Theoretical and Mathematical Physics, **195**(3): 886–915 (2018)

GROUP ANALYSIS OF THE ONE-DIMENSIONAL BOLTZMANN EQUATION: III. CONDITION FOR THE MOMENT QUANTITIES TO BE PHYSICALLY MEANINGFUL

K. S. Platonova* **and A. V. Borovskikh***

We present the group classification of the one-dimensional Boltzmann equation with respect to the function $\mathcal{F} = \mathcal{F}(t, x, c)$ characterizing an external force field under the assumption that the physically meaningful constraints $dx = c\,dt$, $dc = \mathcal{F}\,dt$, $dt = 0$, and $dx = 0$ are imposed on the variables. We show that for all functions $\mathcal{F}$, the algebra is finite-dimensional, and its maximum dimension is eight, which corresponds to the equation with a zero $\mathcal{F}$.

Keywords: Boltzmann equation, symmetry group, gas dynamics equation, equivalence group

DOI: 10.1134/S0040577918060077

1. Boltzmann equation

We perform a group analysis of the one-dimensional Boltzmann equation [1]

$$f_t + c f_x + \mathcal{F} f_c = 0 \quad (t, x, c \in \mathbb{R}), \tag{1}$$

which describes the evolution of a fluid particle distribution (here, t is the time, x is the spatial coordinate, c is the momentum, and $f(t, x, c)$ is the phase density of the particles). We assume that the function $\mathcal{F}(t, x, c)$ responsible for the force field is given (but still arbitrary).

We first say a few words about the original statement of the problem and related problems. We are in fact interested not in the symmetry groups of the Boltzmann equation itself but in their behavior in passing from the Boltzmann equation to a system of equations for the moment functions

$$j^{(n)}(t, x) = \int_{-\infty}^{+\infty} f(t, x, c) c^n \, dc. \tag{2}$$

Of course, it is most promising to consider the three-dimensional equation, but because we here start to consider the statements of absolutely new problems, we first try to do this for the one-dimensional equation.

We stress that the group analysis is always related to a certain transformation of the problem, namely, some arbitrary functions must be introduced. It is well known that equations for which some relations may vary or even be empirical laws are often used in mechanics. These relations can be written analytically only approximately starting from some hypotheses. In this case, the mathematical statements are usually

*Lomonosov Moscow State University, Moscow, Russia, e-mail: kseniya-plat@yandex.ru.

This research was supported by the Russian Foundation for Basic Research (Grant No. 15-01-04066).

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 195, No. 3, pp. 451–482, June, 2018. Original article submitted June 3, 2017; revised August 7, 2017.

based on using arbitrary functions (of course, constraints of the form of positivity, monotonicity, etc., can be imposed on these functions, but they are arbitrary in their origin). In the group analysis, this is simply a rule determined by the technique; this rule works most efficiently in the case of minimal constraints on the class of functional relations that can be varied, while inappropriate, "excessive" assumptions can seriously complicate the analysis. Moreover, using arbitrary functions sometimes results in the mechanical classification of equations being replaced with their mathematical classification, for example, with symmetry groups, as in Ovsyannikov's famous work [2]. We therefore first consider the Boltzmann equation with an arbitrary function $\mathcal{F}(t,x,c)$.

The second specific property of group analysis is a significant extension (compared with the physical statement of the problem) of the class of changes of variables (changes of coordinate systems in the physical language). This drastically simplifies the calculations but generates a problem of interpreting the results from the standpoint of physical meaning. We are sometimes on the edge of this meaning and sometimes even over the edge. The results of group analysis must therefore be constantly reviewed to determine whether the physical meaning is lost, and if so, then why it is lost.

The group analysis of Eq. (1) conducted in [3] is a clearly convincing example of such problems. It is easy to see that Eq. (1) is a linear homogeneous partial differential equation for the function f, which is solved in the standard way consisting in finding a change of variables (t,x,c) converting (1) into the simplest equation $f_t = 0$. Therefore, at the first glance, the group analysis of such an equation is trivial: all equations are equivalent to one another, and the symmetry groups of all equations are isomorphic to the symmetry group of the simplest equation, which is easily found. But there is one oversight here. Admitting arbitrary transformations of the variables (t,x,c), we can very easily lose the same physical meaning because c is the velocity of a particle in a space where the coordinates are given by x and stretching the axis x (while time does not vary) assumes also recalculating c.

We can give an analogous, more clearly convincing example in the three-dimensional case: how can the coordinate system in the space of variables x be rotated without rotating it in the space of velocities? But this is not written "in the equation," and it is impossible to overcome this absurdity using purely mathematical tools. A way out is to impose an additional condition of transformation invariance on the additional relation $dx = c\,dt$ (and simultaneously on the relation $dc = \mathcal{F}\,dt$, with which it forms Newton's second law).

It thus turns out that a group analysis of Eq. (1) without the conditions

$$dx = c\,dt, \qquad dc = \mathcal{F}\,dt \tag{3}$$

(which, as shown in [3], gives a quite different result compared with the same analysis with these conditions taken into account) is senseless from the physical standpoint (by the way, precisely this point was apparently neglected by the authors of [4], where they reduced the left-hand side of the equation to the simplest form).

Similarly, one more condition now related to not the Boltzmann equation itself but moment functions (2) appears "from nowhere" (from the mathematical standpoint). These quantities have physical meanings in an appropriate medium (the function $j^{(0)}$ is interpreted as the mass density, $j^{(1)}$, as the momentum density, and $j^{(2)}$, as the energy density) and are calculated by integrating over the velocities of the corresponding characteristic particles located at a given point of space at a given time, i.e., we formally integrate over the straight line $t = \text{const}$, $x = \text{const}$. But for arbitrary changes of variables, this straight line transforms into an unknown curve in the space (t,x,c), and the integral (together with the corresponding quantity) becomes physically meaningless. To preserve the physical meaning, we must therefore supplement the problem statement with the condition that these straight lines are invariant, i.e., introduce the invariance condition for the relations

$$dt = 0, \qquad dx = 0. \tag{4}$$

There is another factor of physical nature. The quantity

$$f(t, x, c)\, dx\, dc \tag{5}$$

also has the physical meaning of the number of particles located in the phase volume $dx\, dc$ at time t, and this quantity by assumption must be invariant under the change of variables (because a change of variables is only a change of the coordinate system and the set of particles treated as a physical object does not change in this case). One more condition thus arises.

As a result, the original problem becomes nontrivial. We note that some of the listed conditions simplify the analysis, and some (e.g., the last one) strongly complicate it. To take this analysis to an effective result, we must therefore manipulate the conditions, namely, use some of them, temporarily "forget" the others, and return to them when some preliminary classification results are obtained.

Here, we consider Eq. (1) with only the set of conditions (3) and (4), and the situation where the invariance condition for quantity (5) is added will considered in a separate paper.

2. Symmetry groups and equivalence groups of Eqs. (1) and (3)

We first recall the basic definitions and the results obtained in [3], [5].

The *symmetry group* of a differential equation $Y = 0$ is the group G of transformations taking the equation $Y = 0$ to itself. We then say that the equation is *invariant* under the group G.

Each one-parameter subgroup of G is generated by a vector field ξ calculated as the derivative with respect to the parameter of a one-parameter group at the zero value of this parameter. The one-parameter group itself is reconstructed from the vector field as a solution of a system of ordinary differential equations with the right-hand side equal to ξ.

We here deal with a transformation of the space of variables (t, x, c, f), and the corresponding components of the vector field are denoted by $(\tau, \xi, \alpha, \eta)$. A one-parameter subgroup is reconstructed from the vector field as a solution of the system of equations

$$\dot{t} = \tau(t, x, c, f), \qquad \dot{x} = \xi(t, x, c, f), \qquad \dot{c} = \alpha(t, x, c, f), \qquad \dot{f} = \eta(t, x, c, f) \tag{6}$$

(the dot denotes differentiation with respect to the group parameter). With the vector field, we usually associate the differential operator

$$\Xi = \tau(t, x, c, f)\partial_t + \xi(t, x, c, f)\partial_x + \alpha(t, x, c, f)\partial_c + \eta(t, x, c, f)\partial_f, \tag{7}$$

in terms of which the invariance condition for the equation $Y = 0$ becomes succinct, i.e., $\Xi Y|_{Y=0} = 0$ (operator (7) can then be standardly continued to the space of variables that also contain the derivatives; the derivatives are f_t, f_x, and f_c in our case).

The set of all operators generating one-parameter groups forms a Lie algebra, i.e., a linear space invariant under the operation of commutation of operators (satisfying the Jacobi condition). If this space is finite-dimensional, then it can be used to reconstruct the connected component of the group, which contains the identity transformation, as a locally parameterized set (manifold) of the same dimension. Here, we do not construct the group itself (because we do not need this below) but classify the group, as usual, in terms of the Lie algebra of operators (7).

The transformations taking any equation of form (1) to an equation of the same form but with another function $\mathcal{F}$ form a group called the *equivalence group* of the family of equations of form (1). The equivalence group permits dividing the original family into classes of mutually equivalent equations by reducing the classification of all equations to the classification of the set of representatives of the corresponding classes.

This group already acts in the space of five variables $(t, x, c, f, \mathcal{F})$ and is generated, as the symmetry group, by the algebra of operators

$$\widehat{\Sigma} = \tau(t,x,c,f)\partial_t + \xi(t,x,c,f)\partial_x + \alpha(t,x,c,f)\partial_c + \eta(t,x,c,f)\partial_f + \Phi(t,x,c,f,\mathcal{F})\partial_{\mathcal{F}}.$$

As the starting point, we consider two main results in [3] and [5] about the symmetry group and the equivalence group for Eq. (1) supplemented with relations (3).

We let

$$\mathcal{D}_{\mathcal{F}} = \partial_t + c\partial_x + \mathcal{F}\partial_c \tag{8}$$

denote the differential operator generating Eq. (1).

Theorem 1. *The Lie algebra of the group of symmetries of Eq.* (1) *preserving differential relations* (3) *comprises two terms*: *an infinite-dimensional subalgebra of transformations of an unknown function*

$$\Xi_f = \eta(t,x,c,f)\partial_f, \tag{9}$$

where $\eta(t,x,c,f)$ *is an arbitrary function satisfying the condition* $\mathcal{D}_{\mathcal{F}}\eta = 0$, *and an infinite-dimensional subalgebra of transformations of the independent variables*

$$\Xi_{t,x,c} = \tau\partial_t + (\beta + c\tau)\partial_x + (\mathcal{D}_{\mathcal{F}}\beta + \mathcal{F}\tau)\partial_c, \tag{10}$$

where $\tau(t,x,c,f)$ *is an arbitrary function and* $\beta(t,x,c)$ *satisfies the condition*

$$\mathcal{D}_{\mathcal{F}}^2\beta - \mathcal{F}_c\mathcal{D}_{\mathcal{F}}\beta - \mathcal{F}_x\beta = 0. \tag{11}$$

Theorem 2. *The equivalence algebra of Eq.* (1) *preserving differential relations* (3) *comprises two terms*: *an infinite-dimensional subalgebra of transformations* $\widehat{\Xi}_f = \eta(f)\partial_f$ *of the function* f *and an infinite-dimensional subalgebra of transformations of the independent variables* t, x, c, *and* $\mathcal{F}$,

$$\widehat{\Xi}_{t,x,c} = -\beta_c\partial_t + (\beta - c\beta_c)\partial_x + (\beta_t + c\beta_x)\partial_c + \mathcal{D}_{\mathcal{F}}^2\beta\partial_{\mathcal{F}}, \tag{12}$$

where $\beta(t,x,c)$ *is an arbitrary function.*

3. Symmetry groups and equivalence groups of Eqs. (1), (3), and (4)

We now present the main result.

The invariance condition for relations (4) implies the conditions $\tau_c\,dc + \tau_f\,df = \xi_c\,dc + \xi_f\,df = 0$ for any dc and df for both the symmetry algebra and the equivalence algebra, which obviously implies $\tau = \tau(t,x)$, $\xi = \xi(t,x)$.

Therefore, the symmetry subalgebra (10) under which condition (4) is invariant becomes $\Xi_{t,x,c} = \tau\partial_t + \xi\partial_x + (\mathcal{D}_{\mathcal{F}}(\xi - c\tau) + \mathcal{F}\tau)\partial_c$, where $\tau(t,x)$ and $\xi(t,x)$ are arbitrary functions satisfying the equation

$$-\mathcal{F}_t\tau - \mathcal{F}_x\xi - \mathcal{F}_c(\xi_t + c\xi_x - c\tau_t - c^2\tau_x) + \mathcal{F}(\xi_x - 2\tau_t - 3c\tau_x) + \\ + \xi_{tt} + 2c\xi_{tx} + c^2\xi_{xx} - c\tau_{tt} - 2c^2\tau_{tx} - c^3\tau_{xx} = 0, \tag{13}$$

which is obtained from (11) by substituting $\beta = \xi - c\tau$.

For algebra (12), it turns out that $\beta_{cc} = 0$, and the change $\beta = \xi - c\tau$ results in the subalgebra

$$\widehat{\Xi}_{t,x,c} = \tau\partial_t + \xi\partial_x + (\xi_t + c(\xi_x - \tau_t) - c^2\tau_x)\partial_c + \mathcal{D}^2_{\mathcal{F}}(\xi - c\tau)\partial_{\mathcal{F}} \tag{14}$$

with arbitrary functions $\tau = \tau(t,x)$ and $\xi = \xi(t,x)$. As can be seen, this condition quite significantly improves the situation: the equivalence group reduces to the group induced by diffeomorphisms in the space (t,x), and the symmetry group is induced by a subgroup of the group of diffeomorphisms, which is determined by Eq. (11). As is seen below, a sufficiently large equivalence group permits an efficient classification of equations.

Table 1

Case	$\mathcal{F}(t,x,c)$	Basis of the symmetry algebra
1	$\mathcal{F} = 0$	$\Xi_1 = \partial_t$, $\Xi_2 = \partial_x$, $\Xi_3 = t\partial_t$, $\Xi_4 = x\partial_x$, $\Xi_5 = t^2\partial_t + tx\partial_x$, $\Xi_6 = tx\partial_t + x^2\partial_x$, $\Xi_7 = x\partial_t$, $\Xi_8 = t\partial_x$

Representatives of classes of functions for which (1) has an eight-dimensional symmetry group.

Table 2

Case	$\mathcal{F}(t,x,c)$	Basis of the symmetry algebra
2.1	$\mathcal{F} = Ac^a$	$\Xi_1 = \partial_t$, $\Xi_2 = \partial_x$, $\Xi_3 = t\partial_t + \dfrac{a-2}{a-1}x\partial_x$
2.2	$\mathcal{F} = Ae^{ac}$	$\Xi_1 = \partial_t$, $\Xi_2 = \partial_x$, $\Xi_3 = t\partial_t + (x - t/a)\partial_x$
2.3	$\mathcal{F} = A\exp\displaystyle\int \frac{3c+a}{c^2+bc+d}\,dc$	$\Xi_1 = \partial_t$, $\Xi_2 = \partial_x$, $\Xi_3 = (x+(a-b)t)\partial_t + ((a-2b)x-dt)\partial_x$
2.4	$\mathcal{F} = \dfrac{A}{x^3}$	$\Xi_1 = \partial_t$, $\Xi_2 = t\partial_t + \dfrac{x}{2}\partial_x$, $\Xi_3 = t^2\partial_t + tx\partial_x$
2.5	$\mathcal{F} = A\Big(1 + \dfrac{(t+ac)^2}{t^2+2ax}\Big)^{3/2}$	$\Xi_1 = \partial_t - \dfrac{t}{a}\partial_x$, $\Xi_2 = t\partial_t + 2x\partial_x$, $\Xi_3 = -ax\partial_t + (3tx + t^3/a)\partial_x$
2.6	$\mathcal{F} = A\Big(\dfrac{(x-ct)^2+c^2+1}{x^2+t^2+1}\Big)^{3/2}$	$\Xi_1 = (t^2+1)\partial_t + tx\partial_x$, $\Xi_2 = tx\partial_t + (x^2+1)\partial_x$, $\Xi_3 = -x\partial_t + t\partial_x$

Representatives of classes of functions for which (1) has a three-dimensional symmetry group (here A is an arbitrary constant).

Theorem 3. *The symmetry algebra of Eq.* (1) *preserving conditions* (3) *and* (4) *is finite-dimensional for any function* $\mathcal{F}(t,x,c)$. *The finite-dimensional nontrivial symmetry algebras have equations with functions* $\mathcal{F}(t,x,c)$ *belonging to the classes whose representatives are given in Tables* 1–3 (*up to transformations from the equivalence group* (14)). *In each case* 1–4 *shown in the tables, it is assumed that the function* $\mathcal{F}$ *does not belong to the preceding class. For other functions* $\mathcal{F}(t,x,c)$, *Eq.* (1) *does not have nontrivial symmetries preserving conditions* (3) *and* (4).

Table 3

Case	$\mathcal{F}(t,x,c)$	Basis of the symmetry algebra
3.1	$\mathcal{F}=\mathcal{F}(c)$	$\Xi_1=\partial_t,\quad \Xi_2=\partial_x$
3.2	$\mathcal{F}=\dfrac{T(c)}{t}$	$\Xi_1=\partial_x,\quad \Xi_2=t\partial_t+x\partial_x$
4	$\mathcal{F}=\mathcal{F}(t,c)$ $\mathcal{F}=\mathcal{F}(x,c)$	$\Xi_1=\partial_x$ $\Xi_1=\partial_t$

Representatives of classes of functions for which (1) has two- and one-dimensional symmetry groups (here $\mathcal{F}$ and T are arbitrary functions).

4. Proof of the theorem

A direct substitution in expression (13) verifies that all functions $\mathcal{F}(t,x,c)$ listed in the theorem correspond to the indicated symmetry algebras. Therefore, we must in fact show that Eqs. (1) with the corresponding functions do not have wider symmetry algebras than those presented in the theorem.

We first show that the first statement of the theorem holds: a symmetry algebra of Eq. (1) that preserves (3) and (4) cannot be infinite-dimensional (even more, cannot have a dimension greater than eight) for any function $\mathcal{F}(t,x,c)$.

We further use the Lie–Olver classical result, i.e., the classification of Lie algebras of groups of transformations of a two-dimensional space. Lie obtained this result in [6] (for a complex space) and then refined it to the real case in [7]. In fact, we need only a simple consequence of this result: all these Lie algebras except one have two-dimensional subalgebras. The exception is the algebra with the structure $\mathfrak{so}(3)$, which we consider separately.

In what follows, we mainly focus on equations admitting two-dimensional symmetry groups, from which we can distinguish the separate families given in Tables 1 and 2.

4.1. Finite dimensionality of the symmetry algebra. We assume that Eq. (1) with a certain function $\mathcal{F}(t,x,c)$ has a symmetry algebra of dimension at least seven. We can then write at least seven equations of form (13) for each operator $\Xi^i=\tau^i\partial_t+\xi^i\partial_x$, $i=1,\dots,7$. We can regard the obtained system of equations as a linear homogeneous algebraic system for $\mathcal{F}_t$, $\mathcal{F}_x$, $\mathcal{F}_c$, $c\mathcal{F}_c$, $c^2\mathcal{F}_c-3c\mathcal{F}$, $\mathcal{F}$ and the unity. Because this system has a nontrivial solution, its determinant is zero:

$$\begin{vmatrix} \tau^1 & \xi^1 & \xi^1_t & \xi^1_x-\tau^1_t & -\tau^1_x & \xi^1_x & -(\xi^1_{tt}+c(2\xi^1_{tx}-\tau^1_{tt})+c^2(\xi^1_{xx}-2\tau^1_{tx})-c^3\tau^1_{xx}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau^7 & \xi^7 & \xi^7_t & \xi^7_x-\tau^7_t & -\tau^7_x & \xi^7_x & -(\xi^7_{tt}+c(2\xi^7_{tx}-\tau^7_{tt})+c^2(\xi^7_{xx}-2\tau^7_{tx})-c^3\tau^7_{xx}) \end{vmatrix}=0.$$

Because τ^i and ξ^i are independent of c, this condition implies four different determinants each of which is zero. We rewrite the condition that the columns in each of these determinants are linearly dependent as a differential equation that must be satisfied by all pairs (τ^i,ξ^i) (the last component ensures that the solution of this system is nontrivial, and we can hence state that the last column can be expressed in terms of the others). We obtain

$$\xi_{tt}=L^1_{11}(\tau,\xi),\qquad 2\xi_{tx}-\tau_{tt}=L^2_{11}(\tau,\xi),\qquad \xi_{xx}-2\tau_{tx}=L^3_{11}(\tau,\xi),\qquad \tau_{xx}=L^4_{11}(\tau,\xi),\tag{15}$$

where $L^k_{ij}(\tau,\xi)$ denote linear differential operators (generally with variable coefficients) of the order i with respect to derivatives of τ and of the order j with respect to derivatives of ξ, and these differential operators are indexed by k.

The obtained system of differential equations has only a finite-dimensional space of solutions. This can easily be verified as follows. Using the matching conditions for the derivatives of the function ξ in the first three equations, we obtain the two relations

$$\tau_{ttt} = L^5_{21}(\tau, \xi), \qquad \tau_{ttx} = L^6_{21}(\tau, \xi).$$

Supplementing them with the relations

$$\tau_{txx} = L^7_{21}(\tau, \xi), \qquad \tau_{xxx} = L^8_{21}(\tau, \xi),$$

which are obtained by differentiating the last equation in (15) and the first three equations in (15), we obtain a linear system for the functions τ and ξ in normal form such that the dimension of its solution space does not exceed the number of the initial conditions for τ and ξ and their derivatives (derivatives up to the second order with respect to τ and up to the first order with respect to ξ). The total number of such initial conditions is nine, but there are in fact eight of them because the last equation in (15) implies that the initial condition for τ_{xx} is uniquely determined by the others. Therefore, the dimension of the symmetry algebra considered here can be at most eight.

As previously noted, all finite groups of the plane transformations were classified in [6]. The refinement of this classification in [7] showed that all finite real Lie algebras of the groups of transformations of the plane $\mathbb{R}^2$ either are one-dimensional or have a two-dimensional subalgebra or are equivalent to the rotation algebra $\mathfrak{so}(3, \mathbb{R})$. The functions $\mathcal{F}(t, x, c)$ associated with two-dimensional and one-dimensional algebras are considered further below, and in the next section, we determine equations of form (1) associated with the algebra with the structure $\mathfrak{so}(3, \mathbb{R})$.

4.2. Equation admitting the symmetry algebra $\mathfrak{so}(3, \mathbb{R})$. It is convenient to use the $\mathfrak{so}(3)$ realization

$$\Xi_1 = -x\partial_t + t\partial_x, \qquad \Xi_2 = tx\partial_t + (x^2 + 1)\partial_x, \qquad \Xi_3 = (t^2 + 1)\partial_t + tx\partial_x. \tag{16}$$

An advantage of this algebra is that Eq. (13) is homogeneous for it.

We now verify which functions $\mathcal{F}(t, x, c)$ have this algebra. Substituting the operators Ξ_1, Ξ_2, and Ξ_3 in (13), we obtain the system of equations

$$x\mathcal{F}_t - t\mathcal{F}_x - \mathcal{F}_c(1 + c^2) + 3c\mathcal{F} = 0,$$
$$tx\mathcal{F}_t + (1 + x^2)\mathcal{F}_x + c\mathcal{F}_c(x - tc) + 3tc\mathcal{F} = 0,$$
$$(1 + t^2)\mathcal{F}_t + tx\mathcal{F}_x + \mathcal{F}_c(x - tc) + 3t\mathcal{F} = 0.$$

We construct a linear combination to reduce this system to the form

$$x\mathcal{F}_t - t\mathcal{F}_x - \mathcal{F}_c(1 + c^2) + 3c\mathcal{F} = 0,$$
$$(1 + t^2 + x^2)\mathcal{F}_x + \mathcal{F}_c(t + cx) = 0,$$
$$(1 + t^2 + x^2)\mathcal{F}_t + c(1 + t^2 + x^2)\mathcal{F}_x + 3\mathcal{F}(t + cx) = 0.$$

It follows from the last equation that $\mathcal{F} = \Phi(x - ct, c)(1 + x^2 + t^2)^{-3/2}$, and substituting this expression in the first equation yields

$$ac\Phi_a + (1 + c^2)\Phi_c - 3c\Phi = 0,$$

where a denotes the first argument of the function Φ. We hence have

$$\Phi(a,c) = (1+c^2)^{3/2}\Psi\left(\frac{a^2}{1+c^2}\right) \quad\Longrightarrow\quad \mathcal{F} = \frac{(1+c^2)^{3/2}\Psi(\frac{(x-ct)^2}{1+c^2})}{(1+x^2+t^2)^{3/2}}.$$

Finally, substituting the obtained expression in the second equation in the system, we already obtain the ordinary differential equation $2\Psi'(b)(1+b) - 3\Psi = 0$, which obviously implies $\Psi = N(1+b)^{3/2}$ and hence the function

$$\mathcal{F} = N\frac{((x-ct)^2+1+c^2)^{3/2}}{(1+x^2+t^2)^{3/2}},$$

given in the statement in the theorem.

Conversely, the substituting such a function $\mathcal{F}$ in expression (13) immediately implies that the constant term is zero in this equation because of its irrational dependence on c, i.e.,

$$\xi_{tt} = 2\xi_{tx} - \tau_{tt} = \xi_{xx} - 2\tau_{tx} = \tau_{xx} = 0,$$

whence $\tau = At^2 + Btx + \ldots$ and $\xi = Atx + Bx^2 + \ldots$, where the ellipsis denotes linear functions. Substituting τ and ξ in (13) yields the abovementioned algebra. We note that the last reasoning is unnecessary because it follows from the results in [6], [7] that the only finite-dimensional algebra of transformations of $\mathbb{R}^2$ containing $\mathfrak{so}(3)$ is the algebra $\mathfrak{so}(3)$ itself.

4.3. Equations with two-dimensional commutative symmetry algebras. As previously noted, any finite-dimensional symmetry algebra except $\mathfrak{so}(3)$ contains a two-dimensional subalgebra. Therefore, we first determine all equations admitting a two-dimensional symmetry algebra and then seek equations with algebras of greater dimensions among them.

Choosing an appropriate basis, we can reduce any two-dimensional algebra $\Xi = A\Xi_1 + B\Xi_2$ to one of two canonical cases: $[\Xi_1, \Xi_2] = 0$ (commutative algebra) and $[\Xi_1, \Xi_2] = \Xi_2$ (noncommutative algebra). We further consider each of these two cases.

We first assume that the algebra is commutative. Without loss of generality, we can then assume that it has the form $\Xi = A\partial_x + B\partial_t$. Indeed, one of the operators can be reduced to the form $\Xi_1 = \partial_x$ by a change of variables in the equivalence group, and the corresponding change $\bar{t} = \varphi(t,x)$, $\bar{x} = \psi(t,x)$ is then determined as a solution of the system of equations $\Xi_1\varphi = 0$, $\Xi_2\psi = 1$. The commutation conditions for the operator $\Xi_2 = \tau_2\partial_t + \xi_2\partial_x$ then imply that $\tau_2 = \tau_2(t)$ and $\xi_2 = \xi_2(t)$. In this case, we can find a change of variables in the equivalence group taking (τ_2, ξ_2) to $(1, 0)$ and preserving the first vector field,

$$\phi(t) = \int \frac{1}{\tau_2(t)}\,dt, \qquad \psi(t,x) = x - \int \frac{\xi_2(t)}{\tau_2(t)}\,dt.$$

Hence, we can immediately assume that we have an algebra with the basis ∂_t, ∂_x. Substituting the basis components in Eq. (13), we obtain $\mathcal{F} = \mathcal{F}(c)$. Obviously, for any such function $\mathcal{F}$, the corresponding symmetry group contains at least the two-dimensional algebra chosen above. In the set of these functions, we distinguish the functions whose algebra has at least two dimensions.

4.4. Large symmetry algebras containing a two-dimensional commutative algebra. We substitute the function $\mathcal{F}(c)$ in Eq. (13) and simplify the result as

$$-\mathcal{F}_c(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) + \mathcal{F}(\xi_x - 2\tau_t - 3c\tau_x) + \\ + \xi_{tt} + c(2\xi_{tx} - \tau_{tt}) + c^2(\xi_{xx} - 2\tau_{tx}) - c^3\tau_{xx} = 0. \qquad (17)$$

Further using the fact that the function $\mathcal{F}$ is independent of t and x to fix some values of these variables in (17), we can hence solve the obtained ordinary differential equation, obtain the ansatz for $\mathcal{F}$, substitute this function back in Eq. (17), and obtain a system of differential equations for τ and ξ. As a result, there are rather many branchings of the versions leading to the same functions in many cases. Therefore, to reduce the reasoning, we first consider some function classes distinguished in advance and then show that the symmetry algebra cannot have dimension greater than two for the functions that do not belong to these classes.

We note several changes of variables (which are useful below) in the equivalence group of Eqs. (1), (3), and (4) that permit "simplifying" $\mathcal{F}(t,x,c)$ (the words "change of variables takes $\mathcal{F}_1$ to $\mathcal{F}_2$" always mean that the change takes Eq. (1) with $\mathcal{F}_1$ to Eq. (1) with $\mathcal{F}_2$):

1. The change $\bar{t}=t$, $\bar{x}=x+at$, $\bar{c}=c+a$ takes $\mathcal{F}(t,x,c)$ to $\overline{\mathcal{F}}(\bar{t},\bar{x}-a\bar{t},\bar{c}-a)$.

2. The change $\bar{t}=t$, $\bar{x}=x-at^2/2$, $\bar{c}=c-at$ takes $\mathcal{F}=a$ to $\overline{\mathcal{F}}=0$.

3. The change $\bar{t}=x$, $\bar{x}=t$, $\bar{c}=1/c$ takes $\mathcal{F}(t,x,c)$ to $\overline{\mathcal{F}}(\bar{t},\bar{x},\bar{c})=-\bar{c}^3\mathcal{F}(\bar{x},\bar{t},1/\bar{c})$.

4. The change $\bar{t}=x$, $\bar{x}=e^{at}$, $\bar{c}=ae^{at}/c$ takes $\mathcal{F}(c)=ac+bc^2$ to $\overline{\mathcal{F}}(\bar{c})=-b\bar{c}$.

5. The change $\bar{t}=\sin(\mu t)e^{Dx}$, $\bar{x}=\cos(\mu t)e^{Dx}$, $\bar{c}=\frac{D\cos(\mu t)c-\mu\sin(\mu t)}{D\sin(\mu t)c+\mu\cos(\mu t)}$, where $\mu=\sqrt{DG}$, takes $\mathcal{F}=Dc^2+G$ with $DG>0$ to $\overline{\mathcal{F}}=0$.

Hence, we separately consider the symmetry algebras corresponding to Eq. (1) with $\mathcal{F}(c)$ in one of the six classes listed below in Lemmas 1–6. We recall that all further results are formulated up to transformations in the equivalence group.

Lemma 1. *In the set of functions of the form $\mathcal{F}(c)=Qc^5+Ac^4+Bc^3+Dc^2+Ec+G$, the following functions have a symmetry algebra of dimension at least two*:

- $\mathcal{F}(c)=Bc^3+Dc^2+Ec+G$: *Equations with any functions $\mathcal{F}$ of this form are equivalent to the equation with $\mathcal{F}\equiv 0$, whose algebra is eight-dimensional* (*see Table* 1).

- $\mathcal{F}(c)=Qc^5$: *The algebra is three-dimensional in this case, $\Xi_1=\partial_t$, $\Xi_2=\partial_x$, $\Xi_3=t\partial_t+(3/4)x\partial_x$.*

- $\mathcal{F}(c)=Ac^4$: *The algebra is three-dimensional in this case, $\Xi_1=\partial_t$, $\Xi_2=\partial_x$, $\Xi_3=t\partial_t+(2/3)x\partial_x$.*

Proof. We substitute a function $\mathcal{F}$ of the prescribed form in Eq. (17):

$$-(5Qc^4+4Ac^3+3Bc^2+2Dc+E)(\xi_t+c(\xi_x-\tau_t)-c^2\tau_x)+\xi_{tt}+c(2\xi_{tx}-\tau_{tt})+$$
$$+c^2(\xi_{xx}-2\tau_{tx})-c^3\tau_{xx}+(Qc^5+Ac^4+Bc^3+Dc^2+Ec+G)(\xi_x-2\tau_t-3c\tau_x)=0.$$

We obtain a polynomial in c identically equal to zero, and all of its coefficients of powers of c are hence zero,

$$\begin{gathered}2Q\tau_x=0,\qquad 4Q\xi_x-3Q\tau_t-A\tau_x=0,\qquad 5Q\xi_t+3A\xi_x-2A\tau_t=0,\\ 4A\xi_t-B\tau_t+2B\xi_x+D\tau_x+\tau_{xx}=0,\qquad 3B\xi_t+D\xi_x+2E\tau_x-\xi_{xx}+2\tau_{tx}=0,\\ 2D\xi_t+E\tau_t+3G\tau_x-2\xi_{tx}+\tau_{tt}=0,\qquad E\xi_t-G\xi_x+2G\tau_t-\xi_{tt}=0.\end{gathered}\tag{18}$$

1. If $Q \neq 0$, then $\mathcal{F}$ is a polynomial of degree five and always has a real root, denoted here by $c = M$. We change the variables $\bar{t} = t$, $\bar{x} = x - Mt$, and $\bar{c} = c - M$, which takes $\mathcal{F}$ to a function of the form $\mathcal{F} = Qc^5 + Ac^4 + Bc^3 + Dc^2 + Ec$ (with new coefficients). System (18) then becomes

$$\tau_x = 0, \qquad 4Q\xi_x - 3Q\tau_t - A\tau_x = 0, \qquad 5Q\xi_t + 3A\xi_x - 2A\tau_t = 0,$$

$$4A\xi_t - B\tau_t + 2B\xi_x + D\tau_x + \tau_{xx} = 0, \qquad 3B\xi_t + D\xi_x + 2E\tau_x - \xi_{xx} + 2\tau_{tx} = 0,$$

$$2D\xi_t + E\tau_t - 2\xi_{tx} + \tau_{tt} = 0, \qquad E\xi_t - \xi_{tt} = 0.$$

The first relation implies $\tau = \tau(t)$, and then

$$\xi = \frac{3}{4}\tau_t x + n, \qquad 15Q\tau_{tt}x + 20Qn_t + A\tau_t = 0 \quad \Longrightarrow \quad \tau_{tt} = 0, \qquad n = -\frac{Akt}{20Q} + n_2.$$

The remaining four equations are $k(5QB - 2A^2) = 0$, $k(5QD - AB) = 0$, $k(10QE - AD) = 0$, and $AEk = 0$.

The case $k = 0$ leads to a two-dimensional algebra. If $k \neq 0$, then

$$B = \frac{2A^2}{5Q}, \qquad D = \frac{2A^3}{25Q^2}, \qquad E = \frac{2A^4}{250Q^3}.$$

The equation $AEk = 0$ implies $A = 0$, i.e., $\tau = kt + m$, $\xi = 3kx/4 + n_2$ is a three-dimensional algebra for $\mathcal{F} = Qc^5$.

2. Let $Q = 0$ and $A \neq 0$. Then (18) implies the equation $\tau_x = 0$ and the system

$$A(3\xi_x - 2\tau_t) = 0, \qquad 4A\xi_t - B\tau_t + 2B\xi_x = 0, \qquad 3B\xi_t + D\xi_x - \xi_{xx} = 0,$$

$$2D\xi_t + E\tau_t - 2\xi_{tx} + \tau_{tt} = 0, \qquad E\xi_t - G\xi_x + 2G\tau_t - \xi_{tt} = 0.$$

It follows from the first equation in this system that $\xi = 2\tau_t x/3 + n$. Substituting this expression in the next equation, we obtain $8A\tau_{tt}x + 12An_t + B\tau_t = 0$, which implies $\tau = kt + m$ and $n = -Bkt/12A + n_2$. The other equations reduce to the algebraic equations $k(8AD - 3B^2) = k(6AE - BD) = 0$ and $k(16AG - BE) = 0$.

If $k = 0$, then the algebra is two-dimensional. If $k \neq 0$, then

$$D = \frac{3B^2}{8A}, \qquad E = \frac{B^3}{16A^2}, \qquad G = \frac{B^4}{256A^3}.$$

For the function $\mathcal{F} = A(c + B/4A)^4$, we hence find that $\tau = kt + m$, $\xi = k(2x/3 - Bt/12A) + n_2$ is a three-dimensional algebra. A change of variables of form 1 reduces this function to a function of the form $\mathcal{F} = Ac^4$ for which $\tau = kt + m$, $\xi = 2kx/3 + n_2$ is a three-dimensional symmetry algebra.

3. We now consider the case $Q = A = 0$, $B \neq 0$. A change of variables of form 1 reduces the polynomial to the case $G = 0$. We use a change of variables of form 3 to reduce $\mathcal{F} = Bc^3 + Dc^2 + Ec$ to the form $\mathcal{F} = B + Dc + Ec^2$, i.e., to cases 4–6 considered below.

4. Let $Q = A = B = 0$ and $D \neq 0$. The change of variables $\bar{t} = t$, $\bar{x} = x + Et/2D$, $\bar{c} = c + E/2D$ reduces $\mathcal{F}$ to the form $\mathcal{F} = Dc^2 + G$. Further, we have the following cases:

a. Let $DG < 0$. Then we can shift the variable c to reduce $\mathcal{F}$ to the form $\mathcal{F} = Ec + Dc^2$. We use a change of form 4 and obtain $\mathcal{F} = -Dc$. We then apply change 4 repeatedly and obtain $\mathcal{F} = 0$.

b. Let $G = 0$. Then changes of forms 3 and 4 reduce this case to the case $\mathcal{F} = 0$.

c. Let $DG > 0$. Then a change of form 5 reduces this case to the case $\mathcal{F} = 0$.

5. Let $Q = A = B = D = 0$ and $E \neq 0$. With regard to a change of variables of form 1, we can assume that $\mathcal{F} = Ec$. By a change of variables of form 4, a function of this form reduces to $\mathcal{F} = 0$.

6. Let $Q = A = B = D = E = 0$. Then $\mathcal{F} = G$, and a change of variables of form 2 reduces this function to the case $\mathcal{F} = 0$.

7. Let $\mathcal{F} \equiv 0$. System (18) is simplified to $\tau_{xx} = 0$, $\xi_{xx} = 2\tau_{tx}$, $2\xi_{tx} = \tau_{tt}$, and $\xi_{tt} = 0$ in this case. The solution of this system has the form $\tau = k_1tx + k_2x + m_1t^2 + m_2t + m_3$, $\xi = k_1x^2 + m_1tx + n_1x + l_1t + l_2$ and determines the algebra given in Table 1.

The lemma is proved.

Lemma 2. *For functions of the form*

$$\mathcal{F}(c) = Ac^4 + Bc^3 + Dc^2 + Ec + G + \frac{K}{c+M}, \quad K \neq 0, \tag{19}$$

the symmetry algebra is two-dimensional (*except the cases equivalent to those listed in Lemma* 1).

Proof. We first change the variables $\bar{t} = t$, $\bar{x} = x + Mt$, and $\bar{c} = c + M$. This change belongs to the equivalence group and takes $\mathcal{F}$ of form (19) to the form

$$\mathcal{F}(c) = Ac^4 + Bc^3 + Dc^2 + Ec + G + \frac{K}{c}. \tag{20}$$

Substituting this function in (17) and multiplying by c^2, we obtain

$$\begin{aligned}-(4Ac^3 + 3Bc^2 + 2Dc + E)(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x)c^2 + K(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) + \\ + Kc(\xi_x - 2\tau_t - 3c\tau_x) + (\xi_{tt} + c(2\xi_{tx} - \tau_{tt}) + c^2(\xi_{xx} - 2\tau_{tx}) - c^3\tau_{xx})c^2 + \\ + (Ac^4 + Bc^3 + Dc^2 + Ec + G)(\xi_x - 2\tau_t - 3c\tau_x)c^2 = 0.\end{aligned} \tag{21}$$

This polynomial is zero for all c. In particular, we obtain $K\xi_t = 0$ for $c = 0$. The derivative of the polynomial at $c = 0$ is also zero, $K(2\xi_x - 3\tau_t) = 0$. Because $K \neq 0$, we have $\xi_t = 0$ and $2\xi_x = 3\tau_t$. Differentiating the second relation with respect to t, we obtain $\tau_{tt} = 0$, i.e., $\tau = k(x) + tm(x)$, which implies $\xi = (3/2)\int m(x)\,dx + l$. Substituting these formulas in (21), we obtain

$$\begin{aligned}\frac{m'}{2}c^2 + (5Ac^4 + 4Bc^3 + 3Dc^2 + 2Ec + G)\frac{m}{2} + (k'' + tm'')c^3 + \\ + (-Ac^5 + Dc^3 + 2Ec^2 + 3Gc + 4K)(k' + tm') = 0,\end{aligned}$$

and equate the coefficients of powers of c to zero. The obtained system consists of six equations

$$A(k' + tm') = 0, \qquad Am = 0, \qquad 2Bm + (k'' + tm'') + D(k' + tm') = 0,$$

$$3Dm + m' + 4E(k' + tm') = 0, \qquad Em + 3G(k' + tm') = 0, \qquad Gm + 8K(k' + tm') = 0.$$

If $A \neq 0$, then $m = 0$, $k = \text{const}$, and the algebra is two-dimensional for any function of form (20). We hence assume that $A = 0$.

The last of the six equations in this system is a polynomial in t identically equal to zero, and (because $K \neq 0$) its coefficients are hence zero, i.e., $m = \text{const}$ and $k = -(Gm/8K)x + k_1$. We substitute the obtained formulas for k and m in the remaining equations of the system:

$$2Bm = \frac{DGm}{8K}, \qquad 3Dm = \frac{4EGm}{8K}, \qquad Em = \frac{3G^2m}{8K}, \qquad A\left(\frac{5m}{2} + \frac{Gm}{8K}\right) = 0.$$

If $m = 0$, then all these relations hold for any function of the form $\mathcal{F}(c) = Bc^3 + Dc^2 + Ec + G + K/c$, but the algebra is two-dimensional. But if $m \neq 0$, then

$$E = \frac{3G^2}{8K}, \qquad D = \frac{EG}{6K} = \frac{G^3}{16K^2}, \qquad B = \frac{DG}{16K} = \frac{G^4}{256K^3}.$$

We substitute the obtained coefficients in $\mathcal{F}(c)$ and simplify the results. We see that the symmetry algebra is three-dimensional and has the form $\tau = -Mmx + 2mt + k_1$, $\xi = 3mx + l$ for

$$\mathcal{F}(c) = \frac{K}{c}\left(1 + \frac{Gc}{4K}\right)^4 = \frac{K}{c}(1 + Mc)^4.$$

The obtained function $\mathcal{F}$ is indeed equivalent to the polynomial function considered in Lemma 1. To verify this, we use the abovementioned change of variables $\bar{t} = x$, $\bar{x} = t$, $\bar{c} = 1/c$ of form 3 and reduce $\mathcal{F}$ to the form $\mathcal{F} = K(c + M)^4$. Using the change of variables $\bar{t} = t$, $\bar{x} = x + Mt$, $\bar{c} = c + M$ of form 1, we then transform $\mathcal{F}$ to the function $\mathcal{F} = Kc^4$. The lemma is proved.

Lemma 3. *For functions of the form $\mathcal{F}(c) = (c - a)^5/c^2$ with $a \neq 0$, the symmetry algebra is three-dimensional, and its basis is $\Xi_1 = \partial_t$, $\Xi_2 = \partial_x$, $\Xi_3 = (3at + x)\partial_t + 4ax\partial_x$. An equation with this function is equivalent to the equation with the polynomial function $\mathcal{F}(c)$.*

Proof. Substituting $\mathcal{F}(c)$ of the prescribed form in Eq. (17) and multiplying it by c^3, we obtain

$$-5c(c-a)^4(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) + 2(c-a)^5(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) + \\ + (c-a)^5(\xi_x - 2\tau_t - 3c\tau_x)c + (\xi_{tt} + c(2\xi_{tx} - \tau_{tt}) + c^2(\xi_{xx} - 2\tau_{tx}) - c^3\tau_{xx})c^3 = 0. \quad (22)$$

For $c = 0$, we have $a^5\xi_t = 0$. Because $a \neq 0$ by the conditions of the lemma, we derive $\xi = \xi(x)$. We differentiate Eq. (22) with respect to c and again substitute $c = 0$, which implies $3\xi_x = 4\tau_t$ whence $\tau = 3\xi_x t/4 + k(x)$. Substituting the obtained ξ and τ in (22) and simplifying the results, we obtain

$$5(c-a)^4\left(c\frac{\xi_x}{4} - c^2\frac{3\xi_{xx}t}{4} - c^2k_x\right) - 2(c-a)^5\left(\frac{\xi_x}{4} - c\frac{3\xi_{xx}t}{4} - ck_x\right) + \\ + (c-a)^5\left(\frac{\xi_x}{2} + c\frac{9\xi_{xx}t}{4} + 3ck_x\right) + \left(c^2\frac{\xi_{xx}}{2} + c^3\frac{3\xi_{xxx}t}{4} + c^3k_{xx}\right)c^2 = 0.$$

This is a polynomial in t, and its coefficients are equal to zero,

$$(c-a)^4(3c+2a)\left(\frac{\xi_x}{4} - ck_x\right) + (c-a)^5\left(\frac{\xi_x}{2} + 3ck_x\right) + \left(c^2\frac{\xi_{xx}}{2} + c^3k_{xx}\right)c^2 = 0,$$

$$5a(c-a)^4\xi_{xx} - c^4\xi_{xxx} = 0.$$

We obtain $\xi = nx + l$ from the last equation for $c = 0$. We then have $k = k_1x + k_2$ and $n = 4ak_1$ from the first equation for $c = a$. As a result, the symmetry algebra $\tau = k_1(3at + x) + k_2$, $\xi = 4k_1ax + l$ is three-dimensional up to changes of variables in the equivalence group.

It remains to note that a change of form 3 takes our function to $\mathcal{F}(c) = (ac - 1)^5$, and a subsequent change of form 1 takes it to a polynomial function of degree five. The lemma is proved.

Lemma 4. *For functions of the form*

$$\mathcal{F}(c) = A\exp\int \frac{3c+a}{c^2+bc+d}\,dc, \quad A \neq 0,$$

in the case where $\mathcal{F}(c)$ is not a function given in Lemmas 1–3, the symmetry algebra is three-dimensional, and its basis is $\Xi_1 = \partial_t$, $\Xi_2 = \partial_x$, $\Xi_3 = (x+(a-b)t)\partial_t + ((a-2b)x - dt)\partial_x$.

Proof. Using $(c^2+bc+d)\mathcal{F}_c = \mathcal{F}(3c+a)$ and substituting this relation in Eq. (17), we obtain

$$\mathcal{F}(-(a+3c)(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) + (\xi_x - 2\tau_t - 3c\tau_x)(d+bc+c^2)) +$$
$$+ (\xi_{tt} + c(2\xi_{tx} - \tau_{tt}) + c^2(\xi_{xx} - 2\tau_{tx}) - c^3\tau_{xx})(d+bc+c^2) = 0.$$

The following two cases are possible.

1. If the multiplier of $\mathcal{F}$ is nonzero, then $\mathcal{F}$ is a ratio of polynomials: the polynomial in the numerator is of degree five at most, and the polynomial in the denominator is of degree two at most. This is possible only if the denominator in the integrand for the function $\mathcal{F}$ has two real and distinct roots (otherwise, $\mathcal{F}$ is a transcendental function). We obtain $\mathcal{F}$ under the assumption that $c^2+bc+d = (c-m)(c-n)$:

$$\mathcal{F} = A(c-n)^{\frac{a+3n}{n-m}}(c-m)^{-\frac{(a+3m)}{n-m}}.$$

Because the roots are distinct, the degrees of the polynomials must be integer. The sum of the degrees is equal to three, and hence only three variants are possible: the degree of one of the polynomials is equal to zero and the degree of the other is equal to three (then $b = (2a/3)+1$, $d = a(a+3)/9$ or $b = (2a/3)-1$, $d = a(a-3)/9$); the degree of one of the polynomials is -1 and the degree of the other is four (then $b = -a-5n$, $d = n(a+4n)$, $n \neq -a/3$); the degree of one of the polynomials is -2 and the degree of the other is five (then $b = -(a+7n)/2$, $d = n(a+5n)/2$, $n \neq -a/3$). An algebra with such relations between the degrees was already studied in Lemmas 1–3 (the last case reduces to the case described in Lemma 3 by the change $\bar{t} = t$, $\bar{x} = x - nt$, $\bar{c} = c - n$).

2. We assume that the coefficient of $\mathcal{F}$ is zero, i.e.,

$$(3c+a)(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) = (\xi_x - 2\tau_t - 3c\tau_x)(c^2+bc+d). \tag{23}$$

Expanding and equating the coefficients of like powers of c to zero, we obtain the equations

$$2\xi_x - (a-3b)\tau_x = \tau_t, \qquad 3\xi_t + (a-b)\xi_x = (a-2b)\tau_t - 3d\tau_x, \qquad a\xi_t = d\xi_x - 2d\tau_t.$$

We express the derivatives of ξ in terms of the derivatives of τ as

$$\xi_x = \frac{(a-3b)\tau_x + \tau_t}{2}, \qquad \xi_t = \frac{a-3b}{6}((b-a)\tau_x + \tau_t) - d\tau_x, \qquad a\xi_t = \frac{d(a-3b)\tau_x - 3d\tau_t}{2}.$$

We obtain $a\xi_t$ from the second and third relations and equate the obtained expressions to each other. Reducing to a common denominator and grouping, we obtain the equation $(a(a-3b)+9d)((b-a)\tau_x+\tau_t) = 0$.

Because $a(a-3b)+9d \neq 0$ (otherwise, $\mathcal{F}$ is a polynomial), we obtain $(b-a)\tau_x + \tau_t = 0$, i.e., $\tau = \tau(x+(a-b)t)$. In this case, the relations for the derivatives become $\xi_x = (a-2b)\tau_x$, $\xi_t = -d\tau_x$.

It follows from the consistency condition that the equation

$$((a-2b)(a-b)+d)\tau_{xx} = 0$$

must be satisfied. If the coefficient of τ_{xx} is zero, then the denominator in the integrand, which determines $\mathcal{F}$ in the condition of the lemma, has two distinct roots, and $\mathcal{F}(c)$ is a polynomial. But if $(a-2b)(a-b)+d \neq 0$, then $\tau_{xx} = 0$. In this case, $\tau_{tx} = (a-b)\tau_{xx} = 0$ and $\tau_{tt} = (a-b)^2\tau_{xx} = 0$. These relations imply $\tau = m(x+(a-b)t)+k$, $\xi = m((a-2b)x-dt)+l$.

We return to formula (17). If $m = 0$, then we obtain the two-dimensional translation algebra for all $\mathcal{F}$. If $m \neq 0$, then the equation becomes the trivial identity $0 = 0$, and the algebra $\tau = m(x+(a-b)t)+k$, $\xi = m((a-2b)x-dt)+l$ turns out to be three-dimensional. The lemma is proved.

Lemma 5. *For functions of the form $\mathcal{F}(c) = Ac^a$ with $A \neq 0$ and $a \neq -1, 0, \dots, 5$, the symmetry algebra is three-dimensional, and its basis is $\Xi_1 = \partial_t$, $\Xi_2 = \partial_x$, $\Xi_3 = t\partial_t + ((a-2)(a-1)^{-1}x)\partial_x$.*

Proof. Substituting $\mathcal{F}(c)$ of the given form in Eq. (17), we obtain

$$aAc^{a-1}(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) - Ac^a(\xi_x - 2\tau_t - 3c\tau_x) - \xi_{tt} - \\ - c(2\xi_{tx} - \tau_{tt}) - c^2(\xi_{xx} - 2\tau_{tx}) + c^3\tau_{xx} = 0.$$

We have a polynomial in the variable c identically equal to zero, and all its coefficients are therefore zero, including the coefficients of c^{a+1} and c^a, and as a result, $\tau_x = 0$ and $(a-2)\tau_t = (a-1)\xi_x$. This implies $\tau = \tau(t)$ and $\xi = (a-2)(a-1)^{-1}\tau_t x + n(t)$.

We again substitute the obtained formulas for τ and ξ in (17) and equate the coefficients of the remaining powers of c to zero for $a \neq -1, 0, \dots, 5$, which implies $\tau = mt+k$ and $\xi = m(a-2)(a-1)^{-1}x+n$. The obtained algebra is three-dimensional for any function $\mathcal{F}(c) = Ac^a$ with $a \neq -1, \dots, 5$. We note that the cases $a = 0, 1, \dots, 5$ were already considered in Lemma 1 and the case $a = -1$ was considered in Lemma 2. The lemma is proved.

Lemma 6. *For functions of the form $\mathcal{F}(c) = Ae^{ac}$ with $A \neq 0$ and $a \neq 0$, the symmetry algebra is three-dimensional, and its basis is $\Xi_1 = \partial_t$, $\Xi_2 = \partial_x$, $\Xi_3 = t\partial_t + (x - t/a)\partial_x$.*

Proof. Using the relation $\mathcal{F}_c = a\mathcal{F}$ and substituting it in Eq. (17), we obtain

$$\mathcal{F}(\xi_x - 2\tau_t - a\xi_t - c(a\xi_x - a\tau_t + 3\tau_x) + c^2a\tau_x) + \\ + \xi_{tt} + c(2\xi_{tx} - \tau_{tt}) + c^2(\xi_{xx} - 2\tau_{tx}) - c^3\tau_{xx} = 0.$$

Because the function e^{ac} cannot be represented as a ratio of polynomials, we have $A \neq 0$ and $a \neq 0$. We have the relations $\tau = \tau(t)$ and $\xi = \tau_t x + l(t)$ and the equations $\tau_t + a\xi_t = 0$, $\xi_{tt} = 0$, and $2\xi_{tx} - \tau_{tt} = 0$. From the first equation, we derive $\tau = kt+m$, $l = -kt/a + l_1$. The second and third equations are satisfied, i.e., the algebra $\tau = kt+m$, $\xi = kx - kt/a + l_1$ is three-dimensional. The lemma is proved.

4.5. Proof that the symmetry algebra is precisely two-dimensional in other cases. Because for $\mathcal{F} = \mathcal{F}(c)$, Eq. (13) is satisfied for any constant τ and ξ, the symmetry algebra contains a two-dimensional commutative algebra. To complete the consideration of the algebras containing a two-dimensional commutative subalgebra, we must show that the dimension of the symmetry algebra is precisely equal to two in the cases other than the cases considered in the lemmas in the preceding section. For this, we assume that Eq. (17) is satisfied for some nonconstant functions $\tau(t,x)$ and $\xi(t,x)$. We substitute them in the equation that we take as an equation for $\mathcal{F}$:

$$-\mathcal{F}_c A + \mathcal{F}B + D = 0. \tag{24}$$

1. We first assume that $A = 0$. We then have $\tau_x = \xi_t = 0$ and $\tau_t = \xi_x = \text{const}$, and this constant is nonzero by the assumption that at least one of the functions τ and ξ is nonconstant. But it then turns out that B is equal to this constant, $D = 0$, hence $\mathcal{F} = 0$, and the symmetry algebra was already obtained in this case in Lemma 1.

2. We now assume that $A \neq 0$. Differentiating (24) with respect to x and t, we obtain the system of three equations for $\mathcal{F}$ and $\mathcal{F}_c$

$$-\mathcal{F}_c A + \mathcal{F} B + D = 0, \qquad -\mathcal{F}_c A_x + \mathcal{F} B_x + D_x = 0, \qquad -\mathcal{F}_c A_t + \mathcal{F} B_t + D_t = 0. \tag{25}$$

The following three cases are possible: system (25) is inconsistent, all three equations are equivalent to each other, and there are at least two equations nonproportional to each other. We are not interested further in the first case, because there are no functions $\mathcal{F}(c)$ satisfying such a system. We therefore consider the remaining two cases.

2.1. Let $BA_x = B_x A$ and $BA_t = B_t A$. Then the function B is proportional to A with a coefficient depending only on c:

$$\phi(c)(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) = \xi_x - 2\tau_t - 3c\tau_x. \tag{26}$$

2.1.1. Let $\tau_x \not\equiv 0$. Then we fix the corresponding values (t, x) and obtain

$$\phi(c) = \frac{3c + a}{c^2 + bc + d},$$

where a, b, and d are constants. Substituting the obtained expression in (26),

$$(3c + a)(\xi_t + c(\xi_x - \tau_t) - c^2\tau_x) = (\xi_x - 2\tau_t - 3c\tau_x)(c^2 + bc + d),$$

expanding, equating the coefficients of like powers of c, and expressing the derivatives of ξ in terms of derivatives of τ, we obtain

$$\xi_x = \frac{(a - 3b)\tau_x + \tau_t}{2}, \qquad \xi_t = \frac{(a - 3b)}{6}((b - a)\tau_x + \tau_t) - d\tau_x, \qquad a\xi_t = \frac{d(a - 3b)\tau_x - 3d\tau_t}{2}.$$

We can determine $a\xi_t$ from the second and third relations and equate the obtained expressions to each other. Reducing to a common denominator and grouping, we obtain the equation

$$(a(a - 3b) + 9d)((b - a)\tau_x + \tau_t) = 0.$$

2.1.1a. We first assume that $a(a - 3b) + 9d = 0$. If $3b = a$ and consequently $d = 0$, then $2\xi_x = \tau_t$, $3\xi_t + 2b\xi_x = b\tau_t$, and $a\xi_t = 0$. Subtracting the first equation multiplied by b from the second, we obtain $\xi_t = 0$, which implies the third equation. As a result, we have $\xi = \xi(x)$ and $\tau = 2\xi_x t + C(x)$. Substituting these formulas in Eq. (17), we obtain

$$c\mathcal{F}_c(\xi_x + 2c\xi_{xx}t + cC_x) - 3\mathcal{F}(\xi_x + 2c\xi_{xx}t + cC_x) - 3c^2\xi_{xx} - 2c^3\xi_{xxx}t - c^3C_{xx} = 0.$$

The left-hand side contains a polynomial in t. We equate its coefficients to zero,

$$c\mathcal{F}_c\xi_{xx} - 3\mathcal{F}\xi_{xx} - c^2\xi_{xxx} = 0, \qquad c\mathcal{F}_c(\xi_x + cC_x) - 3\mathcal{F}(\xi_x + cC_x) - 3c^2\xi_{xx} - c^3C_{xx} = 0.$$

If the coefficients of $\mathcal{F}_c$ are zero in both equations, then $\xi = \text{const}$, $C = \text{const}$, and the algebra is two-dimensional. If at least one of them is nonzero, then the solution of the equation has the form $\mathcal{F} = Mc^3 - Pc^2$, which was already considered in Lemma 1.

If $3b \neq a$, then we change the variables $\bar{t} = t$, $\bar{x} = x - (3b-a)t/3$, and $\bar{c} = c - (3b-a)/3$. We obtain $\phi(c) = 3/c$, while (17) with the new functions $\bar{\tau}(\bar{t},\bar{x}) = \tau(t,x)$ and $\bar{\xi}(\bar{t},\bar{x}) = \xi - (3b-a)\tau/3$ is taken to itself. Up to changes of variables, we again obtain $\mathcal{F} = Mc^3 - Pc^2$ from the equivalence algebra,

2.1.1b. We now assume that $a(a-3b)+9d \neq 0$ and $(b-a)\tau_x + \tau_t = 0$, i.e., $\tau = \tau(x+(a-b)t)$. We then have $\xi_x = (a-2b)\tau_x$ and $\xi_t = -d\tau_x$. It follows from the consistency condition that the equation $((a-2b)(a-b)+d)\tau_{xx} = 0$ must be satisfied. If the coefficient of τ_{xx} is zero, then $\xi = (a-2b)\tau(x+(a-b)t)+C$. We change the variables $k = a-b$ to simplify the expressions. Then $\tau = \tau(x+kt)$ and $\xi = (k-b)\tau(x+kt)+C$. Substituting the obtained expressions in Eq. (17), we obtain

$$\mathcal{F}_c(d+cb+c^2)\tau' - \mathcal{F}(a+3c)\tau' - (a-b+c)(d+cb+c^2)\tau'' = 0,$$

$$\mathcal{F} = \exp\biggl(\int \frac{a+3c}{(c-a+2b)(c-b+a)}dc\biggr)\biggl(M + \int Q(a-b+c)\,e^{-\int \frac{(a+3c)}{(c-a+2b)(c-b+a)}dc}\,dc\biggr).$$

We apply the change of variables $\bar{t} = t$, $\bar{x} = x + (a-b)t$, $\bar{c} = c + a - b$, which belongs to the equivalence group and takes $\mathcal{F}$ to the form $\mathcal{F}(c) = (M(c+a-b) - Q)c(c+a-b)$, i.e., $\mathcal{F}$ is a polynomial. This case was considered in Lemma 1.

If $(a-2b)(a-b)+d \neq 0$, then $\tau_{xx} = 0$, and we have the equations $\tau_{tx} = (a-b)\tau_{xx} = 0$ and $\tau_{tt} = (a-b)^2\tau_{xx} = 0$, which implies $\tau = m(x+(a-b)t)+K$, $\xi = m((a-2b)x - dt)+C$.

We now return to relation (17). If $m = 0$, then we obtain a two-dimensional translation algebra for all $\mathcal{F}$. If $m \neq 0$, then the equation is simplified to $\mathcal{F}_c(d+bc+c^2) - \mathcal{F}(a+3c) = 0$. Solving this equation, we obtain

$$\mathcal{F} = M\exp\int \frac{a+3c}{d+bc+c^2}\,dc.$$

The symmetry algebra for such a function $\mathcal{F}$ was studied in Lemma 4.

2.1.2. Let $\tau_x \equiv 0$, $\xi_t \not\equiv 0$. Then fixing the values (t,x), we see that Eq. (26) implies $\phi(c) = a/(1+bc)$ and

$$a(\xi_t + c(\xi_x - \tau_t)) = (\xi_x - 2\tau_t)(1+bc).$$

Equating the coefficients of powers of c to zero, we obtain

$$(b-a)\xi_x + (a-2b)\tau_t = 0, \qquad a\xi_t = \xi_x - 2\tau_t.$$

From the second equation, we derive $2\tau_t = \xi_x - a\xi_t$. Substituting this in the first equation, we obtain $a(\xi_x + (a-2b)\xi_t) = 0$.

2.1.2a. In the case $a = 0$, we obtain $\tau = \tau(t)$ and $\xi = 2\tau_t x + C(t)$. Substituting these relations in Eq. (17), we obtain

$$-\mathcal{F}_c(2\tau_{tt}x + C_t + c\tau_t) + 2\tau_{ttt}x + C_{tt} + 3c\tau_{tt} = 0.$$

The left-hand side contains a polynomial in x that is identically zero. This implies $\mathcal{F}_c\tau_{tt} = \tau_{ttt}$ and $\mathcal{F}_c(C_t + c\tau_t) = C_{tt} + 3c\tau_{tt}$. Because $\mathcal{F} = \mathcal{F}(c)$, we obtain $\mathcal{F} = Kc + M$ from these two equations. This case was already studied in Lemma 1.

2.1.2b. In the case $a \neq 0$, we obtain $\tau_t = (b-a)\xi'$ and $\xi = \xi(t + (2b-a)x)$. Because $\tau = \tau(t)$, we have $(b-a)(2b-a)\xi'' = 0$.

2.1.2b(1). In the case $b = a$, we obtain $\tau = C$ and $\xi = \xi(t+bx)$ and have

$$-\mathcal{F}_c(1+cb)\xi' + b\mathcal{F}\xi' + (1+cb)^2\xi'' = 0.$$

If $\xi' = 0$, then the algebra is two-dimensional. Otherwise, separation of variables in this equation gives $\xi'' = K\xi'$ and $\mathcal{F} = (1 + cb)(Kc + M + B)$, i.e., $\mathcal{F}$ is a polynomial. This case was already considered in Lemma 1.

2.1.2b(2). In the case $2b = a$, we obtain $\tau = -b\xi(t) + C$ and $\xi = \xi(t)$ and have

$$-\mathcal{F}_c(1 + cb)\xi_t + 2b\mathcal{F}\xi_t + (1 + cb)\xi_{tt} = 0.$$

If the algebra is not two-dimensional, then separation of variables gives $\xi'' = K\xi'$ as before, and $\mathcal{F}(c)$ is consequently a quadratic trinomial.

2.1.2b(3). Let $\xi'' = 0$, i.e., let $\tau = (b - a)mt + k$ and $\xi = m(t + (2b - a)x) + l$. Substituting these relations in Eq. (17), we obtain $m[\mathcal{F}_c(1 + cb) - \mathcal{F}a] = 0$. The case $m = 0$ corresponds to a two-dimensional algebra. If $m \neq 0$, then we obtain $\mathcal{F} = M(1 + cb)^{a/b}$ for $b \neq 0$ (a change of variables of form 1 can take this function to $\mathcal{F} = Mc^a$ considered in Lemma 5). For $b = 0$, the function reduces to $\mathcal{F} = Pe^{ac}$ considered in Lemma 6.

2.1.3. We assume that $\tau_x \equiv 0$, $\xi_t \equiv 0$. We use the assumption that $A \not\equiv 0$ in (24) and A is the left-hand side of Eq. (26). The condition $\xi_x - \tau_t \not\equiv 0$ must then be satisfied, and we obtain $\phi(c) = a/c$ with regard to (26). Hence, $(a - 1)\xi_x = (a - 2)\tau_t = \text{const}$.

If $a = 2$, then $\xi = l$ and $\tau = \tau(t) \neq \text{const}$. Substituting these relations in Eq. (17), we obtain $c\mathcal{F}_c\tau_t - 2\mathcal{F}\tau_t - c\tau_{tt} = 0$. Because $\tau_t \neq 0$, we have $\tau_{tt} = Q\tau_t$ and $\mathcal{F} = Dc^2 - Qc$. The symmetry algebra for such a function $\mathcal{F}$ was studied in Lemma 1.

Similarly, if $a = 1$, then $\tau = b$ and $\xi = \xi(x)$. Substituting these relations in Eq. (17), we obtain $c\mathcal{F}_c\xi_x - \mathcal{F}\xi_x - c^2\xi_{xx} = 0$ and again have $\mathcal{F} = Dc^2 - Qc$.

If $a \neq 1$ and $a \neq 2$, then we obtain an algebra of the form $\tau = (a - 1)kt + b$, $\xi = (a - 2)kx + l$ and see that $k \neq 0$ (as previously noted, $\xi_x - \tau_t \not\equiv 0$). We return to the original relation, which now becomes $k(c\mathcal{F}_c - a\mathcal{F}) = 0$. This relation obviously implies $\mathcal{F} = Mc^a$, and this case was studied in Lemmas 1 and 5.

2.2. We now assume that at least two equations in system (25) are nonproportional to each other (one of the relations $BA_x = B_xA$ and $BA_t = B_tA$ is not satisfied identically). We can then obtain our ansatz for $\mathcal{F}(c)$. Because $A \neq 0$, it follows from the corresponding pair of equations that $\mathcal{F}$ is a ratio of two polynomials for some fixed values (t, x) (a polynomial of degree five is in the numerator, and a polynomial of degree two is in the denominator). Because $B \neq 0$, it follows from the same pair of equations that $\mathcal{F}_c$ is a ratio of two polynomials for appropriate values (t, x) (a polynomial of degree four is in the numerator, and a polynomial of degree two is in the denominator). Differentiating the expression for $\mathcal{F}$ with respect to c and comparing it with the form of $\mathcal{F}_c$, we see that only the following two cases are possible: the denominators of both functions contain a constant or the fraction is canceled after separation of the integral part, i.e., the possible $\mathcal{F}$ are

$$\mathcal{F}(c) = Qc^5 + Ac^4 + Bc^3 + Dc^2 + Ec + G,$$

$$\mathcal{F}(c) = Ac^4 + Bc^3 + Dc^2 + Ec + G + \frac{K}{c + M}.$$

The algebras corresponding to these functions were studied in Lemmas 1 and 2.

We have thus considered all possible functions $\mathcal{F}(c)$ to which there corresponds an algebra containing a two-dimensional commutative subalgebra. We showed that all cases where this algebra is wider than a two-dimensional one were already considered in Lemmas 1–5 and presented in Tables 1 and 2.

4.6. Equations with two-dimensional noncommutative symmetry algebras. We assume that Eq. (1) with a function $\mathcal{F}(t,x,c)$ is associated with a two-dimensional noncommutative symmetry algebra with the basis $\Xi_1 = \tau_1\partial_t + \xi_1\partial_x$, $\Xi_2 = \tau_2\partial_t + \xi_2\partial_x$. Then, as in the case of a commutative algebra, we find a change of variables $\bar{t} = \varphi(t,x)$, $\bar{x} = \psi(t,x)$ in the equivalence group such that the relations $\tau_1 = 0$ and $\xi_1 = 1$ hold in the new variables. It follows from the commutation relation that $\tau_2 = \tau_2(t)$ and $\xi_2 = -x + D(t)$. We then use the change

$$\varphi(t) = \exp\biggl(-\int\frac{t\,dt}{\tau_2(t)}\biggr), \qquad \psi(x) = x + \psi_1(t), \qquad (\psi_1)_t\tau + D(t) + \psi_1(t) = 0,$$

which belongs to the equivalence group and does not change (τ_1, ξ_1), to reduce the pair of functions (τ_2, ξ_2) to the form $(-t, -x)$. Substituting this in (13), we then see that this group is associated with the function $\mathcal{F}(t,c) = T(c)/t$. We must determine the τ and ξ that correspond to $\mathcal{F}$ of this form. We first express $\mathcal{F}(t,c)$ in (13) in terms of $T(c)$. Reducing it to a common denominator, we obtain

$$\begin{aligned} -T_c t(\xi_t + c\xi_x - c\tau_t - c^2\tau_x) + T(t(\xi_x - 2\tau_t - 3c\tau_x) + \tau) +& \\ + t^2(\xi_{tt} + 2c\xi_{tx} + c^2\xi_{xx} - c\tau_{tt} - 2c^2\tau_{tx} - c^3\tau_{xx}) = 0.& \end{aligned} \tag{27}$$

We further use Eq. (27) together with (13) and discuss the problem in terms of the function $T(c)$.

4.7. Equations with large symmetry algebras containing a two-dimensional noncommutative subalgebra. As in the commutative case, we separately consider the symmetry algebras corresponding to Eq. (1) with functions $T(c)$ in some classes. All further results are formulated up to transformations in the equivalence group.

In this case, we also use changes of variables that belong to the equivalence group of Eqs. (1), (3), and (4) and permit "simplifying" the function $\mathcal{F}(t,x,c)$:

1. The change $\bar{t} = t$, $\bar{x} = x + at$, $\bar{c} = c + a$ takes $\mathcal{F}(t,x,c)$ to $\overline{\mathcal{F}}(\bar{t}, \bar{x} - a\bar{t}, \bar{c} - a)$.
2. The change $\bar{t} = t^{a+1}/(a+1)$, $\bar{x} = x$, $\bar{c} = ct^{-a}$ takes the affine family of functions $\mathcal{F}(t,c) = (Mc^q + ac)/t$ to a linear family of functions $\overline{\mathcal{F}}(\bar{t},\bar{c}) = M\bar{c}^q((a+1)\bar{t})^{\frac{(q-2)a-1}{a+1}}$.
3. The change $\bar{t} = t$, $\bar{x} = x + (t - t\log t)/m$, $\bar{c} = c - \log t/m$ takes $\mathcal{F}(t,c) = \frac{Me^{mc}}{t} + \frac{1}{mt}$ to $\overline{\mathcal{F}}(\bar{c}) = Me^{m\bar{c}}$.
4. The change $\bar{t} = x$, $\bar{x} = t$, $\bar{c} = 1/c$ takes $\mathcal{F}(t,x,c)$ to $\overline{\mathcal{F}}(\bar{t},\bar{x},\bar{c}) = -\bar{c}^3\mathcal{F}(\bar{x},\bar{t},1/\bar{c})$.
5. The change $\bar{t} = t$, $\bar{x} = x - at^2/2$, $\bar{c} = c - at$ takes $\mathcal{F} = a$ to $\overline{\mathcal{F}} = 0$.

Lemma 7. *Functions of the form $T(c) = Qc^5 + Ac^4 + Bc^3 + Dc^2 + Ec + G$ that are reducible by a change of variables to those described in Lemma 1 have a symmetry algebra of dimension greater than two. The function $T(c) = Bc^3 - c/2$ (corresponding to $\mathcal{F} = T(c)/t$) is reducible to $\mathcal{F}(x) = A/x^3$, whose algebra is three-dimensional; its basis is $\Xi_1 = \partial_t$, $\Xi_2 = t\partial_t + (x/2)\partial_x$, $\Xi_3 = t^2\partial_t + tx\partial_x$.*

Proof. We substitute the ansatz $T(c)$ in (27):

$$\begin{aligned} -(5Qc^4 + 4Ac^3 + 3Bc^2 + 2Dc + E)t\xi_t -& \\ - (5Qc^5 + 4Ac^4 + 3Bc^3 + 2Dc^2 + Ec)(t\xi_x - t\tau_t) +& \\ + (2Qc^6 + Ac^5 - Dc^3 - 2Ec^2 - 3Gc)t\tau_x +& \\ + (Qc^5 + Ac^4 + Bc^3 + Dc^2 + Ec + G)(t\xi_x - 2t\tau_t + \tau) +& \\ + t^2(\xi_{tt} + c(2\xi_{tx} - \tau_{tt}) + c^2(\xi_{xx} - 2\tau_{tx}) - c^3\tau_{xx}) = 0.& \end{aligned}$$

Equating the coefficients of all powers of c to zero, we obtain the equations

$$
\begin{aligned}
&Q\tau_x = 0, \qquad Q(3t\tau_t - 4t\xi_x + \tau) + At\tau_x = 0, \qquad 5Qt\xi_t + A(3t\xi_x - 2t\tau_t - \tau) = 0,\\
&4At\xi_t + B(2t\xi_x - t\tau_t - \tau) + Dt\tau_x + t^2\tau_{xx} = 0,\\
&3Bt\xi_t + D(t\xi_x - \tau) + 2Et\tau_x - t^2(\xi_{xx} - 2\tau_{tx}) = 0,\\
&2Dt\xi_t + E(t\tau_t - \tau) + 3Gt\tau_x - t^2(2\xi_{tx} - \tau_{tt}) = 0,\\
&Et\xi_t + G(-t\xi_x + 2t\tau_t - \tau) - t^2\xi_{tt} = 0.
\end{aligned}
\tag{28}
$$

1. Let $Q \neq 0$. Any polynomial of degree five always has a real root. We assume that $T(M) = 0$ and $M \in \mathbb{R}$. We apply the change of variables $\bar{t} = t$, $\bar{x} = x - Mt$, $\bar{c} = c - M$ of form 1, which takes the equation with a function $T(c)$ of the form given in the conditions of the lemma to the equation with $T = Qc^5 + Ac^4 + Bc^3 + Dc^2 + Ec$. For such a $T(c)$, the system of equations becomes

$$
\begin{aligned}
&\tau_x = 0, \qquad 3t\tau_t - 4t\xi_x + \tau = 0, \qquad 5Qt\xi_t + A(3t\xi_x - 2t\tau_t - \tau) = 0,\\
&4At\xi_t + B(2t\xi_x - t\tau_t - \tau) = 0, \qquad 3Bt\xi_t + D(t\xi_x - \tau) - t^2\xi_{xx} = 0,\\
&2Dt\xi_t + E(t\tau_t - \tau) - t^2(2\xi_{tx} - \tau_{tt}) = 0, \qquad E\xi_t - t\xi_{tt} = 0.
\end{aligned}
$$

The first two equations imply $\tau = \tau(t)$ and $\xi = (3\tau_t/4 + \tau/4t)x + C(t)$. From the third, fourth, fifth, and sixth equations, we derive $3t\tau_{tt} + \tau_t - \tau/t = 0$, i.e., $\tau = kt + mt^{-1/3}$ and $\xi = kx + C(t)$. Moreover, we have the equations

$$
15Qt^{4/3}C_t = Am, \qquad 6At^{4/3}C_t = Bm, \qquad 3Bt^{4/3}C_t = Dm, \qquad 9Dt^{4/3}C_t = 2m(3E - 1).
$$

Expressing C from the first of these equations, we obtain $C = -Amt^{-1/3}/5Q + C_1$. The others then become

$$
m\left(\frac{2A^2}{5Q} - B\right) = 0, \qquad m\left(\frac{BA}{5Q} - D\right) = 0, \qquad m\left(\frac{3DA}{5Q} - 2(3E - 1)\right) = 0.
$$

For $m = 0$, we obtain a two-dimensional algebra, and we therefore assume that $m \neq 0$. Then

$$
B = \frac{2A^2}{5Q}, \qquad D = \frac{BA}{5Q} = \frac{2A^3}{25Q^2}, \qquad E = \frac{A^4}{125Q^3} + \frac{1}{3}. \tag{29}
$$

With regard to these relations, we can reduce the expression for T to the form

$$
T(c) = Q\left(c + \frac{A}{5Q}\right)^5 + \frac{1}{3}\left(c + \frac{A}{5Q}\right) - \frac{A}{5Q}\left(\frac{A^4}{5^4Q^3} + \frac{1}{3}\right).
$$

Substituting τ and ξ in the last equation in system (28), we obtain $(3E+4)A = 0$. If $A = 0$, then $B = D = 0$, $E = 1/3$, $C = \text{const}$, and the symmetry algebra $\tau = kt + mt^{-1/3}$, $\xi = kx + C_1$ is three-dimensional for $T(c) = Qc^5 + c/3$. If $A \neq 0$, then

$$
E = -\frac{4}{3}, \qquad T = Q\left(c + \frac{A}{5Q}\right)^5 + \frac{1}{3}\left(c + \frac{A}{5Q}\right),
$$

and we derive $3A^4 + 5^4Q^3 = 0$ from (29). After a similar change of variables, we obtain the function $T(c) = Qc^5 + c/3$, and the corresponding symmetry algebra has the same form as above. A change of variables of form 2 (with $q = 5$ and $a = 1/3$) permits reducing Eq. (1) to the equation with $\mathcal{F} = Qc^5$.

2. Let $Q = 0$ and $A \neq 0$. Then $\tau = \tau(t)$ and $\xi = (2\tau_t/3 + \tau/3t)x + C(t)$, and we have the equations

$$12At\xi_t + B(t\tau_t - \tau) = 0, \qquad 9Bt\xi_t + 2D(t\tau_t - \tau) = 0,$$

$$6Dt\xi_t + 3E(t\tau_t - \tau) - t^2\tau_{tt} - 2t\tau_t + 2\tau = 0, \qquad 3Et\xi_t + 4G(t\tau_t - \tau) - 3t^2\xi_{tt} = 0.$$

Substituting ξ in these equations, we obtain polynomials in x. The first equation implies $2\tau_{tt} + \tau_t/t - \tau/t^2 = 0$, i.e., $\tau = kt + mt^{-1/2}$ $\xi_t = kx + C(t)$. We simplify these relations to

$$8AC_t = Bmt^{-3/2}, \qquad 3BC_t = Dmt^{-3/2},$$

$$8DC_t = 3mt^{-3/2}(2E - 1), \qquad EC_t = 2Gmt^{-3/2} + tC_{tt}.$$

This implies $C = -Bmt^{-1/2}/4A + C_1$, and because $m = 0$ leads to a two-dimensional algebra, we can assume that $m \neq 0$, and hence $D = 3B^2/8A$, $2E - 1 = DB/3A = B^3/8A^2$, and $6G = 3B(2E + 3)/16A$. Therefore,

$$\tau = kt + mt^{-1/2}, \qquad \xi = kx - \frac{Bmt^{-1/2}}{4A} + C_1$$

is a three-dimensional algebra for the function $T = A(c + B/4A)^4 + (c + B/4A)/2$, which is reducible to the form $T = Ac^4 + c/2$ by a change of variables in the equivalence group. The three-dimensional symmetry algebra has the form $\tau = kt + mt^{-1/2}$, $\xi = kx + C_1$ in this case. A change of variables of form 2 can also reduce the corresponding equations to the equation with $\mathcal{F} = Ac^4$.

3. Let $Q = A = 0$ and $B \neq 0$. Then we obtain a polynomial of degree three, which always has a real root. Let $T(M) = 0$, $M \in \mathbb{R}$. We change the variables $\bar{t} = t$, $\bar{x} = x - Mt$, and $\bar{c} = c - M$. This change belongs to the equivalence group and takes T to a function of the form $T = Bc^3 + Dc^2 + Ec$. The system then becomes

$$\begin{aligned} &B(2t\xi_x - t\tau_t - \tau) + Dt\tau_x + t^2\tau_{xx} = 0, \\ &3Bt\xi_t + D(t\xi_x - \tau) + 2Et\tau_x - t^2(\xi_{xx} - 2\tau_{tx}) = 0, \\ &2Dt\xi_t + E(t\tau_t - \tau) - t^2(2\xi_{tx} - \tau_{tt}) = 0, \qquad E\xi_t - t\xi_{tt} = 0. \end{aligned} \tag{30}$$

3.1. Let $E \neq -1, -1/2, 0$. From the two last equations in the system, we then obtain

$$\xi = n(x)\frac{t^{E+1}}{E+1} + l(x), \qquad \tau = k(x)t^{-E} + m(x)t + n_x\frac{t^{E+2}}{(E+1)^2} - n(x)\frac{2Dt^{E+1}}{E(2E+1)}.$$

Substituting these functions in the second equation in system (30), we obtain

$$\begin{aligned} D(l_x - m)t + (2(E+1)m_x - l_{xx})t^2 - Dkt^{-E} + \left(3B + \frac{2D^2}{E(2E+1)}\right)nt^{E+1} - \\ - \frac{D(E+2)(3E+2)}{3E(E+1)^2}n_xt^{E+2} + \frac{3}{E+1}n_{xx}t^{E+3} = 0. \end{aligned} \tag{31}$$

3.1.1. If $E \neq -2, -3/2, 1$, then the powers of the variable t in Eq. (31) are not repeated. Equating the coefficients of distinct powers to zero, we obtain the system

$$D(l_x - m) = 0, \qquad 2(E+1)m_x - l_{xx} = 0, \qquad Dk = 0,$$

$$n\left(3B + \frac{2D^2}{E(2E+1)}\right) = 0, \qquad Dn_x(3E+2) = 0, \qquad n_{xx} = 0.$$

3.1.1a. Let $D = 0$. Then $n = 0$ and $l_x = 2(E+1)m + l_1$. Substituting these relations in the first equation in system (30), we obtain

$$2B((2E+1)m + l_1)t + B(E-1)kt^{-E} + k_{xx}t^{2-E} + m_{xx}t^3 = 0.$$

3.1.1a(1). If $E = -3$, then $k = 0$, $m = l_1/5$, and hence $l = l_1x/5 + l_2$, and the algebra is two-dimensional.

3.1.1a(2). If $E \neq -3$, then all powers of the variable t are distinct, and hence $k = 0$, $m = m_1x + m_2$, and $(2E+1)m_1x + (2E+1)m_2 + l_1 = 0$. Because $E \neq -1/2$, we have $m_1 = 0$ and $l = m_2x + l_2$ and again obtain a two-dimensional algebra.

3.1.1b. Now let $D \neq 0$. Then $k = 0$, $m = m_2$, $l = m_2x + l_1$, $n = n_1x + n_2$, and

$$n\left(3B + \frac{2D^2}{E(2E+1)}\right) = 0, \qquad n_1(3E+2) = 0.$$

3.1.1b(1). If $n = 0$, then the symmetry algebra $\tau = m_2t$, $\xi = m_2x + l_1$ is two-dimensional, and the last equation is satisfied automatically.

3.1.1b(2). If $n \neq 0$ and $n_1 = 0$, then $B = -2D^2/(3E(2E+1))$ and

$$\tau = m_2t - \frac{2Dn_2t^{E+1}}{E(2E+1)}, \qquad \xi = \frac{n_2t^{E+1}}{E+1} + m_2x + l_1.$$

Substituting these relations in the first equation in system (30), we obtain $E = -2$, which is impossible.

3.1.1b(3). Let $n_1 \neq 0$, $n_2 \neq 0$, $E = -2/3$, and $B = -3D^2$. Substituting these conditions in the first equation in system (30), we obtain

$$D^2n_1\left(-\frac{3(E-1)}{(E+1)^2} - \frac{2}{E(2E+1)}\right)t^{E+2} + 2BD\frac{E+2}{E(2E+1)}(n_1x + n_2)t^{E+1} = 0.$$

This implies $n_1 = 0$, which contradicts the above assumption.

3.1.2. We consider the remaining cases.

3.1.2a. For $E = -2$, the last two equations in system (30) imply

$$\xi = -\frac{n(x)}{t} + l(x), \qquad \tau = k(x)t^2 + m(x)t + n_x - \frac{Dn(x)}{3t}.$$

Equation (31) becomes

$$(D(l_x - m) - 3n_{xx})t - (2m_x + l_{xx} + Dk)t^2 + \left(3B + \frac{D^2}{3}\right)\frac{n}{t} = 0.$$

Equating the coefficients of powers of t to zero, we obtain the following two cases.

3.1.2a(1). If $D = 0$, then $n = 0$ and $l_x = -2m + l_1$. Substituting these relations in the first equation in system (30), we obtain $k = 0$, $m = l_1/3$, and $l = l_1x/3 + l_2$, and the algebra is two-dimensional.

3.1.2a(2). If $D \neq 0$ and $n = 0$, then $m = l_x$ and $k = -3l_{xx}/D$. Substituting these relations in the first equation in system (30), we obtain

$$\left(\frac{9B}{D} + D\right)l_{xx}t^2 - 2l_{xxx}t^3 - \frac{3}{D}l_{xxxx}t^4 = 0.$$

This implies $l = l_1x^2 + l_2x + l_3$ or $l_1 = 0$ (the algebra is two-dimensional) or $B = -D^2/9$ and $T(c) = -D^2c^3/9 + Dc^2 - 2c$. The change of variables $\bar{t} = t$, $\bar{x} = x - 3t/d$, $\bar{c} = c - 3/D$ reduces the equation with this function $T(c)$ to the case $T(c) = -D^2c^3/9 + c$. Further, we can again use a change of form 2 (with $q = 3$ and $a = 1$) to reduce it to $\mathcal{F} = Mc^3$.

3.1.2a(3). If $D \neq 0$ and $n \neq 0$, then $B = -D^2/9$. A function of such a form was considered above.

3.1.2b. If $E = -3/2$, then Eq. (31) becomes

$$D(l_x - m)t + (-m_x - l_{xx})t^2 + (-6n_{xx} - Dk)t^{3/2} + \left(3B + \frac{2D^2}{3}\right)nt^{-1/2} - \frac{10D}{3}n_xt^{1/2} = 0.$$

3.1.2b(1). Let $D = 0$. Then $n = 0$ and $l_x = -m + l_1$. Substituting these relations in the first equation in system (30), we obtain

$$2B(-2m + l_2)t - \frac{5B}{2}k(x)t^{3/2} + k_{xx}t^{7/2} + m_{xx}t^3 = 0.$$

Then $m = l_1/2$, $k = 0$, and $l = l_1x/2 + l_2$ is a two-dimensional algebra.

3.1.2b(2). Now let $D \neq 0$. Then $n = n_1$, $k = 0$, $m = m_1$, $l = m_1x + l_1$, and $n_1(9B + 2D^2) = 0$. If $n_1 = 0$, then the algebra is two-dimensional. We therefore assume that $B = -2D^2/9$. We have $\tau = m_1t - 2Dn_1t^{-1/2}/3$ and $\xi = -2n_1t^{-1/2} + m_1x + l_1$ in this case. Substituting these formulas in the last equation in system (30), we obtain $n_1 = 0$. The algebra is two-dimensional in this case.

3.1.2c. If $E = 1$, then Eq. (31) becomes

$$D(l_x - m)t + \left(4m_x - l_{xx} + 3Bn + \frac{2D^2n}{3}\right)t^2 - Dkt^{-1} - \frac{5D}{4}n_xt^3 + \frac{3}{2}n_{xx}t^4 = 0.$$

3.1.2c(1). Let $D = 0$. Then $T(c) = Bc^3 + c$. A function of such a form is equivalent (if a change of form 2 is used) to $\mathcal{F} = Bc^3$.

3.1.c(2). Let $D \neq 0$. Then $l_x = m$, $k = 0$, $n = n_1$, and $l_{xx} = -n_1(B + 2D^2/9)$, i.e.,

$$l = -n_1\left(B + \frac{2D^2}{9}\right)\frac{x^2}{2} + l_1x + l_2, \qquad m = -n_1\left(B + \frac{2D^2}{9}\right)x + l_1.$$

Substituting these relations in the first equation in system (30), we obtain $n_1(B - 2D^2/9) = 0$. If $n_1 = 0$, then the algebra is two-dimensional, and we hence assume that $n_1 \neq 0$. Then $B = 2D^2/9$. For the function $T(c) = 2D^2c^3/9 + Dc^2 + c$, the symmetry algebra

$$\tau = \frac{2}{3}Dn_1t\left(-\frac{2}{3}Dx - t\right) + l_1t, \qquad \xi = n_1\left(\frac{t^2}{2} - \frac{2}{9}D^2x^2\right) + l_1x + l_2$$

is three-dimensional. The change of variables $\bar{t} = t$, $\bar{x} = x + 3t/2D$, $\bar{c} = c + 3/2D$ reduces $T(c)$ to the form $T(c) = 2D^2c^3/9 - c/2$. This is the only case that is not reducible to $\mathcal{F}(c)$, and we distinguished it in the statement of the lemma.

3.2.1. Let $E = -1$. From the last two equations in system (30), we obtain $\xi = n(x)\log t + l(x)$ and $\tau = k(x)t\log t + m(x)t + n_xt\log^2 t - 2Dn(x)$. After the substitution in the second equation in system (30), we obtain

$$(3B + 2D^2)n + D(l_x - m + 4n_x)t + (2k_x - l_{xx})t^2 + D(n_x - k)t\log t - Dn_xt\log^2 t + 3n_{xx}t^2\log t = 0.$$

As a result, we obtain the system

$$n(3B+2D^2)=0,\qquad D(l_x-m+4n_x)=0,\qquad 2k_x-l_{xx}=0,$$

$$D(n_x-k)=0,\qquad Dn_x=0,\qquad n_{xx}=0.$$

3.2.1a. Let $D=0$. Then $n=0$ and $l_x=2k+l_1$. Substituting these relations in the first equation in system (30), we obtain

$$B(3k+2l_1-2m)t-2Bkt\log t+k_{xx}t^3\log t+m_{xx}t^3=0.$$

This implies $m=l_1$, $k=0$, and $l=l_1x+l_2$, and the algebra is two-dimensional.

3.2.1b. Let $D\neq 0$. Then $n=n_2$, $k=0$, $l=l_1x+l_2$, $m=l_1$, and $(3B+2D^2)n=0$. If $n=0$, then the algebra is two-dimensional; otherwise, $B=-2D^2/3$. The last equation in the system implies $n_2=0$, and the algebra in this case is again two-dimensional.

3.2.2. Let $E=-1/2$. Then $\xi=n(x)t^{1/2}+l(x)$. From the third equation in system (30), we obtain $\tau=k(x)t^{1/2}+m(x)t+2n_xt^{3/2}+2Dnt^{1/2}\log t$. Substituting this expression in the second equation in system (30), we obtain

$$\left(\frac{3}{2}Bn-Dk\right)t^{1/2}+D(l_x-m)t+3Dn_xt^{3/2}+$$
$$+(m_x-l_{xx})t^2+3n_{xx}t^{5/2}-2D^2nt^{1/2}\log t=0.$$

3.2.2a. Let $D=0$. Then $n=0$ and $l_x=m+l_1$. Substituting these relations in the first equation in system (30), we obtain

$$B\left(2l_1t-\frac{3}{2}kt^{1/2}\right)+k_{xx}t^{5/2}+m_{xx}t^3=0.$$

This implies $k=0$, $l_1=0$, $m=m_1x+m_2$, and $l=m_1x^2/2+m_2x+l_2$. The symmetry algebra $\tau=m_1xt+m_2t$, $\xi=m_1x^2/2+m_2x+l_2$ is hence three-dimensional for $T(c)=Bc^3-c/2$. We note that the function obtained in item 3.1.2c(2) coincides with this function up to a change of the notation.

3.2.2b. Let $D\neq 0$. Then $n=0$, $k=0$, and $l_x=m$. The first equation in system (30) has the form $Dm_x+tm_{xx}=0$ in this case. Then $m=m_2$, $l=m_2x+l_1$, and the algebra is two-dimensional.

3.2.3. Let $E=0$. Then $\xi=n(x)t+l(x)$, and the third equation in system (30) implies $\tau=k(x)t+m(x)+n_xt^2-2Dnt\log t$. Substituting these relations in the second equation in the system, we obtain

$$-Dm+(3Bn+Dl_x-Dk)t+(-l_{xx}+2k_x-4Dn_x)t^2+3n_{xx}t^3-4Dn_xt^2\log t+2D^2nt\log t=0.$$

3.2.3a. Let $D=0$. Then $n=0$ and $l_x=2k+l_1$. From the first equation in system (30), we obtain $B(2(l_1+k)t-m)+(m_{xx})t^2+t^3k_{xx}=0$. Then $k=-l_1$, $m=0$, $l=-l_1x+l_2$, and the algebra is two-dimensional.

3.2.3b. Let $D\neq 0$. Then $m=0$, $n=0$, $l_x=k=k_2$, and the algebra is two-dimensional.

4. If $Q=A=B=0$ and $D\neq 0$, then the change of variables $\bar{t}=t$, $\bar{x}=x+Et/2D$, $\bar{c}=c+E/2D$ takes $T(c)$ to the form $T=Dc^2+G$. System (28) is then simplified to

$$D\tau_x+t\tau_{xx}=0,\qquad D(t\xi_x-\tau)-t^2(\xi_{xx}-2\tau_{tx})=0,$$

$$2D\xi_t+3G\tau_x-t(2\xi_{tx}-\tau_{tt})=0,\qquad G(-t\xi_x+2t\tau_t-\tau)-t^2\xi_{tt}=0.$$

We derive $\tau = e^{-Dx/t}k(t) + m(t)$ from the first equation, substitute it in the second equation, and obtain

$$\xi = e^{Dx/t}n(t) + l(t) + x\frac{m(t)}{t} + \left(-\frac{k(t)x}{t} + \frac{2k(t)}{D}\right)e^{-Dx/t}.$$

It remains to take the third and fourth equations into account. Substituting the obtained expressions in the third equation, we obtain

$$\frac{2Dn}{t}e^{Dx/t} + \frac{2Dx}{t}\left(m_t - \frac{m}{t}\right) + 2Dl_t - 2m_t + \frac{2m}{t} + tm_{tt} +$$
$$+ \left(-\frac{3D^2x^2k}{t^3} + x\left(\frac{14Dk}{t^2} - \frac{2Dk_t}{t}\right) - \frac{3(GD+2)k}{t} + 10k_t + tk_{tt}\right)e^{-Dx/t} = 0.$$

This relation (because $D \neq 0$) implies $k = n = 0$, $m_t - m/t = 0$, and $2Dl_t - 2m_t + 2m/t + tm_{tt} = 0$. Therefore, $m = m_1 t$, $l = l_1$, and the algebra $\tau = m_1 t$, $\xi = m_1 x + l_1$ is two-dimensional.

5. If $Q = A = B = D = 0$ and $E \neq 0$, then $\tau = k(t)x + m(t)$. Using a change of form 1, we obtain $T(c) = Ec$. A function of such a form is reducible to the zero function by a change of form 2.

6. If $Q = A = B = D = E = 0$, then $T(c) = G$. The equation with a constant function $T(c)$ is reducible to the equation with the zero function by a change of form 3 (with $M = 0$ and $m = 1/G$).

We have thus shown that all functions of the form $\mathcal{F} = (Qc^5 + Ac^4 + Bc^3 + Dc^2 + Ec + G)/t$ with a nonzero symmetry algebra except the function $\mathcal{F} = (Bc^3 - c/2)/t$ with the algebra $\tau = m_1xt + m_2t$, $\xi = m_1x^2/2 + m_2x + l_2$ are taken into account in the commutative case. We try to use admissible changes of variables to simplify the form of the function. We apply changes of forms 2 and 4 to reduce it to $\mathcal{F}(x) = A/x^3$, and the corresponding symmetry algebra has the form $\tau = m_1t^2/2 + m_2t + l_2$, $\xi = (m_1t + m_2)x/2$. The lemma is proved.

Lemma 8. *For functions of the form $T(c) = Ac^4 + Bc^3 + Dc^2 + Ec + G + K/(c+M)$ with $K \neq 0$ that are not reducible by a change of variables to the already considered functions, the symmetry algebra is two-dimensional.*

Proof. As in the commutative case, using a change of form 1, we can assume that $M = 0$. Substituting $T(c)$ of the given form in (27) and reducing to a common denominator, we obtain

$$-c^2t(4Ac^3 + 3Bc^2 + 2Dc + E)(\xi_t + c\xi_x - c\tau_t - c^2\tau_x) + Kt(\xi_t + c\xi_x - c\tau_t - c^2\tau_x) +$$
$$+ c^2(Ac^4 + Bc^3 + Dc^2 + Ec + G)(t\xi_x - 2t\tau_t - 3tc\tau_x + \tau) +$$
$$+ Kc(t\xi_x - 2t\tau_t - 3tc\tau_x + \tau) + t^2c^2(\xi_{tt} + c(2\xi_{tx} - \tau_{tt}) + c^2(\xi_{xx} - 2\tau_{tx}) - c^3\tau_{xx}) = 0. \qquad (32)$$

The left-hand side of the equation contains a polynomial in c identically equal to zero. Equating the coefficients of c^0 and c^1 to zero, we obtain $K\xi_t = 0$ and $K(2t\xi_x - 3t\tau_t + \tau) = 0$. Because $K \neq 0$, we divide the second relation by t and differentiate with respect to t to obtain $3t^2\tau_{tt} - t\tau_t + \tau = 0$, i.e., $\tau = tf(x) + t^{1/3}g(x)$ and $\xi(x) = \int f(x)\,dx$. Substituting these formulas in (32), we again obtain a polynomial in the variable c. We write the coefficients of powers of c as

$$Ag = 0, \qquad A(tf_x + t^{1/3}g_x) = 0, \qquad 4Bg - 3D(t^{5/3}f_x + tg_x) - 3(t^{8/3}f_{xx} + t^2g_{xx}) = 0,$$

$$3Dg = 3t^{5/3}(2E+1)f_x + 2t(3E+1)g_x,$$

$$2(3E+1)g = 27G(t^{5/3}f_x + tg_x), \qquad Gg = 12K(t^{5/3}f_x + tg_x).$$

Let $A \neq 0$. Then $g = f_x = 0$, and the algebra is two-dimensional. Let $A = 0$. The dependence on t is explicitly manifested in the remaining four expressions. The last equation implies $Gg = f_x = g_x = 0$ (because $K \neq 0$). The algebra is two-dimensional for $g = 0$, and we hence assume that $G = 0$. We similarly obtain $E = -1/3$, $D = 0$, and $B = 0$ from the other equations, i.e., the symmetry algebra $\tau = ft + gt^{1/3}$, $\xi = fx + C$ is three-dimensional for $T(c) = -c/3 + K/c$. We use a change of variables of form 2 to reduce the equation with $T(c) = -c/3 + K/c$ to the equation $\mathcal{F}(c) = K/c$ and then use a change of variables of form 4 to reduce the result to the equation with $\mathcal{F}(c) = Kc^4$ whose symmetry algebra was obtained in Lemma 1. The lemma is proved.

Lemma 9. *For functions of the form* $T(c) = A(c^2 + g)^{3/2} + c(c^2 + g)/g$ *with* $A \neq 0$ *and* $g \neq 0$, *the symmetry algebra is three-dimensional, and its basis is* $\Xi_1 = \partial_x$, $\Xi_2 = t\partial_t + x\partial_x$, $\Xi_3 = tx\partial_t + (x^2 - gt^2)/2\partial_x$.

Proof. We note that $3cT = T_c(c^2 + g) - (c^2 + g)$ and substitute T expressed in terms of T_c in (27):

$$T_c(-3t(c\xi_t + c^2(\xi_x - \tau_t) - c^3\tau_x) + (c^2 + g)(t\xi_x - 2t\tau_t - 3tc\tau_x + \tau)) -$$
$$- (c^2 + g)(t\xi_x - 2t\tau_t - 3tc\tau_x + \tau) + 3t^2(c\xi_{tt} + c^2(2\xi_{tx} - \tau_{tt}) + c^3(\xi_{xx} - 2\tau_{tx}) - c^4\tau_{xx}) = 0.$$

Because T_c is not a fractional rational function and is linearly independent of a fractional rational function, the coefficient of T_c is zero, the second term is hence also zero, and we obtain

$$t\xi_x - 2t\tau_t + \tau = 0, \qquad \xi_t + g\tau_x = 0, \qquad 2t\xi_x - t\tau_t - \tau = 0, \qquad g\tau_x + t\xi_{tt} = 0,$$
$$t\xi_x - 2t\tau_t + \tau - 3t^2(2\xi_{tx} - \tau_{tt}) = 0, \qquad \tau_x + t(\xi_{xx} - 2\tau_{tx}) = 0, \qquad \tau_{xx} = 0.$$

It follows from the first, third, and seventh equations in the system that $\tau = k_1xt + k_2t$. We have $\xi = -gk_1t^2/2 + l(x)$ from the second equation. Substituting this relation in the first equation, we obtain $l = k_1x^2/2 + k_2x + l_1$. The three-dimensional symmetry algebra

$$\tau = k_1xt + k_2t, \qquad \xi = \frac{k_1(x^2 - gt^2)}{2} + k_2x + l_1$$

hence corresponds to the function $T(c)$ of the form given in the lemma.

We successively apply changes of variables of the forms 2, 4, and 5 to reduce the equation with $T(c)$ given in the condition of the lemma to the equation with the function $\mathcal{F} = A\bigl(1 + \frac{(t+gc)^2}{t^2+2gx}\bigr)^{3/2}$. The corresponding algebra has the form

$$\tau = -gk_1x + k_2t + l_1, \qquad \xi = k_1\left(3tx + \frac{t^3}{g}\right) + 2k_2x - \frac{l_1t}{g}.$$

The lemma is proved.

Lemma 10. *For functions of the form* $T(c) = Ac^a + pc$ *with* $A \neq 0$ *and* $a \neq -1, 0, \ldots, 5$ *that are not reducible by a change of variables to the already considered functions, the symmetry algebra is two-dimensional.*

Proof. Substituting $T(c)$ of the given form in (27), we obtain

$$-aAc^{a-1}t\xi_t + Ac^a((a-2)t\tau_t - (a-1)t\xi_x + \tau) + (a-3)Ac^{a+1}t\tau_x + (t^2\xi_{tt} - pt\xi_t) +$$
$$+ c(t^2(2\xi_{tx} - \tau_{tt}) - pt\tau_t + p\tau) + c^2(t^2(\xi_{xx} - 2\tau_{tx}) - 2pt\tau_x) - c^3t^2\tau_{xx} = 0.$$

Because the cases $a = -1, 0, 1, \dots, 5$ have already been considered in Lemmas 7 and 8, the powers of c are distinct, and we obtain the system

$$\xi_t = 0, \qquad \tau_x = 0, \qquad (a-2)t\tau_t - (a-1)t\xi_x + \tau = 0, \qquad ;t^2\tau_{tt} + p(t\tau_t - \tau) = 0, \qquad \xi_{xx} = 0.$$

If $p \neq -1$, then we obtain $\tau = kt + mt^{-p}$ and $\xi = nx + l$ from the last two equations. Substituting the result in the third equation, we obtain $(a-1)(k-n)t + (1-(a-2)p)mt^{-p} = 0$. Then $k = n$, $(1-(a-2)p)m = 0$. In this case, either $m = 0$ and the algebra is two-dimensional or $p = 1/(a-2)$ and the symmetry algebra $\tau = kt + mt^{-1/(a-2)}$, $\xi = kx + l$ is three-dimensional for $T(c) = Ac^a + c/(a-2)$. We use a change of variables of form 2, which reduces $T(c) = Ac^a + c/(a-2)$ to the commutative case $\mathcal{F}(c) = Ac^a$.

If $p = -1$, then we obtain $\tau = kt + mt\log t$, $\xi = nx + l$ from the last two equations and substitute these relations in the third equation. As a result, we obtain $((a-1)k + (a-2)m - (a-1)n)t + (a-1)mt\log t = 0$. This implies $m = 0$, $k = n$, and the algebra is two-dimensional. The lemma is proved.

Lemma 11. *For functions of the form $\mathcal{F}(c) = Ae^{ac} + p$, $A \neq 0$, $a \neq 0$, that are not reducible by a change of variables to the already considered functions, the symmetry algebra is two-dimensional.*

Proof. Substituting $T(c)$ of the given form in (27), we obtain

$$Ae^{ac}((t\xi_x - 2t\tau_t + \tau - at\xi_t) - c(at\xi_x - at\tau_t + 3t\tau_x) + ac^2t\tau_x) + pt\xi_x - 2pt\tau_t +$$
$$+ p\tau + t^2\xi_{tt} + c(2t^2\xi_{tx} - t^2\tau_{tt} - 3pt\tau_x) + c^2(t^2\xi_{xx} - 2t^2\tau_{tx}) - c^3t^2\tau_{xx} = 0.$$

Because T is not a fractional rational function and is linearly independent of a fractional rational function, the coefficients of T and the remaining term are zero, and we have

$$t\xi_x - 2t\tau_t + \tau - at\xi_t = 0, \qquad at\xi_x - at\tau_t + 3t\tau_x = 0, \qquad \tau_x = 0,$$
$$pt\xi_x - 2pt\tau_t + p\tau + t^2\xi_{tt} = 0, \qquad 2\xi_{tx} - \tau_{tt} = 0, \qquad \xi_{xx} = 0.$$

This implies $\tau = kt + m$, $\xi = kx + l(t)$ and $l_t = m/at$, $pm + t^2l_{tt} = 0$. Then $l = (m\log t)/a + l_1$ and $m(p - 1/a) = 0$. If $m = 0$, then the algebra is two-dimensional. If $m \neq 0$, then $p = 1/a$, and $\tau = kt + m$, $\xi = kx + m\log t/a + l_1$ is then a three-dimensional algebra for $T(c) = Ae^{ac} + 1/a$. A change of form 3 reduces $T(c) = Ae^{ac} + 1/a$ to the commutative case $\mathcal{F}(c) = Ae^{ac}$. The lemma is proved.

4.8. Proof of the existence of an exact two-dimensional symmetry algebra in other cases. We prove that all functions $T(c)$ with the symmetry algebra of a dimension greater than two were considered in Lemmas 7–11. We consider (27) as an equation for T of the form $-T_cA + TB + D = 0$. The following versions are possible.

1. Let $A = 0$. Then $\tau_x = \xi_t = 0$, and $\tau_t = \xi_x = \text{const}$, $\tau = C_1t + C_3$, and $\xi = C_1x + C_2$. The constant C_1 is nonzero by the assumption that at least one of the functions τ and ξ is not constant. But it then turns out that $B = C_3$, $D = 0$, and hence $T = 0$, which was already considered in Lemma 7.

2. Let $A \neq 0$. We differentiate the relation $-T_cA + TB + D = 0$ with respect to x and t and obtain a system of three equations for $T(c)$ and $T_c(c)$

$$-T_cA + TB + D = 0, \qquad -T_cA_x + TB_x + D_x = 0, \qquad -T_cA_t + TB_t + D_t = 0. \tag{33}$$

Three cases are possible: system (33) is inconsistent, all three equations are equivalent, or at least two of them nonproportional to each other. The first case is not interesting for us, because there do not exist functions $T(c)$ satisfying this system. We consider the remaining two cases.

2.1. Let $BA_x = B_xA$ and $BA_t = B_tA$. Then B is proportional to A with a coefficient depending only on c, $\phi(c)t(\xi_t + c\xi_x - c\tau_t - c^2\tau_x) = t(\xi_x - 2\tau_t - 3c\tau_x) + \tau$.

2.1.1. Let $\tau_x \not\equiv 0$. Then fixing the corresponding values (t, x), we obtain $\phi(c) = (3c+a)/(c^2+bc+d)$, where a, b, and d are constants. Using the obtained ansatz in (27), we obtain

$$t(3c+a)(\xi_t + c\xi_x - c\tau_t - c^2\tau_x) = (c^2+bc+d)((t\xi_x - 2t\tau_t + \tau) - 3tc\tau_x),$$

$$t(a\xi_t + c(3\xi_t + a\xi_x - a\tau_t) + c^2(3\xi_x - 3\tau_t - a\tau_x)) =$$

$$= c^2(t\xi_x - 2t\tau_t + \tau) + bc(t\xi_x - 2t\tau_t + \tau) + d(t\xi_x - 2t\tau_t + \tau) - 3bc^2t\tau_x - 3dct\tau_x.$$

We equate the coefficients of powers of c to zero and as a result obtain

$$2t\xi_x - \tau = t(a-3b)\tau_x + t\tau_t, \qquad 3t\xi_t + t(a-b)\xi_x - b\tau = t(a-2b)\tau_t - 3dt\tau_x,$$

$$at\xi_t = dt\xi_x - 2dt\tau_t + d\tau.$$

Using the change of variables $\bar{t} = t$, $\bar{x} = x - (a-3b)t/3$, $\bar{c} = c - (a-3b)/3$, we obtain new functions $\bar{\tau}(\bar{t}, \bar{x}) = \tau(t, x)$ and $\bar{\xi}(\bar{t}, \bar{x}) = \xi - (a-3b)\tau/3$. Relation (27) is then taken to itself, and the obtained equations become

$$\xi_x = \frac{\tau}{2t} + \frac{\tau_t}{2}, \qquad \xi_t = -\left(\frac{a(a-3b)}{9} + d\right)\tau_x,$$

$$\left(\frac{a(a-3b)}{9} + d\right)\left(2\left(b - \frac{2a}{3}\right)t\tau_x + t\tau_t - \tau\right) = 0.$$

2.1.1a. We first assume that $9d + a(a-3b) = 0$, and then $\xi_x = \tau/2t + \tau_t/2$, $\xi_t = 0$. We write the consistency condition $\tau_{tt} + \tau_t/t - \tau/t^2 = 0$ and find $\tau = k(x)t + m(x)/t$ from it. The formulas for the derivatives of ξ are then simplified: $\xi_t = 0$, $\xi_x = k(x)$, i.e., $\xi = \int k(x)dx$. Substituting these expressions in (27),

$$(m - ct^3k_x - ctm_x)(3T - T_cc) - c^2t^3k_x - c^3t^4k_{xx} - 2cm + 2c^2tm_x - c^3t^2m_{xx} = 0,$$

we obtain a polynomial in t in the left-hand side. We then have

$$k_{xx} = 0, \qquad k_x(3T - T_cc + c) = 0, \qquad m_{xx} = 0, \qquad m_x(3T - T_cc - 2c) = 0, \qquad m(3T - T_cc - 2c) = 0,$$

i.e., $k = k_1x + k_2$, $m = m_1x + m_2$. Let $3T - T_cc - 2c = 0$. Then $T = Mc^3 + c$. If $3T - T_cc - 2c \neq 0$, then $m = 0$ and either $T(c) = Mc^3 - c/2$ or $k = k_2$. As a result, we obtain a two-dimensional algebra. The algebras for such functions $T(c)$ were determined in Lemma 7.

2.1.1b. We assume that $9d + a(a-3b) \neq 0$. Then $2(b - 2a/3)t\tau_x + t\tau_t - \tau = 0$. We determine $\tau = tk(x - 2(b - 2a/3)t)$. Substituting this expression in the formulas for the derivatives of the function ξ, we obtain

$$\xi_x = k - \left(b - \frac{2a}{3}\right)tk', \qquad \xi_t = -\left(\frac{a(a-3b)}{9} + d\right)tk'.$$

We can write the consistency condition as $9(3b-2a)k' = (2(3b-2a)^2 + a(a-3b) + 9d)tk''$, which implies $(2(3b-2a)^2 + a(a-3b) + 9d)k'' = (2a-3b)k' = 0$. By the above assumptions, only the following two cases are possible.

If $k' = 0$, i.e., $k = \text{const}$, then $\xi_x = k$ and $\xi_t = 0$, which means that $\tau = kt$, $\xi = kx + C$ is a two-dimensional algebra.

If $k' \neq 0$, then $k'' = 0$ and $a = 3b/2$, i.e., $k = k_1x + k_2$. We have $\xi_x = k_1x + k_2$ and $\xi_t = -gk_1t$, where $g = d - b^2/4$, which implies $\tau = k_1tx + k_2t$ and $\xi = k_1(x^2 - gt^2)/2 + k_2x + m$. Substituting these expressions in (27), we obtain $k_1[T_c(g + c^2) - 3Tc - (g + c^2)k_1] = 0$. If $k_1 = 0$, then the algebra is two-dimensional. We further assume that $k_1 \neq 0$ and then

$$T(c) = \exp\biggl(\int \frac{3c\,dc}{c^2+g}\biggr)\biggl(M + \int e^{-\int \frac{3c dc}{c^2+g}} dc\biggr) = M(c^2+g)^{3/2} + \frac{c(c^2+g)}{g} \tag{34}$$

(if $g = 0$, then we obtain $T = Mc^3 - c/2$ with an already studied symmetry algebra). The function $T(c)$ of form (34) was considered in Lemma 9.

2.1.2. Let $\tau_x \equiv 0$ and $\xi_t \not\equiv 0$. Then $\phi(c) = a/(1 + bc)$, and we have

$$at(\xi_t + c\xi_x - c\tau_t) = (t(\xi_x - 2\tau_t) + \tau)(1 + bc).$$

Equating the coefficients of powers of c in the obtained polynomial to zero, we obtain $at\xi_t = t\xi_x - 2t\tau_t + \tau$ and $(b - a)t\xi_x + (a - 2b)t\tau_t + b\tau = 0$. Expressing τ_t from the first equation and substituting it in the second equation, we obtain

$$\tau_t = \frac{1}{2}\xi_x - \frac{a}{2}\xi_t + \frac{1}{2t}\tau, \qquad a(t\xi_x + t(a - 2b)\xi_t - \tau) = 0. \tag{35}$$

2.1.2a. In the case $a = 0$, we obtain $\tau = \tau(t)$, $\xi = (2\tau_t - \tau/t)x + C(t)$. Substituting these relations in (27), we obtain

$$T_c\biggl(\Bigl(2t\tau_{tt} - \tau_t + \frac{\tau}{t}\Bigr)x + tC_t + ct\tau_t - c\tau\biggr) - \\ - \Bigl(2t^2\tau_{ttt} - t\tau_{tt} + 2\tau_t - \frac{2\tau}{t}\Bigr)x + t^2C_{tt} + c(3t^2\tau_{tt} - 2t\tau_t + 2\tau) = 0.$$

In the left-hand side, we have a polynomial in x, i.e.,

$$T_c\Bigl(2t\tau_{tt} - \tau_t + \frac{\tau}{t}\Bigr) - \Bigl(2t^2\tau_{ttt} - t\tau_{tt} + 2\tau_t - \frac{2\tau}{t}\Bigr) = 0,$$

$$T_c(tC_t + c(t\tau_t - \tau)) - t^2C_{tt} - c(3t^2\tau_{tt} - 2t\tau_t + 2\tau) = 0.$$

Either we determine $T(c) = Ec + G$ from the first equation, which was already done, or the coefficient of T is zero, $2t^2\tau_{tt} - t\tau_t + \tau = 0$, and then $\tau = kt^{1/2} + mt$. In this case, we consider the second equation, $2T_c(2t^{1/2}C_t - ck) = 4t^{3/2}C_{tt} + ck$. The solvability condition for the equation implies either $k = 0$ and again $T = Ec + G$ or $C_t + 2tC_{tt} = 0$ and $T = -c/2 + G$, which was already considered above.

2.1.2b. In the case $a \neq 0$, we obtain $\xi = x\tau/t + C(t)$ for $a = 2b$. From the first equation in (35), we obtain $2t^2\tau_t = 2t\tau - ax(t\tau_t - \tau) - at^2C_t$. Because $\tau_x = 0$, we have $t\tau_t - \tau = 0$, i.e., $\tau = kt$, $C = \text{const}$, and the algebra is then two-dimensional for any function T. For $a \neq 2b$, we have

$$\xi = \int \frac{\tau\,dt}{t(a - 2b)} + C(t + (2b - a)x), \qquad \tau_t + \frac{b}{(a - 2b)t}\tau = (b - a)C'.$$

If also $a = b$, then $\tau = kt$, $\xi = -kt/a + C(t + ax)$, and relation (27),

$$-T_c(aC' - k)(1 + ac) + aT(aC' - k) + at(1 + ac)^2C'' = 0,$$

implies $T(c) = (1 + ac)(Qc + M)$. This case was already considered above.

If $a \neq b$, then using the relation $\tau = \tau(t)$, we obtain $C = C_1(t+(2b-a)x)+C_2$ and $\tau = kt^{b/(2b-a)}+C_1t$. We also (for $b \neq 0$) have $\xi = -kt^{b/(2b-a)}/b + C_1x + C_2$. Substituting these relations in (27), we obtain

$$k\left(T_c(1+cb) - aT - \frac{(a-b)(1+cb)}{2b-a}\right) = 0.$$

We assume that $k \neq 0$ (otherwise, the algebra is two-dimensional) and obtain

$$T(c) = M(1+cb)^{a/b} - \frac{1+cb}{2b-a}.$$

Up to a change of variables, we obtain a function of the form $T = Mc^p + c/(p-2)$ from the equivalence group, and the symmetry algebra of this function was determined in Lemma 10.

If $b = 0$, then $\tau = k - C_1at$ and $\xi = k\log t/a - C_1ax + C_2$. Substituting these relations in (27), we obtain $T_c - Ta + 1 = 0$, whence we obtain $T(c) = Me^{ac} + 1/a$, which was considered in Lemma 11 and has a three-dimensional symmetry algebra.

2.1.3. Let $\tau_x \equiv 0$ and $\xi_t \equiv 0$. In this case, the condition $\xi_x - \tau_t \not\equiv 0$ is satisfied by the assumption that $A \not\equiv 0$, and we have $\phi(c) = a/c$. Then $(a-1)\xi_x = (a-2)\tau_t + \tau/t = \text{const} = l$. If $a = 2$, then $\tau = lt$, $\xi = lx + m$, and this contradicts the assumption that $\xi_x - \tau_t \not\equiv 0$. If $a = 1$, then $\tau = bt$ and $\xi = \xi(x)$. Substituting these relations in (27), we obtain $(T - T_cc)(\xi_x - b) + tc^2\xi_{xx} = 0$, whence we have either $\xi = lx + m$ or $l = b$ (the algebra is two-dimensional) or $T(c) = Mc$. The symmetry algebra for this function was obtained in Lemma 7.

If $a \neq 1$ and $a \neq 2$, then $\tau = bt^{-1/(a-2)} + lt/(a-1)$, $\xi = lx/(a-1) + m$. We return to relation (27), which now becomes

$$b\left(T_cc - Ta + c\frac{a-1}{a-2}\right) = 0.$$

The case $b = 0$ leads to a two-dimensional algebra. Otherwise, assuming that $b \neq 0$, we obtain $T(c) = Mc^a + c/(a-2)$. The corresponding symmetry algebra was obtained in Lemma 10.

2.2. We assume that there are at least two equations nonproportional to each other in system (33) (one of the relations $BA_x = B_xA$ and $BA_t = B_tA$ is not satisfied identically). We can then determine the ansatz $T(c)$: because $A \neq 0$, the corresponding pair of equations implies that T is a ratio of two polynomials for some fixed (t, x) (a polynomial of degree five is in the numerator, and a polynomial of degree two is in the denominator). And because $B \neq 0$, from the same pair of equations, we find that T_c is a ratio of two polynomials for the corresponding values of (t, x) (a polynomial of degree four is in the numerator, and a polynomial of degree two is in denominator). We differentiate the expression for T with respect to c, compare the result with T_c, and see that only the following two cases are possible. The denominators of both functions contain a constant or the fraction is canceled after separation of the integral part, i.e., the possible functions T are

$$T(c) = Qc^5 + Ac^4 + Bc^3 + Dc^2 + Ec + G, \qquad T(c) = Ac^4 + Bc^3 + Dc^2 + Ec + G + \frac{K}{c}.$$

Such functions were already considered in Lemmas 7 and 8.

We have thus shown that all cases where there is a symmetry algebra of dimension greater than two were considered in the lemmas in the preceding section.

4.9. Equations with one-dimensional symmetry algebras. If the symmetry algebra is one-dimensional, then the generating operator is reducible to the form $\Xi = \partial_x$ by a change of variables, and Eq. (1) with $\mathcal{F}(t, c)$ corresponds to an algebra of such a form. Therefore, up to transformations in the equivalence group, the family of functions $\mathcal{F}$ for which the symmetry algebra of the equation is one-dimensional consists of functions $\mathcal{F}(t, c)$ that are not reducible to functions in the families given in Tables 1–3. We note that the change of variables $\bar{t} = x$, $\bar{x} = t$, which interchanges the roles of the "space" and "time," allows considering $\mathcal{F}(x, c)$ with the one-dimensional algebra and the basis $\Xi = \partial_t$ instead of $\mathcal{F}(t, c)$.

The proof of Theorem 3 is complete.

REFERENCES

1. I. Müller and T. Ruggeri, *Extended Thermodynamics* (Springer Tracts Nat. Phil., Vol. 37), Springer, New York (1998).
2. L. V. Ovsiannikov, "Group properties of the Chaplygin equation [in Russian]," *Prikladnaya Mekhanika i Tekhnicheskaya Fizika*, **1**, No. 3, 126–145 (1960).
3. K. S. Platonova, "Group analysis of the one-dimensional Boltzmann equation I: Symmetry groups," *Differ. Equ.*, **53**, 530—538 (2017).
4. Yu. N. Grigor'ev and S. V. Meleshko, "Complete Lie group and invariant solutions of a system of Boltzmann equations of a multicomponent mixture of gases," *Siberian Math. J.*, **38**, 434–448 (1997).
5. K. S. Platonova, "Group analysis of the one-dimensional Boltzmann equation: II. Equivalence groups and symmetry groups in the special case," *Differ. Equ.*, **53**, 796–808 (2017).
6. S. Lie, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, Teubner, Leipzig (1891).
7. A. González-López, N. Kamran, and P. J. Olver, "Lie algebras of vector fields in the real plane," *Proc. London Math. Soc.*, **s3-64**, 339–368 (1992).