the whole space). This topology is called *Zariski topology.* In the case when $k = \mathbb{C}$ or $k = \mathbb{R}$ we can compare it with the usual topology on $\mathbb{C}^n$ where closed sets are the zeros of continuous functions. Any Zariski closed set is also closed for the usual topology but not vice versa. Hence the Zariski topology is weaker. Another feature is that any open subset of k^n is dense (its closure is the whole k^n).

Definition 4.11. A closed algebraic subset $X \subset k^n$ is irreducible if there is no decomposition $X = X_1 \cup X_2$, where $X_1 \neq X$ and $X_2 \neq X$ are closed algebraic sets.

Proposition 4.7. *A closed algebraic subset $X \subset k^n$ is irreducible iff $I(X)$ is a prime ideal.*

Any closed set has a unique decomposition into a finite union of irreducible subsets $X = \cup_i X_i$ such that $X_i \nsubseteq X_j$ for $i \neq j$ (these X_i's are called the irreducible components of X).

Proof. Suppose X is irreducible. If $fg \in I(X)$, then $X \subset Z(fg) = Z(f) \cup Z(g)$. Therefore, $X = (X \cap Z(f)) \cup (X \cap Z(g))$. Since X is irreducible, we have either $X = X \cap Z(f)$ and $X \subset Z(f)$ or $X = X \cap Z(g)$ and $X \subset Z(g)$. Therefore, either $f \in I(X)$ or $g \in I(X)$, i.e. the ideal $I(X)$ is prime.

Conversely, let $\wp$ be a prime ideal. Suppose that $Z(\wp) = X_1 \cup X_2$. Then $\wp = I(X_1) \cap I(X_2)$, i.e. either $\wp = I(X_1)$ or $\wp = I(X_2)$. Therefore, $Z(\wp) = X_1$ or $Z(\wp) = X_2$, i.e. $Z(\wp)$ is irreducible.

Now let's prove the existence of the finite decomposition of X. Let Σ be the set of non-empty closed subsets of X which *can not be represented* as a finite union of irreducible closed subsets. Suppose that Σ is not empty. Since the ring $k[X_1, \dots, X_n]/I(X)$ is Noetherian, any chain of closed subsets $X \supset Y_1 \supset Y_2 \supset \dots$ stabilizes, i.e. there exists $r > 0$ such that $Y_r = Y_{r+1} = \dots$. Therefore, Σ has a minimal element, say Y. The subset Y can not be irreducible by definition of Σ, hence $Y = Y' \cup Y''$, where Y' and Y'' are proper closed subsets in Y. Since Y is minimal, the sets Y', Y'' can be represented as finite union of closed irreducible subsets. Therefore, Y also can be represented in such a way, a contradiction. Thus, $X = X_1 \cup \dots \cup X_r$, and we can assume w.l.o.g. that $X_i \not\subseteq X_j$ for $i \neq j$ (by deleting proper subsets from the union).

Assume that there exists another representation $X = X_1' \cup \dots X_s'$. Then $X_1' = \cup (X_1' \cap X_i)$. But since X_1' is irreducible, we have $X_1' \subset X_i$ for some i, say $i = 1$. Analogously, $X_1 \subset X_j'$ for some j. Then $X_1' \subset X_j'$ whence $j = 1$ and $X_1 = X_1'$. Now set $Z = \overline{X - X_1}$. Then $Z = X_2 \cup \dots X_r = X_2' \cup \dots X_s'$. By induction on r we obtain the uniqueness of the decomposition. $\square$

Let k be an algebraically closed field. Let us call an ideal $I \subset k[X_1, \dots, X_n]$ *radical* if $\sqrt{I} = I$. A corollary of Hilbert's Nullstellensatz is that radical ideals bijectively (via operations I and Z) correspond to closed algebraic sets. The most important class of radical ideals are prime ideals. Again, by Hilbert's Nullstellensatz, these bijectively correspond to irreducible closed algebraic sets. A particular case of prime ideals are maximal ideals, they correspond to points of k^n.

Definition 4.12. An *affine variety* is a closed irreducible algebraic subset of k^n for some n. The variety k^n will be also denoted $\mathbb{A}_k^n$, and called the affine space of dimension n.

Let $X \subset \mathbb{A}_k^n$ be an affine variety. Let $J = I(X)$ be the corresponding prime ideal. Let us denote $k[X] := k[X_1, ..., X_n]/J$. Then $k[X]$ is an integral k-algebra of finite type. $k[X]$ is called the coordinate ring of X. The fraction field of $k[X]$ is denoted by $k(X)$, and is called the function field of X. Its elements are called *rational functions* as opposed to the elements of $k[X]$ which are called *regular functions*.

The function field $k(X)$ is an important object defined by X. Two affine varieties X and Y are called *birationally equivalent* if $k(X) = k(Y)$. A variety X is called rational if $k(X)$ is a purely transcendental extension of k, that is, $k(X) = k(T_1, \dots, T_l)$.

Zariski topology on $\mathbb{A}_k^n$ induces a topology on a variety $X \subset \mathbb{A}_k^n$. An open subset $U \subset X$ is an intersection of X with an open set of $\mathbb{A}_k^n$. Such sets are called *quasi-affine* varieties.

Definition 4.13. The *dimension* of the topological space X can be defined as the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \dots \subset Z_n$ of distinct closed irreducible subsets of X.

From the Nullstellensatz it follows immediately that $\dim(X) = \dim(k[X])$. By proposition 4.3 the dimension is equal to the transcendence degree of the field $k(X)$.

4.2 Regular functions and morphisms of affine varieties

In order to explain the second component of *spectral data* associated to a commutative ring $B \subset D$, we need to introduce, in particular, the notion of a sheaf of modules on an affine variety (cf. §3 of Wei-Ping Li's lectures on this school!). To do this, we need to introduce the notion of a ring of regular functions on the affine variety, which appears to be naturally isomorphic to the ring B. Also we need to introduce the notion of a morphism of affine varieties, which appear to be defined by a homomorphism of corresponding commutative rings.

Subrings of regular functions on *open subsets* of affine varieties can be obtained with the help of a localisation procedure:

4.2.1 Localization, local rings, DVR

Definition 4.14. Let R be a ring. A subset $S \subset R$ is called *multiplicative* if it is closed under multiplication and contains 1. The *localization* $S^{-1}R$ of R with respect to S is defined as the set of formal fractions $\frac{a}{b}$, with $a \in R$ and $b \in S$, up to the equivalence relation: $\frac{a}{b} = \frac{a_1}{b_1}$ if $(ab_1 - a_1 b)s = 0$ for some $s \in S$.

When R has no zero divisors, the natural map $a \to \frac{a}{1}$ is an injective homomorphism of rings, so that we can think of R as a subset of $S^{-1}R$. Then $S^{-1}R$ is simply the fractions with "restricted denominators". The set formal fractions form a ring with a usual addition and multiplication of fractions. Note that if $I \subset R$ is an ideal, then $S^{-1}I$ is an ideal in $S^{-1}R$.

The localisation of a R-module M with respect to the multiplicative subset $S \subset R$ is defined in analogous way. The $S^{-1}R$-module $S^{-1}M$ is canonically isomorphic to the $S^{-1}R$-module $S^{-1}R \otimes_R M$ ([1, Prop.3.5.]).

Exercise 4.5. Prove that a localization of a Noetherian ring is Noetherian.

Example 4.3. If $S = R\backslash\{0\}$ and R is an integral domain, then $S^{-1}R$ is just the field of fractions.

Another important example: let $S = \{a^n\}$, $n = 0, 1, 2, \dots,$ $a \neq 0$. The localisation $S^{-1}R$ is usually denoted by R_a, and can be understood as the ring of polynomials $R[a^{-1}] \subset \mathrm{Quot}(R)$ if R has no zero divisors. Such rings play important role in algebraic geometry: they form rings of regular functions on open affine sets (see section about affine morphisms of algebraic varieties below). Besides, if R is finitely generated over a field K, then R_a is also finitely generated.

Definition 4.15. Let $\wp \subset R$ be a prime ideal, then $S = R\backslash\wp$ is a multiplicative system. Then $S^{-1}R$ is denoted $R_\wp$ and is called the *localization* of R at $\wp$. The ring $R_\wp$ has a very important property: $S^{-1}\wp$ is its only maximal ideal (every element not in $S^{-1}\wp$ is by definition invertible, hence $S^{-1}\wp$ contains all other ideals). Such rings have a name:

Rings with just one maximal ideal are called *local rings.* If R is a local ring and $\mathcal{M}$ is its maximal ideal then the field $R/\mathcal{M}$ is called the *residue field*. Note that if R is a localisation of a finitely generated ring over an algebraically closed field K at a maximal ideal, then the residue field is a finite extension over K, i.e. it is K.

Note that the ring $R_\wp$ is not finitely generated even if R is finitely generated over K.

Example 4.4.

1. Rational functions in one variable over a field K such that the denominator does not vanish at 0,
2. Formal power series $K[[z]]$,
3. Rational functions in two variables such that the denominator does not vanish at $(0, 0)$.

In all these examples, except the last one, the maximal ideal is principal. Such rings form the simplest class of local rings.

Definition 4.16. Let $\tilde{K}$ be a field. A subring $R \subset \tilde{K}$ is called a *discrete valuation ring* (DVR for short) if there is a discrete valuation v on $\tilde{K}$ such that $R\backslash\{0\} = \{x \in \tilde{K}^* | v(x) \geq 0\}$. In particular, $\tilde{K} = \mathrm{Quot}(R)$.

Remark 4.5. Recall that in our lectures we work with fields over K and K-valuations.

Proposition 4.8. *The following conditions are equivalent*

1. *R is a DVR.*

2. *R is a local ring which is a Noetherian integral domain and whose maximal ideal is principal.*

Moreover, all ideals of R are principal.

Proof. Let R be a DVR. Then it has the ideal $\mathcal{M} = \{x \in R | v(x) > 0\}$. Note that any element of $R \backslash \mathcal{M}$ is invertible, thus $\mathcal{M}$ is a unique maximal ideal. Obviously, $\mathcal{M}$ is generated by any element of valuation one.

To prove the converse statement we need the following lemma.

Lemma 4.3. *Let R be a Noetherian integral domain, $t \in R \backslash R^*$. Then $\cap_{i=1}^{\infty}(t^i) = 0$.*

Proof. For contradiction let $x \neq 0$ be contained in (t^i), for any $i \geq 1$. We write $x = t^i x_i$, then $(x) \subset (x_1) \subset (x_2) \subset \ldots$ is an ascending chain of ideals. Then $(x_{i+1}) = (x_i) = (t x_{i+1})$ for some i. Hence $x_{i+1} = t a x_{i+1} \Leftrightarrow x_{i+1}(1 - ta) = 0$, but this implies $t \in R^*$ since $x_{i+1} \neq 0$ and R is an integral domain, a contradiction. □

Let R be a local ring which is a Noetherian integral domain and whose maximal ideal $\mathcal{M} = (t)$ is principal. By lemma above every $x \in R$, $x \neq 0$, is in $\mathcal{M}^i \backslash \mathcal{M}^{i+1}$ for some $i \geq 0$. Then $x = t^i u$, where $u \in R$ must be a unit. Define $v(x) = i$. If $y = t^j u'$ with $u' \in R^*$, then $xy = t^{i+j} u u'$, hence $v(xy) = v(x) + v(y)$. Suppose that $i \leq j$, then $x + y = t^i(u + t^{j-i} u')$, hence $v(x+y) \geq v(x) = \min\{v(x), v(y)\}$. We now can extend v to the field of fractions $\tilde{K}$ by the formula $v(x/y) = v(x) - v(y)$. The remaining properties are clear.

Now let I be a non-zero ideal of R. Let s be the infinum of v on the ideal $I \subset R$, then there exists $x \in I$ such that $v(x) = s$. Then $\mathcal{M}^s = (x) \subset I$. On the other hand, $v(I \backslash \{0\}) \subset \{s, s+1, \ldots\}$, and $\mathcal{M}^s \backslash \{0\} = \{x \in \tilde{K}^* | v(x) \geq s\}$, hence $I \subset \mathcal{M}^s$. Combining all together we obtain $I = \mathcal{M}^s$.

□

Exercise 4.6. Prove that a DVR is normal (i.e. integrally closed in its field of fractions).

4.2.2 Morphisms of affine varieties

A morphism of affine varieties $X \to Y$, $X \subset \mathbb{A}_k^n$, $Y \subset \mathbb{A}_k^m$, is given by a function representable by m polynomials in n variables (thus affine varieties form a *category*). The varieties X and Y are called isomorphic if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that fg and gf are identities. The following proposition tells that the category of affine varieties is equivalent to the category of finitely generated integral domains over k.

Proposition 4.9. *Let $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ be affine algebraic varieties.*

1. *A morphism $f : X \to Y$ defines a homomorphism of k-algebras $f^* : k[Y] \to k[X]$ via the composition of polynomials.*

2. *Any homomorphism of k-algebras $\varphi : k[Y] \to k[X]$ is of the form $\varphi = f^*$ for a unique morphism $f : X \to Y$.*

3. *$f : X \to Y$ is an isomorphism of affine varieties if and only if $f^* : k[Y] \to k[X]$ is an isomorphism of k-algebras.*

Proof. 1) follows from the fact that the composition of polynomials is a polynomial.

2) Let $x_1, \ldots, x_n$ be the coordinates on X, and $t_1, \ldots, t_m$ be the coordinates on Y. Let Φ be the composition of the following homomorphisms of rings:

$$k[t_1, \ldots, t_m] \to k[Y] = k[t_1, \ldots, t_m]/I(Y) \to k[X] = k[x_1, \ldots, x_n]/I(X)$$

Let $f_i = \Phi(t_i), i = 1, \ldots, m$. The polynomial map $f = (f_1, \ldots, f_m)$ maps X to $\mathbb{A}_k^m$. Let $F(t_1, \ldots, t_m)$ be a polynomial. Since we consider homomorphisms of rings we have

$$F(f_1, \ldots, f_m) = F(\Phi(t_1), \ldots, \Phi(t_m)) = \Phi(F(t_1, \ldots, t_m)).$$

If $F \in I(Y)$, then $\Phi(F) = 0$. Hence all the polynomials from $I(Y)$ vanish on $f(X)$, that is, $f(X) \subset Z(I(Y)) = Y$.

Finally, $f^* = \varphi$, since these homomorphisms take the same values on the generators t_i of the ring $k[Y]$.

3) follows from (1) and (2).

□

Definition 4.17. A rational function $f \in k(X)$ is called *regular* at a point P of X if $f = g/h$, where $g, h \in k[X]$ and $h(P) \neq 0$. A function is regular on an open set $U \subset X$ if it is regular at every point of U.

The ring of regular functions on an open subset $U \subset X$ is denoted by $k[U]$. Since $k[X] \subset k[U] \subset k(X)$, the fraction field of $k[U]$ is $k(X)$. To a rational function $f \in k(X)$ one associates "the ideal of denominators" $D_f \subset k[X]$ consisting of regular functions h such that $hf \in k[X]$ (check this is an ideal!). The set of all points P where f is regular is $X \backslash Z(D_f)$. Indeed, we can write $f = g/h$, $g, h \in k[X]$, $h(P) \neq 0$, if and only if $P \notin Z(D_f)$. An immediate corollary of the Nullstellensatz says that if $I \subset k[X]$ is an ideal, and $f \in k[X]$ vanishes at all the common zeros of I in X, then $f^s \in I$ for some $s > 0$. (Apply the Nullstellensatz to the pre-image of I in $k[x_1, \ldots, x_n]$ under the natural surjective map.) This is a little more general form of the Nullstellensatz.

It can be shown that the open subsets of an affine variety X of the form $h \neq 0$, $h \in k[X]$, form a base of Zariski topology on X. The following lemma gives a connection of these open subsets with the localised ring $k[X]_h$.

Lemma 4.4. *Let X be an affine variety. The subset of $k(X)$ consisting of functions regular at all the points of X is $k[X]$. A function is regular on the open subset given by $h \neq 0$, for $h \in k[X]$, if and only if $f \in k[X]_h$, in other words, if $f = g/h^s$ for some $g \in k[X]$ and $s > 0$.*

Proof. Let f be such a function. Then $Z(D_f) = \emptyset$. By corollary 4.5 D_f must be the whole ring, hence contains 1, hence $f \in k[X]$. This proves the fist statement. To prove the second statement we note that $Z(D_f)$ is contained in the closed set given by $h = 0$. By Nullstellensatz if h vanishes on $Z(D_f)$, then a power of h is in D_f. □

At last, for each point $P \in X$ there is a notion of a *stalk* (cf. also §3 of lectures by Wei-Ping Li):

Definition 4.18. The stalk $\mathcal{O}_{X,P}$ of the ring of regular functions on the variety X at point P is the set of pairs (U, f), where U is a (Zariski) open subset of X, containing P and f is a regular function on U. Two pairs are said to be equivalent if $f = g$ on $U \cap V$.

Exercise 4.7. Prove that $\mathcal{O}_{X,P}$ is a local ring isomorphic to the ring $k[X]_\wp$, where $\wp$ is the maximal ideal corresponding to P by the corollary from Nullstellensatz.

4.3 From algebraic geometry to complex geometry: smooth and singular points of algebraic varieties

If $K=\mathbb{C}$ a natural question arises: when an algebraic variety form a complex manifold? The answer is simple: when algebraic variety is smooth, i.e. all its points are smooth points:

Definition 4.19. Let $X\subset\mathbb{A}^n_K$ is an affine variety and suppose that the ideal of X is generated by polynomials $f_1,\dots,f_m\in K[X_1,\dots,X_n]$. Let $P=(a_1,\dots,a_n)$ be a point of X. The variety X is called *non-simgular* or *smooth* at $P\in X$ if the rank of the matrix $(\partial f_i/\partial x_j)(P)$ is equal to $n-r$, where r is the dimension of X. X is non-singular or smooth if it is smooth at all its points.

Definition 4.20. A Noetherian local ring R with maximal ideal $\mathcal{M}$ and residue field K is regular if $\dim R=\dim_K(\mathcal{M}/\mathcal{M}^2)$.

Theorem 4.8. *Let $X\subset\mathbb{A}^n_K$ be an affine variety. It is smooth at $P\in X$ if and only if the local ring $\mathcal{O}_{X,P}$ $(\simeq k[X]_{\wp})$ is regular.*

Proof. Let $P=(a_1,\dots,a_n)$ be coordinates of P in $\mathbb{A}^n_K$ and let $\wp=(X_1-a_1,\dots,X_n-a_n)$ be the corresponding maximal ideal in $A=K[X_1,\dots,X_n]$. Define a linear map $\theta:A\to K^n$ by setting

$$\theta(f)=\left\langle\frac{\partial f}{\partial X_1}(P),\dots,\frac{\partial f}{\partial X_n}(P)\right\rangle,\quad f\in A.$$

Clearly, the elements $\theta(x_i-a_i)$, $i=1,\dots,n$ form a basis in K^n and $\theta(\wp^2)=0$. Therefore, θ induces an isomorphism $\theta':\wp/\wp^2\simeq K^n$.

Let $I=I(X)$ and $f_1,\dots,f_t$ be its generators. Then the rank of the jacobian matrix $J=(\partial f_i/\partial X_i)(P))$ is equal to the dimension of the subspace $\theta(I)$ and (since θ' is an isomorphism) to the dimension of the subspace $(I+\wp^2)/\wp^2$ in $\wp/\wp^2$. On the other hand, the local ring $\mathcal{O}_{X,P}$ is isomorphic to $(A/I)_{\wp}$. Therefore, if $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{X,P}$, then we have an isomorphism

$$\mathcal{M}/\mathcal{M}^2\simeq\wp/(I+\wp^2).$$

Calculating the dimensions of vector spaces we get the equality

$$\dim\mathcal{M}/\mathcal{M}^2+\operatorname{rk}J=n$$

Now assume $\dim Y=r$. Then the local ring also $\mathcal{O}_{X,P}$ has dimension r (as $\dim\mathcal{O}_{X,P}=ht\mathcal{M}$, $\mathcal{O}_{X,P}\simeq k[X]_{\wp}$, $k[X]_{\wp}/\mathcal{M}\simeq k[X]/\wp\simeq K$ and $ht\wp+\dim K=\dim A=n$ by proposition 4.3). Therefore, $\mathcal{O}_{X,P}$ is regular iff $\dim_K\mathcal{M}/\mathcal{M}^2=r$. But this is equivalent to the equality $\operatorname{rk}J=n-r$, the definition of a smooth point P. □

Remark 4.6. If X is a smooth variety, it can be shown that locally $I(X)$ is generated by $n-r$ functions (try to show it!). Then by the implicit function theorem X is a complex submanifold in $\mathbb{A}^n_K$, i.e. X is a complex manifold.

5 Affine spectral pair and their analytic determination

Now we can define the *affine spectral pair.* It consists of the affine spectral curve and a *spectral sheaf.* Algebraic curves entered for the first time into the theory of commutative subrings of D in the works of Burchnall and Chaundy [5, 6] and in a greater generality in the works of Krichever [13, 14]. Main references for this section (concerning the results about commutative subrings in D) are the papers [19, Section 2] and [29], cf. also [4, Section 1]. Main references concerning results from algebraic geometry are standard books mentioned in section 4.

Before we'll define the notion of a sheaf, let's study basic properties of the spectral module.

Definition 5.1. The rank of the spectral sheaf is the dimension of its localisation at the zero ideal (0) of B:

$$\mathrm{rk}(F) = \dim_{\mathrm{Quot}(B)} F_{(0)} = \dim_{\mathrm{Quot}(B)} F_{\otimes B}\, \mathrm{Quot}(B).$$

Theorem 5.1. *Let $B \subset D$ be a commutative subring of rank r. Then the spectral module F is finitely generated and torsion free B-module of rank r.*

Proof. Since r divides $\mathrm{ord}(P)$ for any $P \in B$, it is easy to see that the elements $1, \partial, \ldots, \partial^{r-1}$ of F are linearly independent over B. Let $F^\circ := \langle 1, \partial, \ldots, \partial^{r-1} \rangle_B \subset F$. It is sufficient to prove that the quotient F/F° is finite dimensional over K. Let $\Sigma := \{d \in \mathbb{Z}_+ \,|\, \text{there exists}\, P \in B \text{ with } \mathrm{ord}(P) = d\}$. Obviously, Σ is a sub–semi–group of $r\mathbb{Z}_+$ (as in proposition 4.1). Moreover, one can find $l \in \mathbb{N}$ such that for all $m \geq l$ there exists some element $P_m \in B$ such that $\mathrm{ord}(P_m) = mr$. One can easily prove that F/F° is spanned over K by the classes of $1, \partial, \ldots, \partial^{lr}$, hence F is finitely generated.

Now note: $F \cdot \mathrm{Quot}(B)$ is torsion free over $\mathrm{Quot}(B)$ and there is an obvious embedding

$$(\mathrm{Quot}(B))^{\oplus r} \hookrightarrow F \cdot \mathrm{Quot}(B), \quad (w_1, \ldots, w_r) \mapsto w_1 \cdot 1 + \ldots + w_r \cdot \partial^{r_1}.$$

Exercise 5.1. Show that $F \cdot \mathrm{Quot}(B) = F \otimes_B \mathrm{Quot}(B) \simeq (\mathrm{Quot}(B))^{\oplus r}$.

□

With every B-module we can associate a "semi-geometric object": the *spectral sheaf.*

Definition 5.2. A sheaf on algebraic variety (or more generally, on a topological space) consists of

a) an abelian group (or a ring or a module) $\mathcal{F}(U)$ (its elements are called *sections* of the sheaf over U) for any open subset $U \subset X$,

b) a homomorphism of abelian groups (rings, modules) $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ (called restriction maps) for any open $V \subset U \subset X$ such that

1. $\mathcal{F}(\emptyset) = 0$
2. $\rho_{U,U} = id$
3. if $W \subset V \subset U$, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$
4. if $U = \cup_i V_i$ and $s \in \mathcal{F}(U)$ is such that $s|_{V_i} = 0$ for any i, then $s = 0$
5. if for any i there are given sections $s_i \in \mathcal{F}(V_i)$ such that $\rho_{V_i, V_i \cap V_j}(s_i) = \rho_{V_j, V_i \cap V_j}(s_j)$ for any i, j, then there exists a section $s \in \mathcal{F}(U)$ (which is unique by item 4) such that $\rho_{U,V_i}(s) = s_i$ for any i.

A *morphism* of sheaves is a collection of groups (rings, modules) homomorphisms $\mathcal{F}(U) \to \mathcal{F}'(U)$ compatible with the restriction maps.

Example 5.1. 1) If we set $\mathcal{F}(U) = K[U]$ for any open U of an algebraic variety X, we get a *sheaf of rings* called the *structure sheaf* and denoted by $\mathcal{O}_X$.

2) If $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module for any open U of a variety X and the maps $\rho_{U,V}$ are compatible with the module structure, then $\mathcal{F}$ is called *a sheaf of $\mathcal{O}_X$-modules.*

For a sheaf we can define the notion of a stalk analogous to the notion of a stalk of (a sheaf) of regular functions:

$$\mathcal{F}_{X,p} = \{(f, U) | \quad f \in \mathcal{F}(U),\, U\text{-open}, p \in U\} / \sim,$$

where two pairs are equivalent $(f, U) \sim (g, V)$, if there exists an open subset $p \in W \subset U \cap V$ such that $\rho_{U,W}(f) = \rho_{U,W}(g)$.

The spectral sheaf $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules. The standard *construction* of a sheaf associated with module is as follows:

Definition 5.3. Let F be a B-module, where $B = K[X]$, X — affine variety. Construct the *associated sheaf of $\mathcal{O}_X$-modules $F^\sim$* as follows:

$$F^\sim(U) = \{\text{set of maps} \quad s : U \to \amalg_{P\in U} F_P \quad \text{s.t. } s(P) \in F_P \ \forall P \in U \text{ and } \forall P \in U$$
$$\exists V \subset U,\ V \ni P \text{ and } \exists f \in F,\ b \in B \text{ s.t. } s(q) = f/b \in F_q \text{ and } b \notin q\ \forall q \in V\} \quad (6)$$

with the obvious module structure (componentwise multiplication) on the maps.

A sheaf $\mathcal{F}$ is called *coherent* if there is a covering $X = \cup_i U_i$ by affine open subsets such that $\mathcal{F}(U_i)$ is a finitely generated $\mathcal{O}_X(U_i)$-module.

The sheaf $F^\sim$ associated to the spectral module F is called the *spectral sheaf.* This sheaf is coherent.

Note that $F^\sim$ is *a torsion free sheaf*, i.e. for any U $F^\sim(U)$ is a torsion free $\mathcal{O}_X(U) = K[U]$-module (i.e. for any $0 \neq m \in F^\sim(U)$ and for any $0 \neq a \in K[U]$ $ma \neq 0$). In particular, $\mathcal{F}_{X,p}$ is a torsion free $\mathcal{O}_{X,p}$-module.

Exercise 5.2. Check that modules $F^\sim(U)$ with usual restriction maps form a sheaf of $\mathcal{O}_X$-modules.

The following properties can be derived easily from definition:

Proposition 5.1. *1. $F^\sim$ is a sheaf of $\mathcal{O}_X$-modules;*

2. for any point $p \in X$ we have $(F^\sim)_{X,p} \simeq F_p$;

3. for any $b \in B$ we have $F^\sim(U_b) \simeq F_b$, in particular, $F^\sim(X) \simeq F$;

Definition 5.4. The sheaf $\mathcal{F}$ is *locally free*, if there exists a covering $X = \cup_i U_i$ such that $\mathcal{F}(U_i)$ is a free $\mathcal{O}_X(U_i)$-module (in particular, for any point $p \in X$ $\mathcal{F}_{X,p}$ is a free $\mathcal{O}_{X,p}$-module, i.e. $\mathcal{F}_{X,p} \simeq \mathcal{O}_{X,p}^{\oplus\dots}$).

If the sheaf is coherent (as we'll consider here), then these free modules are of finite rank.

We'll need one more fact from commutative algebra:

Proposition 5.2. *A torsion free finitely generated module over a Noetherian regular local ring of dimension one (i.e. over an DVR) is free.*

Proof. Let $m_1, \dots, m_s$ be a minimal set of generators of a module M over a ring A from proposition. If M is not free, there is a relation $a_1m_1 + \dots + a_sm_s = 0$ for some $a_1, \dots, a_s \in A$, not all of them are zero. If one of elements a_i is a unit, then m_i belongs to a submodule generated by all other m_j, i.e. $m_1, \dots, m_s$ is not a minimal set of generators. If there are no units, all elements a_i belong to the maximal ideal. Since A is a local ring, this ideal is principal, i.e. for any i we have $a_i = ta_i'$, where t is a generator of the maximal ideal, $a_i' \in A$. Then $a_1m_1 + \dots + a_sm_s = t(a_1'm_1 + \dots + a_s'm_s) = 0$, and, since M is torsion free, we must have $a_1'm_1 + \dots + a_s'm_s = 0$. Repeating our arguments again, we will come to a relation with a unit as one of its coefficients, a contradiction (by lemma 4.3 we will need only finite number of repetitions). □

Remark 5.1. We would like to emphasize that the spectral sheaf is not, in general, locally free. It is locally free only over a smooth locus of the spectral curve. On the other hand, locally free sheaves are closely related to *geometric* objects — vector bundles, see the next section.

5.1 Locally free sheaves and vector bundles

In this subsection we would like to explain the relation between locally free sheaves and vector bundles over algebraic varieties (cf also lectures by Wei-Ping Li at this summer school). Namely, the isomorphism classes of locally free sheaves are in one to one correspondence with the isomorphism classes of vector bundles, i.e. we can "identify" them.

Definition 5.5. Let X be an algebraic variety. A *vector bundle of rank* n over X is a variety Y together with a morphism $f : Y \to X$ and with the following additional structure: there is an open covering $X = \cup_i U_i$ and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{A}^n_K \times U_i$ such that for any i, j and any open affine subset $V = \mathrm{Spec}(A) \subset U_i \cap U_j$ the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of the space $\mathbb{A}^n_K \times V = \mathrm{Spec}\, A[x_1, \dots, x_n]$ is given by a linear automorphism θ of the algebra $A[x_1, \dots, x_n]$, i.e. $\theta(a) = a$ for any $a \in A$ and $\theta(x_i) = \sum a_{ij} x_j$, $a_{ij} \in A$.

An isomorphism $g : (Y, f, \{U_i\}, \{\psi_i\}) \to (Y', f', \{U'_i\}, \{\psi'_i\})$ of vector bundles of rank n is an isomorphism $g : Y \to Y'$ such that $f = f' \circ g$ and Y, f together with the covering of X consisting of all open U_i, U'_i and isomorphisms ψ_i, $\psi'_i \circ g$ also define a structure of a vector bundle on Y.

Construction. Let $\mathcal{F}$ be a locally free sheaf of rank r and let $\{U_i\}$ be a covering $X = \cup_i U_i$ such that $\mathcal{F}(U_i) \simeq \mathcal{O}_X(U_i)^{\oplus r}$. Consider the symmetric algebra $S(\mathcal{F}(U_i)) = (\oplus_{n\geq 0} \mathcal{F}(U_i) \otimes_{\mathcal{O}_X(U_i)} \dots \otimes_{\mathcal{O}_X(U_i)} \mathcal{F}(U_i))/(x \otimes y - y \times x)$ — this is an $\mathcal{O}_X(U_i)$-algebra. If we choose a basis $\{x_1, \dots, x_n\}$ of $\mathcal{F}(U_i)$ over $\mathcal{O}_X(U_i)$, then there is a natural isomorphism

$$S(\mathcal{F}(U_i)) \simeq \mathcal{O}_X(U_i)[x_1, \dots, x_n].$$

Obvious homomorphisms of rings $f^*_{U_i} : \mathcal{O}_X(U_i) \to S(\mathcal{F}(U_i))$ define the morphisms of corresponding affine varieties $f_{U_i} : \mathrm{Spec}\, S(\mathcal{F}(U_i)) \to U_i$, and we have natural isomorphisms $\psi_i : \mathbb{A}^n_K \times U_i = \mathrm{Spec}\, \mathcal{O}_X(U_i) \otimes_K K[x_1, \dots, x_n] \to \mathrm{Spec}\, S(\mathcal{F}(U_i))$.

Symmetric algebras $S(\mathcal{F}(U_i))$ form a sheaf $S(\mathcal{F})$ of $\mathcal{O}_X$-algebras, and varieties $\mathrm{Spec}(\mathcal{F}(U_i))$ can be glued together to form a variety E (it can be not affine, but projective or quasiprojective, see definitions below) with a morphism $E \to X$ — a vector bundle $V(\mathcal{F})$ (all other data are already defined).

Vice versa: If $f : E \to X$ is a vector bundle of rank n, then we can construct the sheaf of sections

$$U \mapsto \{\text{the set of sections of } f \text{ over } U\text{, i.e. morphisms } s : U \to E \text{ s.t. } f \circ s = id_U\}.$$

Direct check shows that it is a sheaf of $\mathcal{O}_X$-modules, called $\mathcal{F}(E)$, which is locally free of rank n.

Remark 5.2. We would like to emphasize that $\mathcal{F}(V(\mathcal{F})) \simeq \mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ (not $\mathcal{F}$ itself!). The sheaf $\mathcal{F}^\vee$ is defined as

$$U \mapsto Hom_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{O}_X|_U)$$

(and it is easy to see that $Hom_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{O}_X|_U)$ are $\mathcal{O}_X(U)$-modules). If we are given $s \in \mathcal{F}^\vee(V)$ for an open affine V, then s defines a homomorphism of $\mathcal{O}_V$-algebras $S(\mathcal{F}^\vee|_V) \to \mathcal{O}_V$ which defines a morphism of affine varieties $V \to f^{-1}(V) = \mathrm{Spec}\, S(\mathcal{F}^\vee|_V)$, i.e. a section of the vector bundle $V(\mathcal{F}) \to X$. This construction establishes the isomorphism $\mathcal{F}(V(\mathcal{F})) \simeq \mathcal{F}^\vee$.

As for vector bundles we can speak about fibres of sheaves: any sheaf on X has a fibre at $p \in X$: $\mathcal{F}|_p := \mathcal{F}_{X,p} \otimes_{\mathcal{O}_{X,p}} K$, where $K \simeq \mathcal{O}_{X,p}/\mathcal{M}_p$ (see definition 4.15). If $\mathcal{F} = F^\sim$, where F is a spectral module, then $\mathcal{F}|_p \simeq F \otimes_B B/p$.

5.2 Geometric meaning of the spectral sheaf

By the Nullstellensatz all points of the affine variety $C_0 = \mathrm{Spec}(B)$ are in one to one correspondence with maximal ideals of the ring B. Any maximal ideal $q \subset B$ gives a K-algebra homomorphism $\chi_q : B \to B/q \simeq K$, and vice versa.

Definition 5.6. Let $q \in C_0$ be any point and $\chi = \chi_q : B \to K$ the corresponding character. We call the K–vector space

$$\mathsf{Sol}\big(B, \chi\big) := \big\{f \in K[[x]] \big| P \circ f = \chi(P) f \text{ for all } P \in B\big\} \tag{7}$$

the *solution space* of the algebra B at the point q. Here, we apply the usual left action $\circ$ of D on $K[[x]]$. Observe, that $\mathsf{Sol}\big(B, \chi\big)$ has a natural B–module structure: $f \in \mathsf{Sol}\big(B, \chi\big) \Rightarrow \forall Q \in B \;\; Q(f) \in \mathsf{Sol}\big(B, \chi\big)$.

The geometric meaning of the B–module F is explained by the next result.

Theorem 5.2. *The following K–linear map*

$$F \stackrel{\eta_\chi}{\to} \mathsf{Sol}\big(B, \chi\big)^*, \quad \partial^i \mapsto \Big(f \mapsto \frac{1}{i!} f^{(i)}(0)\Big) \tag{8}$$

is also B–linear, where $\mathsf{Sol}\big(B, \chi\big)^* = \mathrm{Hom}_K\big(\mathsf{Sol}\big(B, \chi\big), K\big)$ is the vector space dual of the solution space. Moreover, the induced map

$$B/\ker(\chi) \otimes_B F \stackrel{\bar\eta_\chi}{\to} \mathsf{Sol}\big(B, \chi\big)^* \tag{9}$$

is an isomorphism of B–modules.

Proof. First note that the following map

$$\mathop{\mathrm{Hom}}_K\big(F, K\big) \stackrel{\Phi}{\to} K[[x]], \quad \lambda \mapsto \sum_{p=0}^{\infty} \frac{1}{p!} \lambda(\partial^p) x^p \tag{10}$$

is an isomorphism of left D–modules. Let $B \stackrel{\chi}{\to} K$ be a character, then $K = K_\chi := B/\ker(\chi)$ is a left B–module. We obtain a B–linear map

$$\Psi : \mathop{\mathrm{Hom}}_B(F, K_\chi) \stackrel{I}{\to} \mathop{\mathrm{Hom}}_K(F, K) \stackrel{\Phi}{\to} K[[x]], \tag{11}$$

where I is the forgetful map. The image of I consists of those K–linear functionals, which are also B–linear, i.e.

$$\mathsf{Im}(I) = \big\{\lambda \in \mathop{\mathrm{Hom}}_K(F, K) \;\big|\; \lambda(P \circ -) = \chi(P) \cdot \lambda(-) \text{ for all } P \in B\big\}.$$

This implies that $\mathsf{Im}(\Psi) = \mathsf{Sol}(B, \chi)$. Next, we have a canonical isomorphism of B–modules (check it!): $\mathrm{Hom}_B(F, K_\chi) \cong \mathrm{Hom}_K\big(B/\ker(\chi) \otimes_B F, K\big)$. Dualizing again, we get an isomorphism of vector spaces

$$\Psi^* : \mathsf{Sol}(B, \chi)^* \to \big(B/\ker(\chi) \otimes_B F\big)^{**} \cong B/\ker(\chi) \otimes_B F.$$

It remains to observe that Ψ^* is also B–linear and $\big(\Psi^*\big)^{-1} = \bar\eta_\chi$. ☐

Remark 5.3. The isomorphism (9) has the following geometric meaning: if we view F as a coherent sheaf on $C_0 = \mathrm{Spec}(B)$ then for any point $q \in C_0$ (smooth or singular) we have: $F\big|_q \cong \mathsf{Sol}(B, \chi)^*$, where $B \stackrel{\chi}{\to} K$ is the character corresponding to the point q. Because of this fact, F is called *spectral module* of the algebra B.

Corollary 5.1. *Let $B \subset D$ be a commutative subring of rank r. Then for any character $B \stackrel{\chi}{\to} K$ we have: $r \le \dim_K\big(\mathsf{Sol}(B, \chi)\big) < \infty$. Moreover, $\dim_K\big(\mathsf{Sol}(B, \chi)\big) \ge r + 1$ if only if χ defines a* singular point $q \in C_0$ and $F^\sim$ *is* not locally free *at* q.

5.3 Analytic determination of $\mathsf{Sol}(B,\chi)$

For any character $B \overset{\chi}{\to} K$ consider the K–vector space

$$\mathsf{Sol}'\big(B,\chi\big) := \big\{f \in K[[x]] \,\big|\, P \circ f = \chi(P)f \text{ for all } P \in B\big\}. \tag{12}$$

Obviously, $\mathsf{Sol}\big(B,\chi\big) \subseteq \mathsf{Sol}'\big(B,\chi\big)$. However, the following result is true.

Theorem 5.3. *Let $B \subset D$ be a commutative subring of rank r and $B \overset{\chi}{\to} K$ a character. Then we have: $\mathsf{Sol}\big(B,\chi\big) = \mathsf{Sol}'\big(B,\chi\big)$ and there exists a uniquely determined*

$$R_\chi = \partial^m + c_1\partial^{m-1} + \cdots + c_m \in \widetilde{D} \tag{13}$$

such that $\ker(R_\chi) = \mathsf{Sol}'\big(B,\chi\big)$. Moreover, $m \ge r$ and $m = r$ if and only if $\mathcal{F} = F^\sim$ is locally free at the point $q \in C_0$ corresponding to χ.

Proof. Let $P = \partial^n + a_1\partial^{n-1} + \cdots + a_n \in D$. Then the dimension of the K–vector space $\ker(P) \subset K[[x]]$ is n and $\ker(P) \subset K[[x]]$. This implies that $\mathsf{Sol}\big(B,\chi\big) = \mathsf{Sol}'\big(B,\chi\big)$.

For any differential operators $Q_1, \dots, Q_l \in \widetilde{D}$ we denote by $\langle Q_1, \dots, Q_l\rangle \subseteq \widetilde{D}$ the left ideal generated by these elements. Recall that by theorem 3.1 any left ideal $J \subseteq \widetilde{D}$ is principal. Let $P_1, \dots, P_n \in B$ be the algebra generators of B (i.e. $B = K[P_1, \dots, P_n]$) and $\alpha_i = \chi(P_i)$ for all $1 \le i \le n$. Then there exists a uniquely determined $R_\chi \in \widetilde{D}$ as in (13) such that

$$\big\langle P - \chi(P)1 \,\big|\, P \in B\big\rangle = \big\langle P_1 - \alpha_1, \dots, P_n - \alpha_n\big\rangle = \langle R_\chi\rangle. \tag{14}$$

Now let's use Differential Galois Theory: there is the universal Picard–Vessiot extension $PV(K((x)))$ of $K((x))$, see [23, Section 3.2], where any differential operator of order m from $\widetilde{D}$ has exactly m linearly independent solutions with values in $PV(K((x)))$ ($\widetilde{D}$ acts on $PV(K((x)))$).

Obviously, $\ker(R_\chi) = \mathsf{Sol}'\big(B,\chi\big) = \mathsf{Sol}\big(B,\chi\big)$ viewed as subspaces of $PV(K((x)))$. Moreover, $\dim_K\big(\ker(R_\chi)\big) = \mathsf{ord}(R_\chi)$. In virtue of Corollary 5.1, we get the statement about the order of R_χ. □

6 Projective spectral data and classification of commutative rings of ODOs

In this section we explain how to construct a completion of the affine spectral data and explain the classification of commutative subrings of ODOs in terms of their projective spectral data. Main references for this section (concerning results from algebraic geometry) are standard books mentioned in section 4. Main references concerning results about commutative subrings are papers [19] and [17].

6.1 Projective algebraic geometry

From the affine spectral data $(C_0 = \text{Spec}(B), F^\sim)$ we can get a natural completion $(C, \mathcal{F})$, where C is a projective curve and $\mathcal{F}$ is a coherent sheaf on C. In order to explain this we need to give a short introduction to the *projective algebraic geometry.*

It studies algebraic subsets in the *projective space.* The projective space $\mathbb{P}^n_K$ is the set of equivalence classes of points of $\mathbb{A}^{n+1}_K \backslash \{(0, \dots, 0)\}$, where two points are equivalent if they differ by a common non-zero multiple. The equivalence class of $(x_0, x_1, \dots, x_n)$ is denoted by $(x_0 : x_1 : \dots : x_n)$.

The *Zariski topology* on the projective space is defined with the help of *homogeneous* polynomials. The zeros of homogeneous polynomials, denoted also as $Z(T)$, are *closed projective* *sets*. The condition $x_i \neq 0$ defines an open subset of $\mathbb{P}^n_K$ isomorphic to the affine space $\mathbb{A}^n_K$ with coordinates $x_0/x_i, \ldots, x_n/x_i$. We get $n+1$ affine spaces which provide an open covering of $\mathbb{P}^n_K$.

For any affine variety there is a "usual" projective closure defined with the help of *homogenisation* procedure of polynomials. Let $f(T_1, \ldots, T_n)$ be a polynomial of degree d. It can be written as the sum $f = f_0 + \ldots + f_d$, where f_i is a form of degree i. The homogenization of f is the form of degree d in $n+1$ variables given by

$$F(T_0, T_1, \ldots, T_n) = F = T_0^d f_0 + T_0^{d1} f_1 + \ldots + f_d.$$

Now if $X \subset \mathbb{A}^n_K$ is a closed affine set, then associating to polynomials in the ideal of X their homogenizations defines the projective closure of X.

For closed projective sets there is an analogous notion of irreducible set. The irreducible projective sets are called *projective varieties*. *Quasi-projective varieties* are dense open subsets of projective varieties. Projective varieties are simpler than affine, since they are *compact* (also in the usual complex topology if $K = \mathbb{C}$). Geometry of projective varieties is based on the commutative algebra of *graded rings* and *homogeneous ideals*.

Definition 6.1. A ring R is graded if $R = \oplus_{d \geq 0} R_d$, where R_d are abelian groups (called group of homogeneous elements) and $R_d \cdot R_e \subset R_{d+e}$.

Analogously, a *graded R-module*, where R is a graded ring, is a module $M = \oplus_{d \geq 0} M_d$ with $M_d \cdot R_e \subset M_{d+e}$.

An ideal $J \subset R$ is *homogeneous* if $J = \oplus_{d \geq 0}(J \cap R_d)$.

Example 6.1. The ring of polynomials $K[T_0, \ldots, T_n]$ can be considered as graded:
$K[T_0, \ldots, T_n] = \oplus_{d \geq 0} R_d$, where R_d is the group of homogeneous polynomials of degree d. An ideal $J \subset K[T_0, \ldots, T_n]$ is homogeneous if whenever $f \in J$, its homogeneous part $f_i \in J$.

Exercise 6.1. Prove the projective Nullstellensatz: if J is a homogeneous ideal, then

(1) $Z(J) = \emptyset$ if the radical of J contains the ideal $(T_0, \ldots, T_n)$ (the maximal ideal of the zero point in $\mathbb{A}^{n+1}_K$),

(2) if $Z(J) \neq \emptyset$, then $I(Z(J)) = \sqrt{J}$.

For graded rings we have special localisations:

Localisation. If T is a multiplicative system of *homogeneous* elements in a graded ring R, then the localisation with respect to T is

$$T^{-1}R = \{\frac{a}{b}, \quad a, b \in R_d \text{ for some } d \text{ and } b \in T\}.$$

As in the case of usual localisation $T^{-1}R$ is a ring (but not necessarily graded ring).

Analogously, if M is a graded R-module, then

$$T^{-1}M = \{\frac{m}{b}, \quad b \in R_d,\ m \in M_d \text{ for some } d \text{ and } b \in T\}.$$

is a $T^{-1}R$-module.

Example 6.2. 1) If $\wp \subset R$, $\wp \not\supseteq R_+ = \oplus_{d>0} R_d$, then we can take set T to be equal to the set of homogeneous elements in $R \backslash \wp$. Denote by $R_{(\wp)} := T^{-1}R$. This is a local ring (analogue of $R_\wp$ for ordinary rings).

2) If a is a homogeneous element from R, let's take $T = \{a^n\}$, $n \geq 0$ and denote by $R_{(a)} = T^{-1}R$. This ring is an analogue of the ring R_a in affine geometry. In particular, if $a = T_i$ in $R = K[T_0, \ldots, T_n]$, then $R_{(a)} \simeq K[x_1, \ldots, x_n]$.

For projective varieties we have also a dictionary between geometry and commutative algebra, which is analogous to the dictionary in affine geometry.

- Projective variety X defined by homogeneous polynomials $f_1, \ldots, f_k$ corresponds to the graded ring $R = K[T_0, \ldots, T_n]/I$, where $I = (f_1, \ldots, f_k)$ is a prime homogeneous ideal. We'll denote this variety by $\operatorname{Proj} R$.

- $\dim X = \dim R - 1 = trdeg R_{((0))}$

- Open subset of X defined by condition $h \neq 0$, where h is a homogeneous element, corresponds to the ring $R_{(h)}$

- closed subsets are defined by homogeneous radical ideals in R

- Rational functions $K(X)$ on X are defined as elements of the ring $R_{((0))}$, and stalks of regular functions (defined in the same way as for affine varieties) are isomorphic to the local rings $R_{(\wp)}$.

Note that the graded ring $K[T_0, \ldots, T_n]/I$ from the list above is finitely generated over K by the set from its first homogeneous component (by the images of the elements $T_0, \ldots, T_n$). Clearly, if we have a graded ring which is finitely generated by its first homogeneous component over its zero component equal to K, then such a ring is isomorphic to the image of the graded ring $K[T_0, \ldots, T_m]$ for some m, i.e. to the ring $K[T_0, \ldots, T_m]/J$ for some homogeneous ideal J. In particular, this ring determines a projective variety. Now if R is a graded ring finitely generated over K, there is the following important result from commutative algebra:

Proposition 6.1. *[3, Ch.III, § 1.3, prop. 3] Let R be a graded ring finitely generated over $K = R_0$. Then there exists $d > 0$ such that the graded ring $R^{(d)} := \oplus_{i \geq 0} R_{id} \subset R$ is finitely generated by its first graded component $R_1^{(d)} = R_d$ as a $R_0^{(d)} = K$-algebra.*

It can be shown (try to do it or see e.g. [12]) that there is a one to one correspondence between prime homogeneous ideals of the ring R and prime homogeneous ideals of the ring $R^{(d)}$. Thus, if we are given a ring from proposition 6.1, we can find a better ring $R^{(d)}$ and construct a projective variety, which can be thought of as a projective variety constructed by R. We'll denote such a variety also by $\operatorname{Proj} R$.

Morphisms. A *rational map* $f : X \dashrightarrow \mathbb{P}^n_K$ is (a not necessarily everywhere defined function) given by $(F_0, \ldots, F_n)$, where $F_i \in K(X)^*$, defined up to an overall multiple from $K(X)^*$. A rational map f is *regular* at $P \in X$ if there exists a representative $(F_0, \ldots, F_n)$, such that all the F_i s are regular at P, and $(F_0(P), \ldots, F_n(P)) \neq (0, \ldots, 0)$. A *morphism* is an everywhere regular rational map.

6.2 Projective spectral data and the classification

Now we can construct the completion of the affine spectral pair. Notably, our completion of the affine spectral curve will be not usual one.

Construction. Given a commutative ring of ODOs $B \subset D$, we consider the Rees ring $\tilde{B} = \oplus_{d \geq 0} B^{dr} s^d \subset B[s]$.

Exercise 6.2. 1) Show that $gr(B) = \oplus_{d \geq 0} B^{(d+1)}/B^{(d)}$ is a subring of a polynomial ring $K[z]$, therefore it is finitely generated.

2) Deduce from 1) that $\tilde{B}$ is also finitely generated over K.

We define the *projective spectral curve* (a completion of the affine spectral curve C_0) as $C = \operatorname{Proj} \tilde{B}$.

Analogously, we consider the Rees module $\tilde{F} = \oplus_{d\geq 0} F^{(dr-1)} s^d$ (where the homogeneous components are defined by the filtration on the spectral module F), which is a graded torsion free finitely generated $\tilde{B}$-module. Such a module defines a coherent sheaf on C:

$$\mathcal{F}(U) := (\tilde{F})^{\sim}(U) = \{\text{set of maps} \quad s : U \to \amalg_{P\in U}\tilde{F}_{(P)} \quad \text{s.t. } s(P) \in \tilde{F}_{(P)}\ \forall P \in U \text{ and } \forall P \in U$$
$$\exists V \subset U,\, V \ni P \text{ and } \exists f \in \tilde{F}_d,\, b \in (\tilde{B}\backslash P)_d \text{ s.t. } s(q) = f/b \in \tilde{F}_{(q)} \text{ and } b \notin q\ \forall q \in V\}, \quad (15)$$

which is called the *spectral sheaf* (defined on C).

Exercise 6.3. The discrete valuation $-ord$ on D induces a discrete valuation on B and $\mathrm{Quot}(B)$. Now consider the ideal (s) in $\tilde{B}$. Prove that it is prime. Show that $\tilde{B}_{((s))}$ is a DVR with respect to $(-\mathrm{ord})$. So, the ideal (s) defines a smooth point p on C.

Note that $U_s = \mathrm{Spec}\,\tilde{B}_{(s)}$ is just the affine curve C_0. The point p is called the *"divisor at infinity"*.

From the exercise and by proposition 5.2 it follows that $\tilde{F}_{((s))} \simeq \mathcal{O}_{C,p}^{\oplus r}$. Note that our construction gives this isomorphism explicitly:

$$\Phi : \mathcal{O}_{C,p}^{\oplus r} \to \tilde{F}_{((s))} = \mathcal{F}_p \quad (0, \ldots, e_i, \ldots, 0) \mapsto \partial^i, \quad i = 0, \ldots, r-1.$$

Moreover, we have an isomorphism $\pi : \hat{\mathcal{O}}_{C,p} \simeq K[[\partial^{-r}]] \simeq K[[z^r]]$, where $\hat{\mathcal{O}}_{C,p}$ is the completion of the DVR $\mathcal{O}_{C,p}$.

All this can be seen also using the Schur theory: if S is the Schur operator for the ring B, then $S^{-1}BS \subset K((\partial^{-1}))$, therefore we can take the localisation of $A := S^{-1}BS$ in $K((\partial^{-1}))$. It is easy to see that $A \simeq B$ and the A-module $W = F \circ S$ is isomorphic to the B-module F.

Note that $W \subset K((\partial^{-1}))$ too. Thus, the pair (A, W) is a pair of subspaces in $K((\partial^{-1}))$ such that A is a ring, $A \cdot W \subset W$, where $\cdot$ means the usual multiplication in the ring $K((\partial^{-1}))$ and $A \cdot W$ means that we consider the subspace of products of any elements from A and W. Such a pair is called a *Schur pair*.

Summary. Starting from an elliptic commutative ring $B \subset D$ of rank r we have constructed the *(projective) spectral data* $(C, p, \mathcal{F}, \pi, \Phi)$, where $\mathcal{F}$ is a coherent torsion free sheaf of rank r, $\pi : \hat{\mathcal{O}}_{C,p} \simeq K[[z^r]]$ is an isomorphism (it can be thought of as a choice of a local coordinate z^r at the point p) and $\Phi : \mathcal{F}_p \simeq \mathcal{O}_{C,p}^{\oplus r}$ is a trivialisation at p.

Two spectral data $(C_1, p_1, \mathcal{F}_1, \pi_1, \Phi_1)$, $(C_2, p_2, \mathcal{F}_2, \pi_2, \Phi_2)$ are *isomorphic* if there is an isomorphism β of curves such that $\beta(p_1) = p_2$ and an isomorphism $\psi : \mathcal{F}_2 \to \beta_*(\mathcal{F}_1)$ of sheaves on C_2 such that the trivialisations π_i, Φ_j are compatible with the homomorphisms of stalks of structure sheaves and of spectral sheaves (cf. [17, Def. 2.4]).

Given these data we can construct the Schur pair:

Construction (*Krichever correspondence*): Let $(C, p, \mathcal{F}, \pi, \Phi)$ be a spectral datum. Then we have an embedding

$$\mathcal{O}_C(C\backslash p) \hookrightarrow K(C) = \mathrm{Quot}\,\mathcal{O}_{C,p} \hookrightarrow \mathrm{Quot}\,\hat{\mathcal{O}}_{C,p} \simeq K((z^r)).$$

Analogously, since $\mathcal{F}$ is a torsion free sheaf, we have an embedding

$$\mathcal{F}(C\backslash p) \hookrightarrow \mathcal{F} \cdot K(C) \simeq K(C)^{\oplus r} \hookrightarrow (\mathrm{Quot}\,\hat{\mathcal{O}}_{C,p})^{\oplus r} \simeq K((z^r))^{\oplus r} \simeq K((z)), \quad (16)$$

where the last isomorphism is fixed: the generator $(0, \ldots, z^r, \ldots, 0)$ with z^r at i-th place maps to z^{i-r} and $i = 0, \ldots, r-1$. This strange isomorphism is chosen in such a way in order to make this construction compatible with the Sato theory, see below.

We denote by A the image of $\mathcal{O}_C(C\backslash p)$ and by W the image of $\mathcal{F}(C\backslash p)$. Note that (A, W) is a Schur pair in $K((z))$.

Definition 6.2. We define the *picture cohomology* of the pair (A,W) as

$$H^0(W)=W\cap K[[z]],\quad H^1(C,W)=\frac{K((z))}{W+K[[z]]},$$

$$H^0(A)=A\cap K[[z]],\quad H^1(A)=\frac{K((z))}{A+K[[z]]},$$

Remark 6.1. It can be shown that $H^i(W)\simeq H^i(C,\mathcal{F})$, $H^i(A)\simeq H^i(C,\mathcal{O}_C)$ (see the lectures by Wei-Ping Li about cohomologies of sheaves!). To prove it one can use the exact sequence

$$0\to H^0(C,\mathcal{F})\to \mathcal{F}(C\backslash p)\oplus\widehat{\mathcal{F}}_p\to \mathrm{Quot}(\widehat{\mathcal{F}}_p)\to H^1(C,\mathcal{F})\to 0, \tag{17}$$

(and analogous sequence for $\mathcal{O}_C$), which can be found e.g. in [21, Prop. 3], and the fixed isomorphism (16).

Combining all constructions together, we obtain

Theorem 6.1. *There is a one to one correspondence between elliptic normalised commutative subrings $B\subset D$ and spectral data $(C,p,\mathcal{F},\pi,\Phi)$ with $H^0(C,\mathcal{F})=H^0(W)=H^1(C,\mathcal{F})=H^1(W)=0$ up to natural isomorphisms.*

Proof. In one direction (from B to spectral data) we need only to check that $H^0(C,\mathcal{F})=H^0(W)=H^1(C,\mathcal{F})=H^1(W)=0$ (where W is the space associated with $\mathcal{F}$!). Careful check of the constructions shows that $W=z^{-1}\cdot F\circ S$, where S is the Schur operator associated with B. But then it follows directly from definition 6.2. Note that the choice of the Schur operator is not unique, but it is not difficult to check that different choices lead to isomorphic data.

In order to construct the ring B from the spectral data, we apply the construction described above and obtain a Schur pair (A,W) in the space $K((z))\simeq K((\partial^{-1}))$. Now note that the right action of D on $K[\partial]$ can be naturally extended (just by the same rules) to the right action of the ring $E(K[[x]])$ on $K((\partial^{-1}))$. After that we need to apply the Sato theorems 6.2 and 6.2: if S is the Sato operator, then $SAS^{-1}\subset E(K[[x]])$ is a subring of the ring of pseudo-differential operators that stabilises $K[\partial]$, i.e. $K[\partial]\circ SAS^{-1}\subset K[\partial]$. Indeed, $W\cdot A\subset W$, therefore $\partial^{-1}\cdot W\cdot A\subset\partial^{-1}\cdot W$. Hence

$$(K[\partial])\circ SA\subset (K[\partial])\circ S,\quad\text{and}\quad K[\partial]\circ SAS^{-1}\subset K[\partial].$$

By proposition 6.2 $B:=SAS^{-1}\subset D$. It is not difficult to see that these constructions establish a one to one correspondence. For a more detailed proof (which uses however more technical results from algebraic geometry) we refer to the paper [17]. □

Remark 6.2. The construction of the spectral data was rewritten several times by different authors. We used in our lectures an approach offered by Mumford [19] developed further my Mulase [17] (see also recent review in [4, Section 1]).

First classification of commutative rings of ODOs of any rank was proposed by Krichever [13], [14] as an algebro-geometric tool in the theory of integrating non-linear soliton systems and the spectral theory of periodic finite-zone operators (see e.g. review [11]). It used more analytic spectral data (see the next section) and worked for rings B in "generic position", i.e. for rings whose spectral curve is smooth. The advantage of his approach is the explicit formula for the common eigenfunction (the Baker-Akhieser function, see presentation) of a *rank one* subring B. This formula leads in particular to explicit formulae of commuting operators.

Moreover, as it can be seen from the above constructions, the rank one subrings are classified essentially (i.e. up to automorphisms $x\mapsto cx$, $\partial\mapsto c^{-1}\partial$) only by the *geometric data* $(C,p,\mathcal{F})$ (the trivialisations are not important in this case). The higher rank case is much more difficult, see comment 7.1. We would like to note that one theorem due to Makar-Limanov says that the rank of a commutative elliptic subring of the *first Weyl algebra* must be greater that one. By the Schur theory, the maximal subring in D containing such a ring will belong again to A_1. Thus, we see that the most "easy" coefficient ring is the most difficult to study.

Remark 6.3. If there are two commutative subrings $B \subset B' \subset D$, then on the geometric side this means that the spectral sheaf $\mathcal{F}$ on C is isomorphic to the direct image of the sheaf $\mathcal{F}'$ on C'. This can be derived directly from the constructions above.

6.3 Sato's theory

The following statements are due to M. Sato [24], cf. [17, Appendix].

Proposition 6.2. *If $G \subset E(K[[x]])$ is a subring that stabilises $K[\partial]$, i.e. for each operator $P \in E(K[[x]])$ $K[\partial] \circ P \subset K[\partial]$, then $G \subset D$.*

Proof. Obviously, every differential operator $P \in D$ preserves $K[\partial]$. In order to prove the converse, we need the valuation topology on the ring $K[[x]]$, i.e. the topology induced by the metric associated with the proper discrete K-valuation v such that $v(x) = 1$. Denote by $E := E(K[[x]])$ and let $P \in E$. Let

$$P_- = \sum_{n=1}^{\infty} f_n(x)\partial^{-n} \tag{18}$$

be the $E^{\le -1}$-part of P (see theorem 3.2). The condition $K[\partial] \circ P \subset K[\partial]$ implies that $D \circ P \mod xE \subset D \mod xE$, i.e.

$$(QP)_- \in xE \tag{19}$$

for every $Q \in D$. Therefore, $P_- \in xE$ because $1 \circ P \mod xE \in D \mod xE$. Thus $v(f_n) \ge 1$ for all $n \ge 1$. So let f_m be the coefficient of (18) with the lowest valuation and let $v(f_m) = l \ge 1$. Consider the operator $(\partial^l P)_-$. Then we have

$$(\partial^l P)_- = (\partial^l P_-)_- = (\sum_{n=1}^{\infty} \partial^l f_n \partial^{-n})_- = (\sum_{n=1}^{\infty}\sum_{i=0}^{l} C_l^i f_n^{(i)} \partial^{-n+l-i})_- =$$
$$(\sum_{j=1}^{\infty}\sum_{i=0}^{l} C_l^i f_{j-i}^{(i)} \partial^{l-j})_- = \sum_{j=l+1}^{\infty}\sum_{i=0}^{l} C_l^i f_{j-i}^{(i)} \partial^{l-j}. \tag{20}$$

Since $f_n^{(i)}(0) = 0$ for $0 \le i < l$, we have

$$\partial^l \circ P = \sum_{j=l+1}^{\infty} f_{j-l}^{(l)}(0)\partial^{l-j} = \sum_{j=1}^{\infty} f_j^{(l)}(0)\partial^{-j}.$$

But $(\partial^l P)_- \in xE$ by (19). Thus $f_n^{(l)}(0) = 0$ for all $n \ge 1$. This means that $v(f_m) > l$, a contradiction with our assumption. Therefore, none of the coefficient f_n can have the lowest valuation. Namely, $f_n(x) = 0$ for all $n \ge 1$, i.e. P is a differential operator. □

Theorem 6.2. *Let W be a subspace in the space $K((z)) \simeq K((\partial^{-1}))$ with $H^0(W) = H^1(W) = 0$. Then there exists a unique Sato operator, i.e. a zero-th order invertible operator $S = 1 + s_1\partial^{-1} + \dots$, such that $W = \partial \cdot (K[\partial]) \circ S$.*

Proof. Since $H^0(W) = H^1(W) = 0$, we can choose a basis $\{w_n\}_{n\geq 0}$ for $z\cdot W$ in the following form for every $n\geq 0$ (we identify here ∂^{-1} and z):

$$w_n = z^{-n} + \sum_{l=1}^{\infty} a_{nl} z^l.$$

Then the equation

$$w_0 = 1\circ S = 1 + \sum_{l=1}^{\infty} s_l(0) z^l$$

determines all the constant terms of the coefficients as $s_l(0) = a_{0l}$, $l\geq 1$. Now let's assume that we know $s_l^{(i)}(0)$ for all $l\geq 1$ and $0\leq i<n$. Note that we have

$$z^{-n}\circ S = \sum_{m=0}^{\infty}\sum_{i=0}^{n} C_n^i s_m^{(i)}(0)\partial^{n-m-i} = \partial^n + \sum_{l=1}^{\infty}\sum_{i=0}^{n} C_n^i s_{l-i}^{(i)}(0)\partial^{n-l} =$$
$$\partial^n + \sum_{l=1}^{n-1}\sum_{i=0}^{l} C_n^i s_{l-i}^{(i)}(0)\partial^{n-l} + \sum_{i=0}^{n-1} C_n^i s_{n-i}^{(i)}(0) + \sum_{l=n+1}^{\infty}\sum_{i=0}^{n} C_n^i s_{l-i}^{(i)}(0)\partial^{n-l} =$$
$$z^{-n} + \sum_{l=1}^{n-1}\sum_{i=0}^{l} C_n^i s_{l-i}^{(i)}(0) z^{-n+l} + \sum_{i=0}^{n-1} C_n^i s_{n-i}^{(i)}(0) + \sum_{l=1}^{\infty}\sum_{i=0}^{n} C_n^i s_{n+l-i}^{(i)}(0) z^l. \quad (21)$$

The non-negative order terms of the above expression exactly coincides with

$$w_n + \sum_{l=1}^{n-1}\sum_{i=0}^{l} C_n^i s_{l-i}^{(i)}(0) w_{n-l} + \sum_{i=0}^{n-1} C_n^i s_{n-i}^{(i)} w_0,$$

which contains only known quantities. Therefore, the equation

$$z^{-n}\cdot S = z^{-n} + \sum_{l=1}^{n-1}\sum_{i=0}^{l} C_n^i s_{l-i}^{(i)}(0) z^{-n+l} + \sum_{i=0}^{n-1} C_n^i s_{n-i}^{(i)}(0) + \sum_{l=1}^{\infty}\sum_{i=0}^{n} C_n^i s_{n+l-i}^{(i)}(0) z^l =$$
$$w_n + \sum_{l=1}^{n-1}\sum_{i=0}^{l} C_n^i s_{l-i}^{(i)}(0) w_{n-l} + \sum_{i=0}^{n-1} C_n^i s_{n-i}^{(i)} w_0 \quad (22)$$

determines $s_l^{(n)}(0)$ for all $l\geq 1$. Thus we have obtained $s_l(x) = \sum_{n=0}^{\infty}\frac{1}{n!}s_l^{(n)}(0)x^n$. Now the operator $S = 1 + \sum_{l=1}^{\infty} s_l(x)\partial^{-l}$ satisfies $\partial\cdot K[\partial]\circ S = W$ as required. □

7 Efficiency of the classification

7.1 Analytic theory: Baker-Akhieser function

In this subsection we explain the Krichever classification of commutative subrings of ODO's from [13, 14], the main ingredient of which is the notion of Baker-Akhieser function. If the spectral curve is smooth and the rank of the sheaf (which locally free in this case, hence corresponds to a vector bundle) is one, then there are Krichever explicit formulae for the Baker-Akhieser functions. This is an important case where algebra, geometry and analysis meet together and produce beautiful effective results.

Assume the spectral curve C is a smooth curve of genus g (i.e. C is a sphere with g handles). Then the spectral sheaf $\mathcal{F}$ is locally free, i.e. corresponds to a vector bundle. If we take the sheaf $\mathcal{F}(p)$ instead of $\mathcal{F}$, it will have r *global sections* $\eta_1,\dots\eta_r$ (cf. the axiomatic description of the sheaf $\mathcal{F}(p)$ in [4, Th. 1.17]). They are linear dependent at rg points $\gamma_1, \ldots, \gamma_{rg}$ (which form the determinantal divisor on the curve C, i.e. the divisor corresponding to the determinantal line bundle of $\mathcal{F}(p)$, cf. lectures by Wei-Ping Li, where the connection between line bundles and divisors was explained):

$$\eta_l(\gamma_i) = \sum_{j=1}^{r-1} \alpha_{ij}\eta_j(\gamma_i).$$

The set (γ, α) is called *Tyurin parameters*. These are "local coordinates" of the moduli space of rank r and degree rg vector bundles on C.

Krichever used another form of spectral data to classify commuting ODOs:

$$\{C, p, z, \gamma_1, \ldots, \gamma_{rg}, \alpha_1, \ldots, \alpha_{rg}, \omega_1(x), \ldots, \omega_{r-1}(x)\}$$

where z is a local coordinate near p, $z(p) = 0$, $\omega_i(x)$ are some functions.

He showed that these data determines a unique (up to a nonzero constant) *Baker-Akhieser function*:

Definition 7.1. A *vector Baker–Akhiezer function* is a function $\psi(x, P) = (\psi_0(x, P), \ldots, \psi_{r-1}(x, P))$, $P \in C$ on the curve C depending on a formal parameter x (which can be thought of as a local coordinate in some neighbouhood of $0 \in \mathbb{C}$) which satisfies the following conditions:

1. $\psi(x, P) = \left(\sum_{s=0}^{\infty} \xi_s(x) z^s\right) \Psi_0(x, P)$, $\xi_0 = (1, 0, \ldots, 0)$, $\frac{d}{dx}\Psi_0 = A\Psi_0$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ z^{-1} + \omega_1(x) & \omega_2(x) & \omega_3(x) & \ldots & \omega_{r-1}(x) & 0 \end{pmatrix},$$

2. on $C - \{p\}$ ψ is meromorphic with the simple poles at $\gamma_1, \ldots, \gamma_{rg}$
3. $\mathrm{Res}_{\gamma_i}\psi_j = \alpha_{ij}\mathrm{Res}_{\gamma_i}\psi_{r-1}$.

If we know the BA-function, we can reconstruct the ring B as follows: if $f(P)$ is meromorphic function with the pole in p of order n, then there exist ODO $L(f)$ (note: it can be easily and effectively constructed!) such that

$$L(f)\psi(x, P) = f(P)\psi(x, P), \ \mathrm{ord}L(f) = rn.$$

Example 7.1. (i) Let $C = \mathbb{C}P^1$, $p = \infty$. Then the Baker–Akhiezer function is $\psi = e^{xz^{-1}}$, and for any $f = z^n + c_{n-1}z^{n-1} + \cdots + c_0$ we have

$$\partial_x^n\psi + c_{n-1}\partial_x^{n-1}\psi + \cdots + c_0\psi = f\psi.$$

(ii) If C is an elliptic curve (i.e. it is a torus), $C = \mathbb{C}/\{\Lambda\}$, $p = 0$, then

$$\psi = e^{-x\zeta(z)}\frac{\sigma(z + x)}{\sigma(x)\sigma(z)},$$

where σ and ζ are the Weierstrass elliptic functions. Then for the Wallenberg operators we have

$$(\partial_x^2 - 2\wp(x))\psi(x, z) = \wp(z)\psi(x, z),$$
$$\left(\partial_x^3 - 3\wp(x)\partial_x - \frac{3}{2}\wp'(x)\right)\psi(x, z) = \frac{1}{2}\wp'(z)\psi(x, z).$$

7.2 Theta-functions

The explicit formulae of Krichever use special functions, namely the theta-functions. In this subsection we give a short review of them. The main reference is the book of Mumford [20].

Let U be a complex symmetric matrix of order g with $Im(U)$ positive defined. Set

$$\theta(z,U) = \sum_{n\in\mathbb{Z}^g} \exp(\pi i n^t U n + 2\pi i n^t z), \quad z = (z_1,\ldots,z_g)^t \in \mathbb{C}^g$$

Then θ is a holomorphic function on $\mathbb{C}^g$ and is U-quasi-periodic, i.e. $\forall m \in \mathbb{Z}^g$

$$\theta(z+m,U) = \theta(z,U)$$

$$\theta(z+Um,U) = \exp(-\pi i m^t U m - 2\pi i m^t z)\theta(z,U).$$

Theta-functions are used, in particular, for the embedding of a complex torus $X_B = \mathbb{C}^g/\mathbb{Z}^g + U\mathbb{Z}^g$ to the projective space. The Weierstrass functions mentioned above can be also represented with the help of theta-functions. There are formulae for solutions of polynomial equations of degree greater than four that use theta-functions.

7.3 Krichever's formulae

If the rank of the bundle is 1, there are explicit formulae for the BA-function in terms of theta-functions of the *Jacobian* of the spectral curve.

Take a basis $\{a_i,b_i\}$ of $H_1(C,\mathbb{Z})$ such that $a_ib_j = \delta_{ij}$, $a_ia_j = b_ib_j$, and choose a basis $\{w_1,\ldots,w_g\}$ in $H^0(C,\Omega^1_C)$ s.t. $\int_{a_i} w_j = \delta_{ij}$. Then the matrix $U = (U_{ij})$, where $U_{ij} = \int_{b_i} w_j$, is symmetric and $Im(U)$ is positively defined.

The torus $\mathbb{C}^g/(\mathbb{Z}^g + U\mathbb{Z}^g)$ is called the Jacobian $Jac(C)$ of the curve C.

Now fix a differential $\Omega = d(z^{-1}) + \ldots$. Set $2\pi i A_0 = (\int_{b_1}\Omega_0,\ldots,\int_{b_g}\Omega_0)$. Set

$$j(P) = (\int_p^P w_1,\ldots,\int_p^P w_g).$$

The map j is the *Abel-Jacobi* map $j: C \to Jac(C)$.

By the Riemann theorem there exists a vector $\zeta \in \mathbb{C}^g$ such that $\theta(j(P)+\zeta,U)$ has zeros exactly at the points $\gamma_1,\ldots,\gamma_g$ (here the function θ is considered as a function on C minus cycles a_i,b_j, $i,j = 1,\ldots g$).

Theorem 7.1 (Krichever's formula)**.** *Let*

$$\{C,p,z,\gamma_1,\ldots,\gamma_{rg},\alpha_1,\ldots,\alpha_{rg},\omega_1(x),\ldots,\omega_{r-1}(x)\}$$

be spectral data as above with C smooth of genus g. Then the Baker-Akhieser function (defined up to a non-zero constant) is

$$\psi(x,P) = e^{xz^{-1}}\frac{\theta(j(P)+A_0x+\zeta,U)}{\theta(j(P)+\zeta,U)}.$$

Remark 7.1. The BA function ψ of rank one spectral data is equal to $S(\exp^{-xz^{-1}}$, where S is the Sato operator from theorem 6.2. For higher rank data the connection is more complicated, cf. [18].

Comment 7.1. The Krichever formulae for the Baker-Akhieser functions exist in a broader context: in particular, they can be extended in order to define a function on the product $Jac(C) \times C$. These formulae lead to explicit algebro-geometric solutions of the *KP hierarchy* and, in particular, to the explicit solutions of non-linear equations like KdV (Korteweg de Vries equation from the introduction) or KP (Kadomtsev-Petviashvili). They played an important role in the solution of the Schottky problem by T. Shiota, see e.g. [20, Appendix], [27] or the short special course for students [9].

Explicit formulae for the Baker-Akhieser functions exist also in the case when C is a *rational* singular curve (and the rank of the spectral sheaf is one), see [31]. In other cases there are no such formulae.

If the rank of the spectral sheaf is greater than one, there are no general explicit formulae for any curves. Though there are many explicit examples of higher rank commutative operators (see the references in the extra reference list), there are still many open questions.

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