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COVARIANT SINGLE-TIME EQUATIONS
FOR A SYSTEM OF N SPINOR PARTICLES

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I. Introduction

Description of the relativistic bound states in quantum field theory represents one of the central problems in quantum field theory. There exist various approaches to that problem. One of them is based on the four-dimensional Bethe-Salpeter equation \(^1\). This equation is employed for describing quantum electromagnetic systems \(^2,3\) and quark systems, the latter being not only two-particle systems \((q\bar{q})\) but also systems of the type \((qq\bar{q})\) \(^4,5\).

Another approach is based on three-dimensional relativistic equations derived within quantum field theory (see a review \(^6\)). This approach is widely used for the description of two- and N-particle systems. For instance, in refs. \(^7\), multiquark systems of the type \((qq\bar{q})\) are described with the help of wave functions in the three-dimensional approach at the light-front \(^8\). Note is to be made that this three-dimensional approach having a lot of merits is in form essentially different from the conventional well-elaborated apparatus of the nonrelativistic theory of N-particle bound systems.

At the same time, the covariant single-time three-dimensional approach to the description of bound states within the framework of quantum field theory \(^9-11\) may be considered as a direct generalization of the nonrelativistic potential theory.

It is specified by the absence of the relative time and by the probabilistic interpretation of the wave function. This method was earlier applied to determine the explicit form of relativistic interaction of two-particle systems: scalar \(^12\), scalar and spinor \(^13\), two spinor particles \(^14,15\).

In this paper, we consider a system of N particles with spin 1/2 which we assume to be distinguishable, so to say ignoring the identity principle. Two-time Green functions are determined perturbatively, and then they are used to calculate a basic quantity of the covariant single-time approach, the kernel (quasipotential) of an integral equation, i.e. the operator of interaction of spinor particles through the photon or (pseudo)scalar boson exchange. For a colourless bound three-quark system (the baryon) we determine quasipotentials responsible for the gluon exchange. Also we
establish a parametric dependence of all the quasipotentials on the total energy of the N-particle system.

2. The Green functions of N-particles system.

The covariant two-time Green function of a system of N spinor particles in the momentum representation is defined as follows:

$$\tilde{G}^{(\omega)}_{p}(p'; p) \chi = \int \exp \left\{ i \sum_{j=1}^{N} \left( \frac{p_j p_j}{2} - k_j y_j \right) \right\} \cdot G^{(\omega)}_{p}(x) \cdot G^{(\omega)}_{p}(y) \delta^{(N)}(\chi_{ \{ p \}} \cdot \chi_{ \{ y \}}) \cdot (d\chi) \cdot (d\chi)' \cdot (d\chi) \cdot (d\chi)' \cdot (d\chi) \cdot (d\chi)'$$

$$= \int \left\{ \prod_{j=1}^{N} \Lambda_{ \{ p \}}(j) \right\} \tilde{G}^{(\omega)}_{p}(p' \cdot p) \cdot \Lambda'_{ \{ p \}}(j) \cdot \Lambda'_{ \{ p \}}(j) \cdot (d\chi) \cdot (d\chi)' \cdot (d\chi) \cdot (d\chi)' \cdot (d\chi) \cdot (d\chi)'$$

(2.1)

Here we use the abbreviations:

$$(d) = (\omega_1, \ldots, \omega_N); \quad (p) = (p_1, \ldots, p_N); \quad (x) = (x_1, \ldots, x_N); \quad (d\chi) = d\chi_1 \ldots d\chi_N$$

for spinor indices $(\omega)$, momenta $(p)$ and $(x)$, coordinates $(x)$ and $(y)$ of particles, and elements of integration $(d\chi)$ and $(d\chi)'$.

The Green function in the coordinate representation in (2.1) is the expectation value of the time-ordered product of Heisenberg operators of spinor fields of the form

$$\tilde{G}^{(\omega)}_{p}(x) \cdot (y) = - i \langle 0 | T \left[ \chi_{ \{ x \}} \cdot \chi_{ \{ y \}} \right] | 0 \rangle$$

(2.2)

A time-like unit vector $\lambda$ specifies the frame of reference in which times of all particles $1, 2, \ldots, N$ are made equal to each other since $\lambda x = \lambda' x'$, where $L_\lambda$ is the Lorentz transformation with the property $L_\lambda \lambda = \lambda'$. The covariant two-time Green function $\tilde{G}$ may also be expressed in terms of the conventional Fourier transform $\tilde{G}^{(\omega)}_{p}(p; k) \delta^{(N)}(\chi_{ \{ p \}} \cdot \chi_{ \{ k \}}) \delta^{(N)}(\chi_{ \{ p \}} \cdot \chi_{ \{ k \}})$ of the 2N-time function (2.3) in the form

$$\tilde{G}^{(\omega)}_{p}(p; k) \lambda = \frac{i}{(2\pi)^{2N}} \int \tilde{G}^{(\omega)}_{p}(p; k; x; p') \delta^{(N)}(\chi_{ \{ x \}} \cdot \chi_{ \{ p \}}) \delta^{(N)}(\chi_{ \{ x \}} \cdot \chi_{ \{ p \}}) \cdot (d\chi) \cdot (d\chi)'$$

(2.3)

Here $(dx)(dp) = dx_1 \ldots dx_N dp_1 \ldots dp_N$, i.e., integration runs over $2N$ one-dimensional variables, and $(p; \lambda) = (p_1, p_2, \ldots, x_1, x_2, \ldots)$.

It is not difficult to determine the Lorentz transformation of the Green function (2.4) \cite{16}:

$$\tilde{G}^{(\omega)}_{p}(p; k; \lambda) = \left[ \prod_{j=1}^{N} \Lambda_{ \{ p \}}(j) \right] \tilde{G}^{(\omega)}_{p}(p' \cdot p; k' \cdot k) \left[ \prod_{j=1}^{N} \Lambda_{ \{ p \}}'(j) \right]$$

(2.5)

where $\Lambda_{ \{ p \}}(j)$ are matrices of a bispinor representation of the Lorentz group. Choosing $L_\lambda$ to be the transformation $L_\lambda k = L_\lambda p + L_\lambda k$, we arrive at the Green function $\tilde{G}^{(\omega)}_{p}(p; k; \lambda)$ which, as follows from (2.4), may be written in the form

$$\tilde{G}^{(\omega)}_{p}(p; k; \lambda) = \tilde{G}^{(\omega)}_{p}(\bar{p}; \bar{k}; \bar{\lambda})$$

(2.6)

The function (2.6) depends only on spatial components of the particles' momenta $\bar{p} = p_1, \ldots, p_N$ and on the total initial and final energies $\bar{k} = k_1, \ldots, k_N$. In this aspect, it is similar to the nonrelativistic Green function. We, however, for keeping the relativistic covariance at all stages of consideration, will employ in calculations the general expression (2.4) rather than (2.6).

In the free case, the 2N-time Green function of a system of N spinor particles in the momentum representation is of the form

$$\tilde{G}^{(\omega)}_{p}(p; k; \lambda) = \left[ \prod_{j=1}^{N} \Lambda_{ \{ p \}}(j) \right] \tilde{G}^{(\omega)}_{p}(p; k; \lambda) \delta^{(N)}(\chi_{ \{ p \}} \cdot \chi_{ \{ k \}})$$

(2.7)

where $\delta^{(N)}$ are propagators of the spinor particles given by

$$S_{\{ p \}}(p; \lambda) = \frac{m_1 + p_1 + \lambda_1 + \lambda}{(p_1 + \lambda_1)^2 - m_1 + i 0} \cdot \hat{p}_1 = \hat{p}_1 \hat{\chi}_1 \hat{\chi}_1 ,$$

(2.8)

and $\hat{\chi}_1$ are the Dirac matrices. Substituting (2.7) into (2.4) we may integrate over $(4N)(4N)$ and determine the function $\tilde{G}^{(\omega)}_{p}(p; k; \lambda)$. However, it is known \cite{16} that there is no an inverse function which is necessary for construction of the interaction operator. For this reason, the two-time Green function (2.1) and (2.4) is projected onto...
positive-frequency states. This projection is usually made after the transition to the "single-time" formalism, i.e., after (2.6). Let us perform this projection in an arbitrary frame of reference, and before we should present a set of covariant definitions.

We expand each of the 4-momenta \( p \) and \( \kappa \) into components longitudinal and transverse to the vector \( \lambda \) as follows:

\[
\hat{p} = \lambda \hat{p}_\lambda + \hat{p}_\perp ; \quad \hat{\lambda} \hat{p}_\lambda = 0 \tag{2.10}
\]

and introduce the on-shell 4-momenta\(^x\)

\[
\hat{\rho} = \hat{p}_\perp + \lambda \hat{\omega}(\hat{p}_\perp) = \hat{p} - \lambda (\hat{\lambda} \hat{p}_\perp) + \lambda \hat{\omega}(\hat{p}_\perp) \tag{2.11}
\]

where

\[
\omega(\hat{p}_\perp) = \sqrt{m^2 - \hat{p}_\perp^2} = \sqrt{m^2 - \hat{p}_\perp^2 + (\lambda \hat{p}_\perp)^2} ; \quad \hat{p}_\perp^2 = m^2.
\]

It is then not difficult to verify that the spinor propagator (2.8) can be represented in the following covariant form

\[
S(p + \varepsilon \lambda) \hat{\lambda} \hat{U}_\varepsilon^\dagger(p) = \frac{1}{p \lambda + \varepsilon - \omega(\hat{p}_\perp) - i 0} \hat{U}_\varepsilon^\dagger(p) \tag{2.12}
\]

Positive- and negative-frequency spinors \( \hat{U}_\varepsilon^\dagger(p) \) and \( \hat{U}_\varepsilon^\dagger(p) \) (\( \varepsilon \) is the polarization index) will be defined as solutions to the equations

\[
(m - \hat{\rho}) \hat{U}_\varepsilon^\dagger(p) = 0 ; \quad (m + \hat{\rho}) \hat{U}_\varepsilon^\dagger(p) = 0 \tag{2.13}
\]

which, according to (2.10), may be written in the form

\[
(m - \hat{\rho}) \hat{U}_\varepsilon^\dagger(p) = 0 ; \quad (m + \hat{\rho}) \hat{U}_\varepsilon^\dagger(p) = 0. \tag{2.14}
\]

These spinors will be normalized by the following invariant condition\(^x\)

\[
\text{We stress that the momenta } p \text{ and } \kappa \text{ are, generally, off the mass shell: } p^2 + m^2 ; \ k^2 + m^2.
\]

Since

\[
S(p + \varepsilon \lambda) \hat{\lambda} \hat{U}_\varepsilon^\dagger(p) = \frac{1}{p \lambda + \varepsilon - \omega(\hat{p}_\perp) - i 0} \hat{U}_\varepsilon^\dagger(p) \tag{2.17}
\]

and

\[
\hat{U}_\varepsilon^\dagger(p) S(p + \varepsilon \lambda) \hat{\lambda} = \frac{1}{p \lambda + \varepsilon - \omega(\hat{p}_\perp) + i 0} \hat{U}_\varepsilon^\dagger(p) \tag{2.18}
\]

we get

\[
G_{(\varepsilon)(\gamma)}(p; q) \lambda = \prod_{j=1}^{n} \hat{U}_\varepsilon^\dagger(\hat{p}_j) \hat{U}_{\gamma}(\hat{q}_j) \tag{2.19}
\]

Note that the operation of time ordering, which in the momenta space means integration in (2.4) over \( (x) \) and \( (\gamma) \) (in fact, over relative energies), and the operation of projecting (2.18), which is performed with the use of the spinors \( \hat{U}_\varepsilon^\dagger(p) \) independent of \( (x) \) and \( (\gamma) \), commute with each other. Therefore, at first performing the projection (2.16) of the function (2.7) and using the properties (2.17), and then integrating over \( (x) \) and \( (\gamma) \) according to (2.4), we get

\[
G_{(\varepsilon)(\gamma)}(p; q) \lambda = \frac{2 \pi}{n} \delta(p \lambda - K) \prod_{j=1}^{n} \delta_{g_j}(p \lambda - K) \tag{2.20}
\]
Here \( \mathbf{K} = \sum_j \mathbf{k}_j \) and \( \mathbf{P} = \sum_{j \neq k} \mathbf{p}_j \) are the total initial and final momenta of the \( \mathbf{N} \)-particle system, respectively.

Note also that the relationships

\[
\prod_{j \neq i} \delta^{(4)}(\mathbf{p}_j - \mathbf{k}_j) = \delta^{(4)}(\mathbf{p}_i - \mathbf{K}) \prod_{j \neq z} \delta^{(4)}(\mathbf{p}_z - \mathbf{k}_z) \quad (2.20)
\]

\[
\delta^{(4)}(\mathbf{p}_i - \mathbf{K}) \delta^{(4)}(\mathbf{P} - \mathbf{K}) = \delta^{(4)}(\mathbf{P} - \mathbf{K})
\]

are valid, where the transverse momenta \( \mathbf{P}_i \) and \( \mathbf{K}_z \) are defined as follows: \( \mathbf{P}_i = \mathbf{P} - \mathbf{\lambda}(\mathbf{P}) \) and \( \mathbf{K}_z = \mathbf{K} - \mathbf{\lambda}(\mathbf{K}) \). Consequently, the two-time Green function \( \mathbf{G}_z \) contains the function \( \delta^{(4)}(\mathbf{P} - \mathbf{K}) \) that represents the energy-momentum conservation law and does not depend upon the vector \( \mathbf{\lambda} \). The two-time Green function \( (2.4) \) possesses a similar property that is a result of translational invariance.

Now let us calculate the covariant two-time Green function in the second order of perturbation theory. For this reason we shall rewrite (2.3) in the interaction representation. We start with the consideration of quantum electrodynamics when the interaction between spinor particles is mediated by photons (the interaction Lagrangian has a standard form). In the second order the function \( \mathbf{G}_z \) may be determined as the sum

\[
\mathbf{G}_z((\mathbf{p}_i);(\mathbf{k}_z)) = \sum_{l + l = \mathbf{k}_z} \mathbf{G}_{z}^{-}\mathbf{G}_{z}^{+}(\mathbf{p}_i);(\mathbf{k}_z) \quad \text{(Fig. 1)}
\]

![Diagram](Image)

Fig. 1. The Green function \( \mathbf{G}_z \) in the second order of the perturbation theory.

The term corresponding to the exchange by a photon between particles \( \mathbf{p}_i \) and \( \mathbf{k}_z \) is (spinor indices are omitted)

\[
\mathbf{G}_z^{\xi\eta}(\mathbf{p}_i;\mathbf{k}_z) = \mathbf{\bar{e}} \mathbf{e} \mathbf{\sigma}^{2\eta} \mathbf{\sigma}^{(4)} \{ \mathbf{p}_i + \mathbf{p}_z - \mathbf{k}_z \} \quad (2.21)
\]

\[
\left( \prod_{j \neq i} \mathbf{S}(\mathbf{p}_j;\mathbf{k}_j) \right) \mathbf{S}(\mathbf{p}_z;\mathbf{k}_z) \right) \mathbf{D}_{\mathbf{p}_z;\mathbf{k}_z} \]

\[
\left[ \prod_{j \neq i, z} \left( \mathbf{\sigma}^{(4)} \mathbf{\sigma}^{(4)} \right) \mathbf{S}(\mathbf{p}_j;\mathbf{k}_j) \right] \mathbf{S}(\mathbf{p}_z;\mathbf{k}_z)
\]

\[
\prod_{j \neq i, z} \left( \mathbf{\sigma}^{(4)} \mathbf{\sigma}^{(4)} \right) \mathbf{S}(\mathbf{p}_j;\mathbf{k}_j) \right) \mathbf{S}(\mathbf{p}_z;\mathbf{k}_z)
\]

Here \( \mathbf{\sigma}_i \) is the electric charge of a particle \( \mathbf{p}_i \) and \( \mathbf{D}_{\mathbf{p}_z;\mathbf{k}_z} \) is the photon propagator.

\[
\mathbf{D}_{\mathbf{p}_z;\mathbf{k}_z} = \frac{\mathbf{g}_{\mathbf{p}_z;\mathbf{k}_z}}{\mathbf{q}^2 + i\mathbf{q}} + (\mathbf{q}^2 + i\mathbf{q}) \frac{\mathbf{g}_{\mathbf{p}_z;\mathbf{k}_z}}{\mathbf{q}^2 + i\mathbf{q}} \quad (2.22)
\]

Performing integration of the function \( (2.21) \) in accordance with \( (2.4) \) and covariant projection according to \( (2.18) \), we obtain the corresponding term of the function \( \mathbf{G}_z^{\xi\eta} \) in the form

\[
\mathbf{G}_z^{\xi\eta}(\mathbf{p}_i;\mathbf{k}_z) = -\mathbf{\bar{e}} \mathbf{e} \mathbf{\sigma}^{2\eta} \mathbf{\sigma}^{(4)} \{ \mathbf{p}_i + \mathbf{p}_z - \mathbf{k}_z \} \quad (2.23)
\]

\[
\left[ \prod_{j \neq i, z} \mathbf{\bar{u}}^{(4)}(\mathbf{p}_j;\mathbf{p}_z) \mathbf{u}^{(4)}(\mathbf{k}_j;\mathbf{k}_z) \right] \mathbf{S}(\mathbf{p}_z;\mathbf{k}_z)
\]

\[
\prod_{j \neq i, z} \left( \mathbf{\sigma}^{(4)} \mathbf{\sigma}^{(4)} \right) \mathbf{S}(\mathbf{p}_j;\mathbf{k}_j) \right) \mathbf{S}(\mathbf{p}_z;\mathbf{k}_z)
\]

\[
\prod_{j \neq i, z} \left( \mathbf{\sigma}^{(4)} \mathbf{\sigma}^{(4)} \right) \mathbf{S}(\mathbf{p}_j;\mathbf{k}_j) \right) \mathbf{S}(\mathbf{p}_z;\mathbf{k}_z)
\]

\[
\prod_{j \neq i, z} \left( \mathbf{\sigma}^{(4)} \mathbf{\sigma}^{(4)} \right) \mathbf{S}(\mathbf{p}_j;\mathbf{k}_j) \right) \mathbf{S}(\mathbf{p}_z;\mathbf{k}_z)
\]

Here we made use of the notation

\[
\mathbf{R}_p = \left( \mathbf{p}_i - \sum_{j \neq i} \mathbf{\omega}(\mathbf{p}_j) + i\mathbf{q} \right)^{-2} \quad (2.24)
\]

\[
\mathbf{R}_k = \left( \mathbf{p}_i - \sum_{j \neq i} \mathbf{\omega}(\mathbf{k}_j) + i\mathbf{q} \right)^{-2}
\]
In a form similar to (2.23), the covariant two-time Green function may also be written in mesodynamics, the theory in which exchange between spinor particles is realized either by a scalar or a pseudoscalar boson.

1. Covariant equation for the single-time wave function of an N-particle system

Now we shall explicitly derive the equation for the single-time wave function given by the expression

\[
\Psi_{\text{sm}}^{(n)}(Q;\{\lambda\}|\lambda) = \prod_{j=1}^{N} \left[ \frac{-1}{i} \sum_{\nu} \left( \Omega^{\nu}_{j} \right) \lambda_{\nu} \right] \int \left( d\xi \right) \exp \left\{ i \sum_{\nu} p_{\nu} x_{\nu} \right\} \]

\[
\Psi_{\text{sm}}^{(n)}(Q;\{\xi\}) = \prod_{j=1}^{N} \delta(\xi_{j} - \xi_{j})
\]

where \( \Psi_{\text{sm}}^{(n)}(Q;\{\lambda\}) \) is the N-time Bethe-Salpeter amplitude

\[
\Psi_{\text{sm}}^{(n)}(Q;\{\xi\}) = \langle \lambda | \prod_{j=1}^{N} \chi_{\nu_{j}}(x_{\nu_{j}}) | Q, M; J, m \rangle
\]

The vector \( \{Q, M; J, m\} \) describes a bound state of an N-particle system with the 4-momentum \( P_{1}, \ldots, P_{N} \), mass \( M \), spin \( J \), and its projection \( m \) onto a fixed axis.

Translational invariance allows us to pick the factor \( \delta^{(n)}(P-Q) \) out of the wave function (1.1), where \( P_{1} = \sum_{j=1}^{N} P_{j} \). It is also not difficult to establish the Lorentz transformation for the function (1.1):

\[
\Psi_{\text{sm}}^{(n)}(L^{*}Q;\{L^{*}p\}) = \prod_{j=1}^{N} \left\{ V_{\lambda_{\nu}}^{j} (L^{*}P_{j}, \bar{P}_{j}) \right\}
\]

\[
\Psi_{\text{sm}}^{(n)}(L^{*}Q;\{L^{*}p\}) = \prod_{j=1}^{N} \left\{ V_{\lambda_{\nu}}^{j} (L^{*}Q, Q) \right\}
\]

where \( L \) is an arbitrary Lorentz transformation, and \( D\{V_{\lambda_{\nu}}^{j} (L^{*}P_{j}, \bar{P}_{j})\} \) is a 2x2 matrix of the Wigner rotation given by

\[
V_{\lambda_{\nu}}^{j} (L^{*}Q, Q)
\]
\[ \Lambda (L^\xi \lambda) \Lambda (\tilde{p}) = \Lambda (L^\xi \tilde{p}) \left[ I \otimes D \left\{ V_k^{-1} (L^\xi \tilde{p}, \tilde{p}) J \right\} \right], \] (3.4)

\[ \Lambda (\tilde{p}) \] being the boost matrix.

The integral equation for the wave function (3.1) can be deduced from the behaviour of the total Green function (2.18) around the pole of the bound state with the 4-momentum \( Q \) and mass \( M \), from the identity \( G^{-1} \tilde{G} = \tilde{I} \), and from the definition of the interaction operator \( V \) by the relation \( \tilde{G}^{-1} = \tilde{C}_0^{-1} - V \). It is of the form \( I \otimes D \) (the indices \( J \) and \( m \) are omitted):

\[ \int \left[ \tilde{C}_0^{-1} \left( \sigma \right) (p; (k) | \lambda) - V (\sigma) (p; (k) | \lambda) \right] \psi (q; (k) | \lambda) (d^4 k) d \lambda = 0. \] (3.5)

(3.5)

Now, instead of the momenta of particles \((p)\) and \((k)\), we shall introduce relative momenta of the system \((\tilde{p})\) and \((\tilde{k})\) with the relations

\[ p_j = \tilde{p}_j \tilde{p}_j; \quad \tilde{p}_j \equiv \tilde{p} \] (3.6)

and in a like manner, for \((k)\). The explicit form of the \( \tilde{C}_0^{-1} \) matrix of the coefficients \( \tilde{c} \) may be chosen in various ways but in any case \( \det \tilde{c} = 1 \) from which it follows that \( d^4 k_j d K_j = d^4 K_j H_j d^4 \tilde{k}_j \).

From (3.6) we may also get the following relations

\[ p_{j\lambda} = \tilde{p}_{j\lambda} \tilde{p}_{j\lambda}; \quad p_j = \left( \tilde{c} \right)_{j\lambda} \tilde{p}_{\lambda}; \quad \tilde{p}_j = \left( \tilde{c} \right)^{-1}_{j\lambda} \tilde{p}_{\lambda}. \] (3.7)

Each of the functions \( \tilde{C}_0^{-1}, V, \psi \) in (3.5) includes the \( \delta \) functions of the total momenta:

\[ V (\sigma) (p; (k) | \lambda) = \delta (p - K) V (\sigma) (p; (k) | \lambda), \] (3.8)

\[ \psi (\sigma) (q; (k) | \lambda) = \delta (q - K) \psi (\sigma) (q; (k)_{j\lambda} | \lambda). \] (3.9)

Using (3.8), (3.9) and (2.19) we may in (3.5) single out the motion of the center of masses and derive the equation

\[ \left[ Q \lambda - \tilde{\omega} (p_{j\lambda} - \ldots - \omega (p_{j\lambda} \right] \psi^{(\sigma)} (q; p_{j\lambda}, \ldots, p_{j\lambda} | \lambda) = \int \psi^{(\sigma)} \left( q; (p_j; (k)_{j\lambda} \right) \psi^{(\sigma)} (q; (k)_{j\lambda}, \ldots, (k)_{j\lambda} | \lambda) \tilde{p}_j d \tilde{k}_j \lambda. \] (3.10)

The transverse momenta in this equation, \( p_{j\lambda} \) and \( k_{j\lambda} \), are expressed through \( p_{j\lambda} \) and \( k_{j\lambda} \) by the relations (3.7) in which \( p_{j\lambda} = p_{j\lambda} \) \( = k_{j\lambda} = k_{j\lambda} = Q \lambda \). We stress that (3.10) is an equation 3-dimensional in each of relative momenta; the total momentum of the system, \( Q \), is a parameter.

Now consider an essential part of the integral equation (3.10), the operator kernel \( V \). Proceeding from the definition \( V = \tilde{C}_0^{-1} - \tilde{G}_0^{-1} \) and expanding \( V \) and \( \tilde{G} \) into perturbative series

\[ V = \sum_{k=0}^{\infty} V_{2k}; \quad \tilde{G} = \sum_{k=0}^{\infty} \tilde{G}_{2k}, \] (3.11)

we derive the following formula for \( V_{2k} \):

\[ V_{2k} = \tilde{G}_0^{-1} \tilde{G}_{2k} \tilde{G}_0^{-1} - \sum_{p=1}^{k} \psi_{2k-2p} \tilde{G}_{2p} \tilde{G}_0^{-1}. \] (3.12)

To the second order of perturbation theory, we have \( V_2 = \tilde{G}_0^{-1} \tilde{G}_2 \tilde{G}_0^{-1} \). Therefore, using (2.19) for \( \tilde{G}_0 \), we may separate the \( \delta \) function representing the 4-momentum conservation we obtain the kernel of eq. (3.10) in the form

\[ V^{(\sigma)} \left( q; (p_j; (k)_{j\lambda} \right) = \sum_{j=1}^{n} \sum_{t \in \sigma} \left[ \tilde{p}_j - \tilde{K}_j \right] \psi_{j\lambda} \tilde{G}_{j\lambda} \] (3.13)

\[ \cdot \left[ \prod_{j=1}^{n} \delta_{j\lambda} \delta^{(3)} \left( p_{j\lambda} - k_{j\lambda} \right) \right] V_{j\lambda, t \in \sigma} \left( q; (p_j; (k)_{j\lambda} \right), \]
where the functions \( V_{\ell;\ell'} \) are given by the relations (2.29)–(2.30).

The kernel of the interaction operator (3.13) was obtained on the basis of quantum electrodynamics. Without repeating the above considerations we present the result for the corresponding kernels in mesodynamics, the theory with the Lagrangian \( \mathcal{L} = \sum_{i} \mathcal{L}_i \), where \( \mathcal{L}_i = \mathcal{L}_i \) (\( \mathcal{L}_i = \mathcal{L}_i \)) for the exchange by a (pseudo)scalar boson \( \gamma \) of the mass \( m \):

\[
V_{\ell;\ell'}^{(1)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots = \sum_{i=1}^{n} \frac{g_{\ell;\ell'}^{(1)} g_{\ell;\ell'}^{(1)}}{(2\pi)^3} \left[ \prod_{j=1}^{n} \mathcal{L}_j \right] \mathcal{U}_j \mathcal{U}_j^{*} \mathcal{L}_j \mathcal{L}_j^{*},
\]

\[
V_{\ell;\ell'}^{(2)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots = \prod_{j=1}^{n} \mathcal{L}_j \mathcal{L}_j^{*} \mathcal{L}_j \mathcal{L}_j^{*},
\]

where

\[
V_{\ell;\ell'}^{(2)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots = \prod_{j=1}^{n} \mathcal{L}_j \mathcal{L}_j^{*} \mathcal{L}_j \mathcal{L}_j^{*},
\]

The functions \( R_{\ell;\ell'}^{(1)}(m) \) and \( R_{\ell;\ell'}^{(2)}(m) \) differ from (2.29) and (2.29) by the change of \( W_{\ell;\ell'}^{(1)} \) and \( W_{\ell;\ell'}^{(2)} \) to the quantity

\[
W_{\ell;\ell'}^{(1)}(m) = \sqrt{-\left( P_{\ell;\ell'} - K_{\ell;\ell'} \right)^2 + m^2} = \sqrt{-\left( P_{\ell;\ell'} - K_{\ell;\ell'} \right)^2 + m^2} = W_{\ell;\ell'}^{(1)}(m).
\]

Note the connection of the quasipotential amplitudes (3.13) and (3.14) with the quasipotential scattering amplitude defined, generally, off the energy shell (2.9) by the relation \( G = \tilde{G} + \tilde{G}^* \tilde{G} \), from which it follows that to the second order of perturbation theory \( V_{\ell;\ell'} = \tilde{G}_{\ell;\ell'}^{(1)} \tilde{G}_{\ell;\ell'}^{(1)} \). The expressions (3.13) and (3.14) we have found not only for a bound system of \( N \) particles but also for scattering states with \( Q^2 > \left( \frac{\alpha}{\mu} m \right)^2 \). In this case, on the energy shell

\[
P\lambda = Q\lambda = \sum_{j=1}^{N} \omega(p_{\lambda j}) = \sum_{j=1}^{N} \omega(k_{\lambda j}),
\]

the quasipotential amplitudes \( V_{\ell;\ell'} = \tilde{V}_{\ell;\ell'} \) (3.13), (3.14) exactly coincide with the corresponding amplitudes in electrodynamics and mesodynamics:

\[
\tilde{V}_{\ell;\ell'}^{(1)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots = \tilde{V}_{\ell;\ell'}^{(1)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots
\]

Besides, due to \( \delta^{(0)} \) -functions in (3.13), from (3.17) it follows that

\[
\tilde{V}_{\ell;\ell'}^{(2)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots = \tilde{V}_{\ell;\ell'}^{(2)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots
\]

Therefore, from (2.29) and (2.29) on the energy shell we have

\[
R_{\ell;\ell'}^{(1)} = \tilde{R}_{\ell;\ell'}^{(1)} = \tilde{R}_{\ell;\ell'}^{(1)} = \tilde{R}_{\ell;\ell'}^{(1)},
\]

and, on the basis of (2.26), we get

\[
V_{\ell;\ell'}^{(1)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots = \tilde{V}_{\ell;\ell'}^{(1)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots
\]

And finally, it may be verified that the momenta on the mass shell defined by the (2.9)–(2.11) obey the equality

\[
\left( P_{\ell;\ell'} - K_{\ell;\ell'} \right) = \left( P_{\ell;\ell'} + \omega(p_{\ell;\ell'}) - K_{\ell;\ell'} \right) = \left( P_{\ell;\ell'} - \omega(k_{\ell;\ell'}) \right).
\]

Thus, using (3.22) in (3.21), we derive from (3.19) on the energy shell (3.17) the following formula

\[
\tilde{V}_{\ell;\ell'}^{(2)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots = \tilde{V}_{\ell;\ell'}^{(2)}(Q; \ell; \ell'; (\ell'; k_\ell); (\ell' k_{\ell})); \ldots
\]
\[
\left[ V_\varepsilon^{(g)}(Q:\hat{p}_z;\kappa_\lambda;\kappa_\lambda) \right]_{\varepsilon S} = -\sum_{l=1}^{L} \left( \frac{e_l e_q}{(2\pi)^3} \right) \int \frac{d^4 k_\lambda}{(2\pi)^4} \frac{1}{(p_{\varepsilon} - k_\lambda)^2 + i\epsilon},
\] (3.23)

This expression does represent the physical amplitude of scattering in an \( N \)-particle system to the second order of perturbation theory. Each term in the sum (3.23) describes the exchange by a photon between particles (1,1), the remaining particles are not interacting. We note that for the latter, the equality \( \hat{p}_j = k_j \) results in the equality \( \hat{p}_j = k_j \).

Thus, we have proved the validity of the relation \( \left[ \hat{R}_j \right]_{\varepsilon S} = \left[ V_\varepsilon \right]_{\varepsilon S} = T_{\varepsilon \nu} \hat{M}_{\nu} \), in electrodynamics. For mesodynamics the proof goes in a similar manner.

For the wave function of a bound state \( \Psi(Q; p_{\omega 1}, ..., p_{\omega L}|\lambda) \) besides (3.10), we may derive a normalization condition (117'). Its explicit form depends on the choice of normalization of the vectors \( Q, M \rangle = \left( \frac{Q, M | J, M_j \rangle}{(Q, M | J, M_j \rangle)_{\lambda}} \right) \) in the definition (3.2) which we subject to the invariant conditions:

\[
\left< Q', M' | Q, M \right> = \delta_{\mu\nu} \delta_{Q, Q'} \delta^{(4)}(\hat{Q}' - \hat{Q}) = \delta_{\mu\nu} \delta(Q|Q') \delta^{(4)}(Q'_\lambda - Q_\lambda),
\] (3.24)

For the explicit form of the normalization condition it is important to know the explicit dependence of the quasi-potential on the parameter \( Q_\lambda \) (the total energy in the frame of reference where \( \lambda = (1, \hat{0}) \)). Indeed, the normalization condition looks as follows (we omit the derivation):

\[
\sqrt{Q^2 + M^2} = \sum_{(\alpha)} \int \Psi^{(\alpha)}(Q; p_{\omega 1}, ..., p_{\omega L}|\lambda) \Psi^{(\alpha)}(Q; p_{\omega 1}, ..., p_{\omega L}|\lambda)_{p_{\varepsilon}} d^4 p_{\varepsilon},
\] (3.25)

Therefore the second term in the r.h.s. of (3.25) can only with the expressions (3.13) and (3.14). Their parametric dependence on \( Q_\lambda \) is quite definite; their derivatives with respect to \( Q_\lambda \) can easily be computed (we do not write the results in view of their cumbersome form).

To complete the section, we should note the following. In most papers on the quasipotential approach the vector \( \lambda \) characterizing a system in which times of particles are equated is taken in the form \( \lambda = \lambda_q = Q/\sqrt{Q^2} = Q/M \), i.e. the unit vector \( \hat{\lambda} \) is directed along the total 4-momentum of the system. In this case, passing to the c.m.s. with the help of transformations of (3.3) we arrive at the following properties: \( Q_* = M; \bar{Q} = 0; \hat{\lambda}_0 = (1, \hat{0}); \hat{\lambda}_\bar{e} = M \). Consequently, the splitting (2.9) of all vectors into longitudinal and transverse components in fact means the splitting into time and spatial components so that

\[
\hat{p}_e = (0, \hat{p}_e) = \hat{p}_e \in \omega(p); \quad \hat{p} = (\omega(p), \hat{p}).
\] (3.26)

In the considered case (3.10) assumes the form

\[
\left[ M - \sum_{j=1}^{L} \omega(p_j) \right] \Psi^{(\alpha)}(M; \hat{p}_e, ..., \hat{p}_e) = \psi^{(\alpha)}(M; \hat{p}_e, ..., \hat{p}_e) \]

\[
= \int \left< V^{(\alpha)}(M; \hat{p}_e, \hat{k}_e) \Psi^{(\alpha)}(M; \hat{p}_e, ..., \hat{p}_e) \right> d^3 \hat{p}_e d^3 \hat{k}_e,
\] (3.27)

and it is clear that \( M = Q_\lambda \) is a spectral parameter. With this choice of the vector \( \lambda \) one may rewrite the quasipotentials (3.13), (3.14) and the normalization condition (3.25) using the expression (3.26).
4. A covariant equation for the single-time wave function of a 3-quark system

Now consider a bound system of three quarks (the baryon), interacting via gluon exchange. The interaction Lagrangian in QCD is of the form (see, e.g., ref. 16/):

$$\mathcal{L}_g = g \sum_{j \bar{d}_j} \overline{\psi}_j \gamma^a \gamma^j \psi_j ,$$  \hspace{1cm} (4.1)

where the index $j$ labels the quark flavour ($u, d, s, \ldots$), $\gamma^a_j, \gamma^j_d$ are colour indices, $\gamma^a_j$ are the gluon fields, $\gamma^j_d$ are the generators of the group SU(3) in the fundamental representation.

The covariant single-time wave function of a 3-quark system is defined by relations of the type (3.1) and (3.2) in which the spinor fields have one more, colour index. Therefore the wave function $\Psi^{(\alpha)}(Q; (p_1), \ldots, (p_3))$ depends both on the polarisation indices $(\nu) = (\nu_1, \nu_2, \nu_3)$ and on the colour indices $(c) = (c_1, c_2, c_3)$. To determine the operator $\mathbf{V}$, we should compute the Green functions of a 3-quark system $G_3$ and $G_2$. The free Green function has the form (2.19), where now $N=3$ and only the product of three $\mathbf{S}$-symbols of the colour indices is added. When calculating the Green function $G_3$ we shall take advantage of the free propagator of the gluon field in the covariant gauge being of the form 16/:

$$D_{\mu\nu}^{\alpha\beta}(q) = \frac{\mathbf{S}_{\alpha\beta}}{q^2 + i0} \left( g_{\mu\nu} + (\mathbf{d}-q) \frac{g_{\mu\nu}}{q^2 + i0} \right) .$$  \hspace{1cm} (4.2)

Upon calculating, for the interaction operator to the one-gluon exchange approximation we get the following expression

$$\mathbf{V}^{(\alpha)}(Q; (p_1), (p_2), (p_3)) = \sum_{i, j, k} \frac{g^2}{(2\pi)^3} \left[ \mathbf{S}_{\alpha\beta} \mathbf{S}_{\gamma\delta} \delta^i_{c_1} \delta^j_{c_2} \delta^k_{c_3} \right] \left( p_{i1} - q \right) \left( p_{j2} - q \right) \left( p_{k3} - q \right) .$$  \hspace{1cm} (4.3)

where summation over $i, j, k$ runs over the cycle $(1, 2, 3)$ and $\mathbf{V}_1(Q; (p_1), (p_2), (p_3))$ is given by the relations (2.25) and (2.3) in which $N=3$.

To write an equation for $\Psi^{(\alpha)}(Q_1, Q_2)$ we shall, considering the baryon to be a colourless object, define the wave function being singlet in colour

$$\Psi^{(\alpha)}(Q_1, Q_2, Q_3) = \frac{1}{\sqrt{6}} \mathbf{E}_{c_1 c_2 c_3} \Psi^{(\alpha)}(Q_1, Q_2, Q_3) .$$  \hspace{1cm} (4.4)

Multiplying the equation for $\Psi^{(\alpha)}(Q_1, Q_2)$ by the antisymmetric tensor $\mathbf{E}_{c_1 c_2 c_3}$ and considering the property of generators of the group SU(3)/8/, we arrive at the following equation for the singlet wave function

$$\left[ Q \lambda - \omega(p_1) - \omega(p_2) - \omega(p_3) \right] \Psi^{(\alpha)}(Q_1, Q_2, Q_3) = 0 .$$  \hspace{1cm} (4.5)

we shall take advantage of the free propagator of the gluon field in the covariant gauge being of the form 16/:

$$D_{\mu\nu}^{\alpha\beta}(q) = \mathbf{S}_{\alpha\beta} \mathbf{S}_{\gamma\delta} \delta^i_{c_1} \delta^j_{c_2} \delta^k_{c_3} \left( p_{i1} - q \right) \left( p_{j2} - q \right) \left( p_{k3} - q \right) .$$  \hspace{1cm} (4.2)

Upon calculating, for the interaction operator to the one-gluon exchange approximation we get the following expression

$$\mathbf{V}^{(\alpha)}(Q_1, Q_2, Q_3) = \sum_{i, j, k} \frac{g^2}{(2\pi)^3} \left[ \mathbf{S}_{\alpha\beta} \mathbf{S}_{\gamma\delta} \delta^i_{c_1} \delta^j_{c_2} \delta^k_{c_3} \right] \left( p_{i1} - q \right) \left( p_{j2} - q \right) \left( p_{k3} - q \right) .$$  \hspace{1cm} (4.3)

Now let us examine the change of the quasipotential in sign due to the factor $(-2/3)$. Unlike the case of quantum electrodynamics when in a system of three particles with the same electric charge the interaction (3.13) is repulsive, i.e. practically there are no bound states, in the case of quantum chromodynamics in a system of three quarks with the same colour charges $\mathbf{g}$ the interaction is attractive.

We have determined the interaction operator (4.3) to the second perturbation order; however, as distinct from quantum electrodynamics, this approximation in quantum chromodynamics is not likely satisfactory. Therefore it seems reasonable also to consider higher
orders of perturbation theory, as well, which we shall make as follows.

In a number of papers (see, e.g., ref.19 and references therein), serious arguments were adduced in favour of the total (not free) gluon propagator in the covariant gauge when \( q^2 \to 0 \), behaves in the following manner:

\[
\mathcal{D}^{\alpha\beta}_{\mu\nu}(q) = \frac{\Lambda^2 \delta^{\alpha\beta}}{(q^2 + i\eta)^2} \left\{ g_{\mu\nu} + (\alpha-1) \frac{q_\mu q_\nu}{q^2 + i\eta} \right\},
\]

(4.7)

where \( \Lambda \) is a constant with the mass dimensionality. Moreover, expression (4.7) was assumed as the total propagator for all \( q^2 \).

Now we shall calculate the covariant two-time Green functions of a \( 3 \)-quark system using the propagator (4.7) instead of (4.2), which is well-founded since the integrals over \( (d\omega) \) and \( (dp) \) in (2.4) are computed on the basis of the theory of residues where just the pole behaviour of the propagator (4.7) is important in the infrared region \( q^2 \to 0 \). Using the propagator (4.7) for computing the Green function \( G_\alpha^\beta \) (the approximation of the "dressed"-gluon exchange), for the interaction operator we obtain an expression of the form (4.1), but the functions \( V_{\mu+\nu}^{\alpha\beta} \) are replaced by the functions

\[
V_{\mu+\nu}^{\alpha\beta}(\lambda,p;\lambda',p') = \Lambda^2 \Lambda^{\alpha\beta} \Lambda_{\mu+\nu}^{\alpha\beta} \left( \lambda,p;\lambda',p' \right) + \Lambda^2 (\alpha-1) \Lambda^{\alpha\beta} \Lambda_{\mu+\nu}^{\alpha\beta} \left( \lambda,p;\lambda',p' \right),
\]

(4.9)

where, in turn,

\[
V_{\mu+\nu}^{\alpha\beta}(\lambda,p;\lambda',p') = \frac{1}{4 W_\mu W_\nu} \left( R_{\mu+\nu}^2 + R_{\mu-\nu}^2 \right) + \frac{1}{4 W_\mu W_\nu} R_{\mu+\nu}' R_{\mu-\nu}' \left( \lambda,p;\lambda',p' \right),
\]

(4.9)

\[
V_{\mu+\nu}^{\alpha\beta}(\lambda,p;\lambda',p') = R_{\mu+\nu}^{-1} \left\{ \frac{5}{16 W_\mu W_\nu} \left( R_{\mu+\nu}' R_{\mu-\nu}' - R_{\mu+\nu} R_{\mu-\nu} \right) \right\} - \frac{5}{16 W_\mu W_\nu} \left( R_{\mu+\nu}' R_{\mu-\nu}' \right) + \frac{4}{16 W_\mu W_\nu} \left( R_{\mu+\nu}' R_{\mu-\nu}' \right).
\]

(4.10)

The quantities \( R_{\mu+\nu} \), \( R_{\mu+\nu}' \) and \( W_{\mu+\nu} \) are given by the same formulae, (2.28) and (2.30) in which now the number of particles, \( N^3 \).

It is also not difficult to derive an equation for the singlet wave function of the baryon in the considered approximation of the "dressed"-gluon exchange. It is of the form (4.6) with the change of \( \nu \leftrightarrow \mu \) to \( \nu \leftrightarrow \alpha \).

5. Conclusion

Based on the relativistic quantum field theory, we have derived 3(\( N+1 \))-dimensional covariant single-time equations for a bound system of \( N \) relativistic particles with spin \( 1/2 \). Unlike the 3(\( N+1 \))-dimensional equations in terms of the light-front variables, our formalism has a close analogy with the apparatus of nonrelativistic potential theory, which allows a larger clarity for it. Based on the field-theoretical equal-time Green functions, we have explicitly found the kernels of those equations that are the relativistic generalization of three-dimensional potentials of quantum mechanics. We have shown that the dependence is determined explicitly, which allows us to study its influence on the physical characteristics of a bound system, mass spectra, decay widths, etc. The normalization condition is established for the single-time wave function of a bound state that takes into account the dependence of the quasipotential on the total energy of a composite system. This dependence is determined explicitly, which allows us to study its influence on the physical characteristics of a bound system, mass spectra, decay widths, etc. The normalization condition is established for the single-time wave function of a bound state that takes into account the dependence of the quasipotential on the total energy of a composite system. A 3-quark system is analyzed for which the corresponding quasipotential of quark interaction is determined in the framework of QCD.

The results obtained here can easily be generalized to systems consisting of spinor particles and antiparticles, for instance, of quark-antiquark systems of the type \( (q\bar{q}) \), \( (q\bar{q}q\bar{q}) \), \( (qqqqq) \). For this case if the particle with the number \( j \) is replaced by the antiparticle, then one should use the spinor \( \tilde{U}(\tilde{F}) \) instead of the spinor \( U(F) \) while projecting the Green function (2.16). As a result, the matrix element

\[
\tilde{U}(\tilde{F}) \tilde{Y}(\tilde{F}) U(F)
\]

in the Green function (2.23) and the interaction operator (3.13) must be substituted by

\[
\tilde{U}(\tilde{F}) \tilde{Y}(\tilde{F}) U(F)
\]

Further publications will be devoted to the application of the solutions of the relativistic equations for the wave functions for describing the processes of interactions of the bound states of \( N \)-particles with other objects as well as the processes of their decays or their productions.
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References

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