Existence of Liouvillian Solutions in The Problem of Motion of a Rotationally Symmetric Body on a Sphere

Alexander S. Kuleshov[1,a)] and Vera A. Katasonova[1,b)]

[1] *Department of Mechanics and Mathematics, Lomonosov Moscow State University, Russian Federation, 119991, Moscow, Leninskie Gory, Main building of MSU, Department of Mechanics and Mathematics.*

[a)] Corresponding author: kuleshov@mech.math.msu.su
[b)] vera.katasonova@icloud.com

Abstract. The problem of rolling without slipping of a rotationally symmetric rigid body on a sphere is considered. The rolling body is assumed to be subjected to the forces, the resultant of which is directed from the center of mass G of the body to the center O of the sphere, and depends only on the distance between G and O. In this case the solution of this problem is reduced to solving the second order linear differential equation over the projection of the angular velocity of the body onto its axis of symmetry. Using the Kovacic algorithm we search for liouvillian solutions of the corresponding second order differential equation in the case, when the rolling body is a dynamically symmetric ball.

STATEMENT OF THE PROBLEM

The problem of rolling without slipping of a rotationally symmetric body on a fixed supporting surface is a classical problem of nonholonomic mechanics. In 1897 S. A. Chaplygin in his paper [1] proved that the problem of motion of a heavy rotationally symmetric body on a horizontal plane can be reduced to solving the second order linear differential equation over the projection of the angular velocity of the body onto its symmetry axis. In 1909 P. V. Woronetz has shown [2] that the results obtained by Chaplygin can be extended to the problem of motion of a rotationally symmetric body on a sphere, if the body is subjected to the forces, the resultant of which is directed from the center of mass G of the body to the center O of the sphere and depends only on the distance between G and O. In this case the solution of this problem is also reduced to solving the second order linear differential equation. Following Woronetz, let us prove here this fact. We will introduce four systems of coordinates (the unit vectors of the axes are indicated in brackets):

$Ox_1y_1z_1$ $\left(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\right)$ is a fixed system of coordinates with origin at the center of the supporting sphere;

$Gxyz$ $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a system of coordinates, rigidly connected with the rolling body; its origin G is chosen to be at the center of mass of the body, while the axes are directed along the principal axes of inertia of the system;

$Puvn$ $(\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_n)$ is a moving system of coordinates with origin at the point of contact P of the rolling body with the supporting sphere, the axes of which are directed along the tangent to the coordinate lines and along the normal to the surface of the rolling body;

$Pu_1v_1n_1$ $(\mathbf{e}_{u_1}, \mathbf{e}_{v_1}, \mathbf{e}_{n_1})$ is a moving system of coordinates, the axes of which are directed along the tangent to the coordinate lines and along the normal to the supporting sphere.

The position of the contact point P on the surface S of the body is determined by the radius – vector

$$\boldsymbol{\rho} = \overrightarrow{GP} = x\,(u, v)\,\mathbf{e}_1 + y\,(u, v)\,\mathbf{e}_2 + z\,(u, v)\,\mathbf{e}_3,$$

where u and v are the Gaussian coordinates of the point P on the surface S. We will denote the coefficients of the first two quadratic forms of the surface of the rolling body by E, F, G and L, M, N respectively. We will assume that the coordinate lines of the surface S are lines of curvature, therefore

$$F = 0, \quad M = 0.$$

The supporting sphere S_1 of the radius R_1 is defined by the radius – vector

$$\boldsymbol{\rho}_1 = \overrightarrow{OP} = x_1\mathbf{e}_x + y_1\mathbf{e}_y + z_1\mathbf{e}_z = R_1 \sin u_1 \cos v_1 \mathbf{e}_x + R_1 \sin u_1 \sin v_1 \mathbf{e}_y + R_1 \cos u_1 \mathbf{e}_z,$$

where u_1 and v_1 are Gaussian coordinates of the point P on the sphere. For the unit basis vectors $\mathbf{e}_u$, $\mathbf{e}_v$, $\mathbf{e}_n$ and $\mathbf{e}_{u_1}$, $\mathbf{e}_{v_1}$, $\mathbf{e}_{n_1}$ we have the formulae

$$\begin{aligned}
&\mathbf{e}_u = \frac{1}{\sqrt{E}}\frac{\partial \boldsymbol{\rho}}{\partial u}, \quad \mathbf{e}_v = \frac{1}{\sqrt{G}}\frac{\partial \boldsymbol{\rho}}{\partial v}, \quad \mathbf{e}_n = [\mathbf{e}_u \times \mathbf{e}_v]\,;\\
&\mathbf{e}_{u_1} = \frac{1}{R_1}\frac{\partial \boldsymbol{\rho}_1}{\partial u_1}, \quad \mathbf{e}_{v_1} = \frac{1}{R_1 \sin u_1}\frac{\partial \boldsymbol{\rho}_1}{\partial v_1}, \quad \mathbf{e}_{n_1} = [\mathbf{e}_{u_1} \times \mathbf{e}_{v_1}]\,.
\end{aligned} \tag{1}$$

Mutual orientation between two coordinate systems $Gxyz$ and $Puvn$ is defined by the direction cosine table

	x	y	z
u	c_{11}	c_{12}	c_{13}
v	c_{21}	c_{22}	c_{23}
n	c_{31}	c_{32}	c_{33}

where the coefficients c_{ij} are easily calculated using (1).

Following Woronetz [2], we will define the position of the rolling body by Gaussian coordinates u, v, u_1, v_1, and by the angle θ between the Pu and Pv_1 axes. We will assume that the body rolls along the supporting sphere without slipping. This means that two nonholonomic constraints are imposed on the system. The equations of these constraints have the form:

$$R_1\dot{u}_1 = -\sqrt{E}\dot{u}\sin\theta + \sqrt{G}\dot{v}\cos\theta, \qquad R_1\dot{v}_1 \sin u_1 = \sqrt{E}\dot{u}\cos\theta + \sqrt{G}\dot{v}\sin\theta. \tag{2}$$

Let the velocity $\boldsymbol{w}$ of the center of mass G and the angular velocity vector $\boldsymbol{\omega}$ of the body are specified in the coordinate system $Gxyz$ by the components w_1, w_2, w_3 and ω_1, ω_2, ω_3 respectively. From the condition, that the point of contact P of the body with the sphere be instantaneously at rest with respect to the sphere, we obtain the following equations:

$$w_1 + \omega_2 z - \omega_3 y = 0, \quad w_2 + \omega_3 x - \omega_1 z = 0, \quad w_3 + \omega_1 y - \omega_2 x = 0, \tag{3}$$

and for components ω_1, ω_2, ω_3 of the vector $\boldsymbol{\omega}$ we have the following formulae:

$$\omega_1 = c_{11}\tau\dot{v} + c_{21}\sigma\dot{u} + c_{31}n, \quad \omega_2 = c_{12}\tau\dot{v} + c_{22}\sigma\dot{u} + c_{32}n, \quad \omega_3 = c_{13}\tau\dot{v} + c_{23}\sigma\dot{u} + c_{33}n, \tag{4}$$

$$\tau = -\left(\frac{N}{G} - \frac{1}{R_1}\right)\sqrt{G}, \quad \sigma = \left(\frac{L}{E} - \frac{1}{R_1}\right)\sqrt{E},$$

$$n = -\dot{\theta} + \frac{1}{2\sqrt{EG}}\left(\frac{\partial E}{\partial v}\dot{u} - \frac{\partial G}{\partial u}\dot{v}\right) - \dot{v}_1\cos u_1. \tag{5}$$

We will assume that the rolling rigid body is subjected to the potential forces with the potential energy depending only on the coordinates u, v of the point P: $V = V(u, v)$. This case takes place, for example, when the body is subjected to the forces, the resultant of which is directed from the center of mass G of the body to the center O of the sphere, and depends only on the distance between G and O ("central forces"). Thus, we will assume that $V = V(u, v)$.

Let $\Theta = \Theta(\dot{u}, \dot{v}, u, v, n)$ be the kinetic energy of the system, derived using (2)-(4). It can be represented in the following form:

$$2\Theta(\dot{u}, \dot{v}, u, v, n) = K_{33}n^2 + 2(K_{13}\dot{u} + K_{23}\dot{v})n + K_{11}\dot{u}^2 + 2K_{12}\dot{u}\dot{v} + K_{22}\dot{v}^2, \tag{6}$$

where the coefficients K_{ij} are functions of u and v. If we denote by m the mass of the rolling body and by ρ and ε the distance from the center of mass G to the point of contact P and to the tangent plane to the surface S of the body at P

$$\rho^2 = x^2 + y^2 + z^2, \quad \varepsilon = xc_{31} + yc_{32} + zc_{33},$$

then equations of motion of the body can be written as follows:

$$\frac{d}{dt}\left(\frac{\partial\Theta}{\partial\dot u}\right)-\frac{\partial\Theta}{\partial u}=\sqrt{EG}\left(\frac{LN}{EG}-\frac{1}{R_1^2}\right)\frac{\partial\Theta}{\partial n}\dot v+\frac{\sqrt{E}}{R_1}\frac{1}{\tau}\frac{\partial\Theta}{\partial\dot v}n-m\rho\frac{\partial\rho}{\partial u}n^2-m\varepsilon\sqrt{EG}\left(\frac{N}{G}-\frac{1}{R_1}\right)n\dot v-\frac{\partial V}{\partial u},$$

$$\frac{d}{dt}\left(\frac{\partial\Theta}{\partial\dot v}\right)-\frac{\partial\Theta}{\partial v}=-\sqrt{EG}\left(\frac{LN}{EG}-\frac{1}{R_1^2}\right)\frac{\partial\Theta}{\partial n}\dot u+\frac{\sqrt{G}}{R_1}\frac{1}{\sigma}\frac{\partial\Theta}{\partial\dot u}n-m\rho\frac{\partial\rho}{\partial v}n^2+m\varepsilon\sqrt{EG}\left(\frac{L}{E}-\frac{1}{R_1}\right)n\dot u-\frac{\partial V}{\partial v},\tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial\Theta}{\partial n}\right)=-\frac{\sqrt{G}}{R_1}\frac{1}{\sigma}\frac{\partial\Theta}{\partial\dot u}\dot v-\frac{\sqrt{E}}{R_1}\frac{1}{\tau}\frac{\partial\Theta}{\partial\dot v}\dot u+m\rho\left(\frac{\partial\rho}{\partial u}\dot u+\frac{\partial\rho}{\partial v}\dot v\right)n-m\varepsilon\frac{LG-NE}{\sqrt{EG}}\dot u\dot v.$$

Equations (7) together with equation (5) and equations of nonholonomic constraints (2) form the complete system of equations for determining the six unknown functions u, v, n, θ, u_1, v_1.

Now we assume that the rigid body rolling on a supporting sphere is a rotationally symmetric body, i.e. its moments of inertia A_1 and A_2 with respect to the axes Gx and Gy are equal to each other ($A_1 = A_2$) and the surface S of the body is defined by equations

$$x = f(u)\cos v,\quad y = f(u)\sin v,\quad z = g(u).\tag{8}$$

Then the following conditions are valid

$$\frac{\partial\Theta}{\partial v}=0,\quad \frac{\partial\rho}{\partial v}=0,\quad \frac{\partial V}{\partial v}=0.$$

Moreover in the expression (6) for the kinetic energy Θ we will have

$$K_{12}=0,\quad K_{13}=0$$

and the remaining coefficients K_{ij} will be functions of only u. In this case two last equations of the system (7) give:

$$\frac{d}{dt}(K_{23}n+K_{22}\dot v)=(a_1n+b_1\dot v)\dot u,\quad \frac{d}{dt}(K_{33}n+K_{23}\dot v)=(a_2n+b_2\dot v)\dot u,\tag{9}$$

where the coefficients K_{22}, K_{23}, K_{33}, a_1, a_2, b_1, b_2 are functions of only u. Passing to the new independent variable u instead of t we can transform the system (9) to the form

$$K_{23}\frac{dn}{du}+K_{22}\frac{d\dot v}{du}=a_1'n+b_1'\dot v,\quad K_{33}\frac{dn}{du}+K_{23}\frac{d\dot v}{du}=a_2'n+b_2'\dot v,\tag{10}$$

where a_1', b_1', a_2', b_2' are functions of only u. This system of two first order linear differential equations can be transformed to the second order linear differential equation. If we find the general solution of this equation, the problem can be solved by quadratures.

Note that for the body bounded by the surface (8) we have $c_{23} = 0$ and therefore $\dot v$ can be expressed through n and ω_3 using (4). Therefore the system (10) can be represented as the system of two linear differential equations for determining the two unknown functions n and ω_3.

THE CASE OF A DYNAMICALLY SYMMETRIC BALL

Let us assume now that the rolling body is a nonhomogeneous dynamically symmetric ball. Let R be the radius of the ball and a is the distance from the center of mass G of the ball to its geometric center. In this case formulae (8) can be written as follows:

$$x = R\sin u\cos v,\quad y = R\sin u\sin v,\quad z = R\cos u + a.$$

The system of two first order linear differential equations over n and ω_3 has the form:

$$\frac{dn}{du}=a_1n+a_2\omega_3,\quad \frac{d\omega_3}{du}=b_1n+b_2\omega_3,$$

$$a_1 = -\frac{mR^2\left(\left(A_3 - A_1\right)R\cos u + A_3 a\right)\sin u}{\left(A_1A_3 + A_1mR^2\sin^2 u + A_3m\left(R\cos u + a\right)^2\right)R_1}, \quad a_2 = \frac{\left(R_1 + R\right)\left(\left(A_3 + mR^2\right)\left(A_3 - A_1\right) - A_3ma^2\right)\sin u}{\left(A_1A_3 + A_1mR^2\sin^2 u + A_3m\left(R\cos u + a\right)^2\right)R_1},$$

$$b_1 = \frac{mR^3A_1\sin u}{\left(A_1A_3 + A_1mR^2\sin^2 u + A_3m\left(R\cos u + a\right)^2\right)R_1}, \quad b_2 = \frac{mR\left(R_1 + R\right)\left(\left(A_3 - A_1\right)R\cos u + A_3 a\right)\sin u}{\left(A_1A_3 + A_1mR^2\sin^2 u + A_3m\left(R\cos u + a\right)^2\right)R_1}.$$

This system can be reduced to the second-order linear differential equation over ω_3:

$$\frac{d^2\omega_3}{du^2} + d_1\frac{d\omega_3}{du} + d_2\omega_3 = 0, \tag{11}$$

$$d_1 = -\frac{2mR^2\left(A_3 - A_1\right)\sin^2 u\cos u + \left(3 - \cos^2 u\right)mRaA_3 + A_3\left(A_1 + mR^2 + ma^2\right)\cos u}{\left(A_1A_3 + A_1mR^2\sin^2 u + A_3m\left(R\cos u + a\right)^2\right)\sin u},$$

$$d_2 = \frac{mR^2\left(R_1^2 - R^2\right)\left(A_3 - A_1\right)\sin^2 u}{\left(A_1A_3 + A_1mR^2\sin^2 u + A_3m\left(R\cos u + a\right)^2\right)R_1^2}.$$

Note that under condition $R_1 = R$ (i.e. when the radius of the ball is equal to the radius of the supporting sphere), equation (11) has a particular solution

$$\omega_3 = \omega_3^0 = \text{const}.$$

We will change the independent variable in equation (11) by the formula $\cos u = x$. Then equation (11) can be written as follows:

$$\frac{d^2\omega_3}{dx^2} + d_1\frac{d\omega_3}{dx} + d_2\omega_3 = 0, \tag{12}$$

$$d_1 = \frac{3\left(2x - x_1 - x_2\right)}{2\left(x - x_1\right)\left(x - x_2\right)}, \qquad d_2 = \frac{\left(R_1^2 - R^2\right)}{\left(x - x_1\right)\left(x - x_2\right)R_1^2}.$$

Here $d_1, d_2 \in \mathbb{C}(x)$ are rational functions of x, and x_1 and x_2 are roots of the equation

$$A_1A_3 + A_1mR^2\left(1 - x^2\right) + mA_3\left(Rx + a\right)^2 = 0.$$

In order to reduce the equation (12) to a simpler form, the following transformation is made

$$y = \omega_3\exp\left(\frac{1}{2}\int d_1(x)\,dx\right),$$

then equation (12) becomes

$$\frac{d^2y}{dx^2} = \left(\frac{1}{2}\frac{d\left(d_1\right)}{dx} + \frac{d_1^2}{4} - d_2\right)y = S(x)\,y, \tag{13}$$

$$S(x) = \frac{R_1^2 + 8R^2}{8R_1^2\left(x_1 - x_2\right)\left(x - x_1\right)} - \frac{3}{16\left(x - x_1\right)^2} - \frac{R_1^2 + 8R^2}{8R_1^2\left(x_1 - x_2\right)\left(x - x_2\right)} - \frac{3}{16\left(x - x_2\right)^2}.$$

Equation (13) is the second-order linear differential equation with rational coefficients. Therefore we can use the Kovacic algorithm [3] to find liouvillian solutions of this differential equation. The direct application of the Kovacic algorithm to the equation (13) gives the following results.

Theorem 1 *Equation* (13) *has a liouvillian solution of the form*

$$y = \exp\left(\int \omega(x)\,dx\right),$$

where $\omega(x)$ is a rational function $\omega(x) \in \mathbb{C}(x)$, when the condition

$$\frac{R}{R_1} = \frac{N}{2}$$

is valid. Here N is a natural number.

For example, when $R/R_1 = 1/2$ the general solution of equation (12) has the form

$$\omega_3 = \frac{c_1}{\sqrt{x - x_1}} + \frac{c_2}{\sqrt{x - x_2}}.$$

When $R/R_1 = 3/2$ the general solution of equation (12) has the form:

$$\omega_3 = \frac{c_1 (4x - x_1 - 3x_2)}{\sqrt{x - x_2}} + \frac{c_2 (4x - 3x_1 - x_2)}{\sqrt{x - x_1}}.$$

Theorem 2 *In general case equation* (13) *has a liouvillian solution of the form*

$$y = \exp\left(\int \omega(x)\, dx\right),$$

where $\omega(x)$ is algebraic over $\mathbb{C}(x)$ of degree 2.

The general solution of equation (12) for arbitrary values of parameters of the problem has the form

$$\omega_3 = \frac{c_1 \left(\sqrt{x - x_1} + \sqrt{x - x_2}\right)^{\frac{2R}{R_1}}}{\sqrt{(x - x_1)(x - x_2)}} + \frac{c_2 \left(\sqrt{x - x_1} + \sqrt{x - x_2}\right)^{-\frac{2R}{R_1}}}{\sqrt{(x - x_1)(x - x_2)}}.$$

Thus we proved that the general solution of equation (12) are liouvillian for all values of parameters. Therefore the problem of motion of a dynamically symmetric ball on a spherical surface under the action of potential forces with the potential energy $V = V(u)$ are integrable in liouvillian functions.

ACKNOWLEDGMENTS

This research was supported financially by the Russian Foundation for Basic Researches (grants no. 16-01-00338 and no. 17-01-00123).

REFERENCES

[1] S.A. Chaplygin, Regul. Chaotic Dyn. **7**, 119–130 (2002) English translation of the Russian paper published in 1897.
[2] P.V. Woronetz, Univ. Proc., St. Vladimir University. 1–11 (1909) (in Russian).
[3] J.J. Kovacic, J. Symb. Comput. **2**, 3–43 (1986)