MOMENT-ANGLE MANIFOLDS, 2-TRUNCATED CUBES
AND MASSEY OPERATIONS

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Abstract. We construct a family of manifolds having a nontrivial Massey $n$-product in their cohomology for any given $n$. These manifolds turn out to be smooth closed 2-connected manifolds with a compact torus $\mathbb{T}^m$-action called moment-angle manifolds $Z_{P}$, whose orbit spaces are simple $n$-dimensional polytopes $P$ obtained from a $n$-cube by a sequence of truncations of faces of codimension 2 only ($2$-truncated cubes). Moreover, the polytopes $P$ are flag nestohedra but not graph-associahedra. For the latter class of flag polytopes $Q$ we prove a formula for $\beta^{-1,2(i+1)}(Q)$ and relate it to the structure of the loop homology (Pontryagin algebra) $H_*(\Omega Z_{Q})$.

1. Introduction

Denote by $K$ a simplicial complex of dimension $n-1$ on the vertex set $[m] = \{1, \ldots, m\}$ and by $k$ the base field or the ring of integers. Let $k[v_1, \ldots, v_m]$ be the graded polynomial algebra on $m$ variables, $\deg(v_i) = 2$. The Stanley–Reisner ring (or the face ring) of $K$ over $k$ is the quotient ring

$$k[K] = k[v_1, \ldots, v_m]/I_K$$

where $I_K$ is the ideal generated by square free monomials $v_{i_1} \cdots v_{i_k}$ such that $\{i_1, \ldots, i_k\}$ is not a simplex in $K$. The monomial ideal $I_K$ is called the Stanley–Reisner ideal of $K$. Then $k[K]$ has a structure of a $k$-algebra and a module over $k[v_1, \ldots, v_m]$ via the quotient projection.

In what follows we denote by $P$ a simple $n$-dimensional convex polytope with $m$ facets (i.e faces of codimension 1) $F_1, \ldots, F_m$. Such a polytope $P$ can be defined as a bounded intersection of $m$ halfspaces:

$$P = \{ x \in \mathbb{R}^n : \langle a_i, x \rangle + b_i \geq 0 \text{ for } i = 1, \ldots, m \}, \quad (*)$$

where $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. We assume that the hyperplanes defined by the equations $\langle a_i, x \rangle + b_i = 0$ are in general position, that is, at most $n$ of them meet at a single point. We also assume that there are no redundant inequalities in $(*)$, that is, no inequality can be removed from $(*)$ without changing $P$. Then the facets of $P$ are given by

$$F_i = \{ x \in P : \langle a_i, x \rangle + b_i = 0 \}, \quad \text{for } i = 1, \ldots, m.$$
Let \( A \) be the \( m \times n \) matrix of row vectors \( a_i \), and denote by \( b \) be the column vector of scalars \( b_i \in \mathbb{R} \). Then we can rewrite (\( * \)) as

\[
P = \{ x \in \mathbb{R}^n : A x + b \geq 0 \}.
\]

Consider the affine map

\[
i_P : \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(x) = A x + b.
\]

which embeds \( P \) into

\[
\mathbb{R}^m_+ = \{ y \in \mathbb{R}^m : y_i \geq 0 \text{ for } i = 1, \ldots, m \}.
\]

**Definition.** We define the space \( Z_P \) as a pull-back in the following commutative diagram [4, Construction 3.1.8]:

\[
\begin{array}{ccc}
Z_P & \xrightarrow{i_Z} & \mathbb{C}^m \\
\downarrow & & \downarrow \mu \\
P & \xrightarrow{i_P} & \mathbb{R}^m_+
\end{array}
\]

where \( \mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2) \). The latter map may be thought of as the quotient map for the coordinatewise action of the standard torus

\[
T^m = \{ z \in \mathbb{C}^m : |z_i| = 1 \text{ for } i = 1, \ldots, m \}
\]

on \( \mathbb{C}^m \). Therefore, \( T^m \) acts on \( Z_P \) with quotient \( P \), \( i_Z \) is a \( T^m \)-equivariant embedding with a trivial normal bundle and \( Z_P \) is embedded into \( \mathbb{C}^m \) as a nondegenerate intersection of Hermitian quadrics. One can easily see that \( Z_P \) has a structure of a smooth closed manifold of dimension \( m + n \), called the *moment-angle manifold* corresponding to \( P \).

Suppose \((X, A) = \{(X_i, A_i)\}_{i=1}^m\) is a set of topological pairs. The following notion appeared firstly in the monograph by V.Buchstaber and T.Panov [4] and then was studied intensively and generalized in the works of J.Grbić and S.Theriault [9], K.Iriye and D.Kishimoto [11], A.Bahri, M.Bendersky, F.Cohen, S.Gitler [1] and others.

**Definition.** A *polyhedral product* is a topological space:

\[
(X, A)^K = \bigcup_{I \in K} (X, A)^I,
\]

where \((X, A)^I = \prod_{i=1}^m Y_i \) for \( Y_i = X_i \), if \( i \in I \), and \( Y_i = A_i \), if \( i \notin I \). Particular cases of a polyhedral product \((X, A)^K\) include *moment-angle complexes* \( Z_K = (\mathbb{D}^2, S^1)^K \) and *real moment-angle complexes* \( R_K = (\mathbb{D}^1, S^0)^K \).

Denote by \( K_P \) the boundary \( \partial P^* \) of the dual simplicial polytope. It can be viewed as a \((n-1)\)-dimensional simplicial complex on the set \([m] \), whose simplices are subsets \( \{i_1, \ldots, i_k\} \) such that \( F_{i_1} \cap \ldots \cap F_{i_k} \neq \emptyset \) in \( P \). By [3, Theorem 6.2.4], \( Z_P \) is \( T^m \)-equivariantly homeomorphic to the moment-angle complex \( Z_{K_P} \).

The Tor-groups of \( K \) acquire a topological interpretation by means of the following result due to Buchstaber and Panov.
Theorem 1.1 ([3, Theorem 4.5.4] or [14, Theorem 4.7]). The cohomology algebra of the moment-angle complex \( Z_K \) is given by the isomorphisms
\[
H^\bullet (Z_K; k) \cong \text{Tor}^\bullet_{k[v_1, \ldots, v_m]}(k[K], k) \\
\cong H [\Lambda [u_1, \ldots, u_m] \otimes k[K], d] \\
\cong \bigoplus_{I \subset [m]} \tilde{H}^\bullet (K_I),
\]
where bigrading and differential in the cohomology of the differential bigraded algebra are defined by
\[
\text{bideg } u_i = (-1, 2), \quad \text{bideg } v_i = (0, 2); \quad du_i = v_i, \quad dv_i = 0.
\]

In the third row, \( \tilde{H}^\bullet (K_I) \) denotes the reduced simplicial cohomology of the full subcomplex \( K_I \) of \( K \) (the restriction of \( K \) to \( I \subset [m] \)). The last isomorphism is the sum of isomorphisms
\[
\tilde{H}^p (Z_K) \cong \sum_{I \subset [m]} \tilde{H}^p - |I| - 1 (K_I),
\]
and the ring structure is given by the maps
\[
\tilde{H}^p - |I| - 1 (K_I) \otimes \tilde{H}^q - |J| - 1 (K_J) \to \tilde{H}^p + q - |I| - |J| - 1 (K_{I \cup J}),
\]
which are induced by the canonical simplicial maps \( K_{I \cup J} \hookrightarrow K_I * K_J \) (join of simplicial complexes) for \( I \cap J = \emptyset \) and zero otherwise.

Additively the following theorem due to M.Hochster holds.

Theorem 1.2 ([10]). For any simplicial complex \( K \) on \( m \) vertices we have:
\[
\text{Tor}_{k[v_1, \ldots, v_m]}^{i, 2j} (k[K], k) \cong \bigoplus_{J \subset [m], |J| = j} \tilde{H}^{j-i-1} (K_J).
\]

The ranks of the bigraded components of the Tor-algebra
\[
\beta^{i, 2j} (k[K]) = \text{rk}_k \text{Tor}_{k[v_1, \ldots, v_m]}^{i, 2j} (k[K], k)
\]
are called the bigraded Betti numbers of \( k[K] \) or \( K \), when \( k \) is fixed. A particular case of the Hochster result is the following one. If \( j = i + 1 \) then
\[
\beta^{i, 2(i+1)} (P) = \sum_{J \subset [m], |J| = i+1} (\text{cc}(P_J) - 1),
\]
where \( P_J = \cup_{j \in J} F_j \) and \( \text{cc}(P_J) \) equals the number of connected components of \( P_J \).

Due to [3, Theorem 3.2.9] the Tor-algebra of \( K \) acquires a multigrading and the multigraded components can be calculated in terms of full subcomplexes:

Theorem 1.3. For any simplicial complex \( K \) on \( m \) vertices we have:
\[
\text{Tor}_{k[v_1, \ldots, v_m]}^{i, 2J} (k[K], k) \cong \tilde{H}^{i, |J|-i-1} (K_J),
\]
where \( J \subset [m] \).

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2. Main results

We begin with a definition of a family of simple polytopes called nestohedra and state the result of Buchstaber and Volodin on geometric realization of flag nestohedra.

**Definition.** Let \( S = \{1, 2, \ldots, n+1\}, n \geq 2 \). A building set on \( S \) is a family of subsets \( B = \{B_k \subseteq S\} \), such that: 1) \( \{i\} \in B \) for all \( 1 \leq i \leq n+1 \), 2) if \( B_i \cap B_j \neq \emptyset \), then \( B_i \cup B_j \in B \). A building set is connected if \( [n+1] \in B \).

Then a nestohedron is a simple convex \( n \)-dimensional polytope \( P_B = \sum_{B_k \in B} \Delta B_k \), where in the Minkowski sum one has \( \Delta B_k = \text{conv}\{e_j | j \in B_k\} \subset \mathbb{R}^{n+1} \). Note that facets of \( P_B \) are in 1-1 correspondence with proper elements in \( B \).

**Theorem 2.1** ([5]). A nestohedron \( P_B \) is a flag polytope if and only if it is a 2-truncated cube.

More precisely, if \( P_B \) is a flag polytope, then there exists a sequence of building sets \( B_0 \subset B_1 \subset \cdots \subset B_N = B \), where \( P_{B_0} \) is a combinatorial cube, \( B_i = B_{i-1} \cup \{S_i\} \), and \( P_{B_i} \) is obtained from \( P_{B_{i-1}} \) by 2-truncation at the face \( F_{S_{j_1}} \cap F_{S_{j_2}} \subset P_{B_{i-1}} \), where \( S_i = S_{j_1} \cup S_{j_2} \), and \( S_{j_1}, S_{j_2} \in B_{i-1} \).

The following family of polytopes introduced by Carr and Devadoss [6], are flag nestohedra and therefore by Theorem 2.1 can be realized as 2-truncated cubes.

**Definition.** A graphical building set \( B(\Gamma) \) for a graph \( \Gamma \) on the vertex set \( S \) consists of such \( B_k \) that \( \Gamma_{B_k} \) is a connected subgraph of \( \Gamma \).

Then \( P_{\Gamma} = P_{B(\Gamma)} \) is called a graph-associahedron.

**Example 2.2.** The following families of graph-associahedra are of particular interest in convex geometry, combinatorics and representation theory.

- \( \Gamma \) is a complete graph on \([n+1]\).
  Then \( P_{\Gamma} = Pe^n \) is a permutohedron.
- \( \Gamma \) is a stellar graph on \([n+1]\).
  Then \( P_{\Gamma} = St^n \) is a stellahedron.
- \( \Gamma \) is a cycle graph on \([n+1]\).
  Then \( P_{\Gamma} = Cy^n \) is a cyclohedron (or Bott-Taubes polytope).
- \( \Gamma \) is a chain graph on \([n+1]\).
  Then \( P_{\Gamma} = As^n \) is an associahedron (or Stasheff polytope).

In order to determine the nerve-complex \( K_P \) of a graph-associahedron \( P = P_{\Gamma} \) we should describe the combinatorial structure of its face poset. The following is a reformulation of the general property stated in [3, Theorem 1.5.13].

**Proposition 2.3.** Facets of \( P_{\Gamma} \) are in 1-1 correspondence with non maximal connected subgraphs of \( \Gamma \).

Moreover, a set of facets corresponding to such subgraphs \( \Gamma_{i_1}, \ldots, \Gamma_{i_s} \) has a nonempty intersection iff:

1. For any two subgraphs \( \Gamma_{i_k}, \Gamma_{i_l} \), either they do not have a common vertex or one is a subgraph of another;
(2) If any two of the subgraphs \( \Gamma_{k_1}, \ldots, \Gamma_{k_l}, l \geq 2 \) do not have common vertices, then their union graph is disconnected.

To compute bigraded Betti numbers \( \beta^{-i,2(i+1)}(P) \) of graph-associahedra \( P = P_\Gamma \) we need the following crucial statement due to A.Fenn.

**Lemma 2.4** ([7, Theorem 4.6.4]). Suppose \( P \) is a nestohedron, \( \sigma \) – a subset in its building set \( B \). Consider the following set:

\[
\sigma = \sigma \cup \{ V \in 2^{[n+1]} \setminus B \mid \exists \omega \in \sigma, \omega \subset V \}.
\]

Then \( Pe_\sigma^\perp \) and \( P_\sigma^\perp \) are homotopy equivalent.

**Theorem 2.5.** Let \( P = P_\Gamma \) be a graph-associahedron of dimension \( n \geq 3 \) on a connected graph \( \Gamma \). Then for \( i > i_{\text{max}} \)

\[
\beta^{-i,2(i+1)}(P) = 0,
\]

where \( i_{\text{max}} \) is a maximal possible number of connected subgraphs in \( \Gamma \), each of them either having a nontrivial intersection with a given connected subgraph or having no common vertices and the union of the 2 subgraphs being a connected subgraph in \( \Gamma \).

If \( i_{\text{max}} \) is achieved on a given connected subgraph, the latter will be called **special**. Denote the number of special subgraphs in \( \Gamma \) by \( s \). Then

\[
\beta^{-i_{\text{max}},2(i_{\text{max}}+1)}(P) = s.
\]

**Proof.** By Theorem 1.2 and Proposition 2.3 it is sufficient to prove the following:

(a) We have: \( \text{cc}(P_J) \leq 2 \) if \( |J| > i_{\text{max}} \). Moreover, if \( J = J_1 \sqcup J_2 \) with \( J_1, J_2 \geq 2 \), then there exists another \( J' \subset [m] \) such that \( J' = J'_1 \sqcup J'_2 \) with \( J'_1 = 1 \) and \( |J'| > |J| \).

(b) Suppose \( \text{cc}(P_J) = 2 \) if \( J = J_1 \sqcup J_2, |J| > i_{\text{max}} \). Then either \( |J_1| = 1 \) or \( J_2 = 1 \). Moreover, if \( |J| = i_{\text{max}} + 1 \) then \( J_1 \) consists of a special subgraph of \( \Gamma \) and \( J_2 \) consists of all the \( i_{\text{max}} \) connected subgraphs in \( \Gamma \) that either nontrivially intersects this special subgraph or have no common points with it and in the latter case their union graph is connected (this means, the corresponding facets of \( P \) do not intersect the facet, corresponding to the special subgraph).

(c) Suppose \( |J| > i_{\text{max}} + 1 \). Then \( \text{cc}(P_J) = 1 \).

But in the case of associahedra \( As^n \) the statement (a) follows from [13, Lemmas 2.13, 2.14], the statement (b) follows from [13, Lemmas 2.15, 2.16] and the statement (c) follows from [13, Lemma 2.17]. The general case can now be obtained using Lemma 2.4. \( \square \)

As an application of Theorem 2.5 the following examples can be computed explicitly in terms of the structure of the graph \( \Gamma \), using the definition of the corresponding graph-associahedron \( P = P_\Gamma \) and a simple combinatorial enumeration.

**Corollary 2.6.** For the 4 classical series of graph-associahedra the theorem gives the following values of \( i_{\text{max}} \) and \( s \).
\[ \beta^{-q,2(q+1)}(A^n) = \begin{cases} n + 3, & \text{if } n \text{ is even;} \\ \frac{n+3}{2}, & \text{if } n \text{ is odd;} \end{cases} \]

\[ \beta^{-i,2(i+1)}(A^n) = 0 \quad \text{for } i \geq q + 1, \]

where \( q = q(n) \) is:

\[ q = q(n) = \begin{cases} \frac{n(n+2)}{4}, & \text{if } n \text{ is even;} \\ \frac{n^2}{4}, & \text{if } n \text{ is odd.} \end{cases} \]

\[ \beta^{-q,2(q+1)}(Cy^n) = \begin{cases} 2n + 2, & \text{if } n \text{ is even;} \\ n + 1, & \text{if } n \text{ is odd;} \end{cases} \]

\[ \beta^{-i,2(i+1)}(Cy^n) = 0 \quad \text{for } i \geq q + 1, \]

where \( q = q(n) \) is:

\[ q = q(n) = \begin{cases} \frac{n(n+2)-2}{2}, & \text{if } n \text{ is even;} \\ \frac{(n+1)^2-2}{2}, & \text{if } n \text{ is odd.} \end{cases} \]

\[ \beta^{-q,2(q+1)}(Pe^n) = \binom{n + 1}{2} \]

\[ \beta^{-i,2(i+1)}(Pe^n) = 0 \quad \text{for } i \geq q + 1, \]

where \( q = q(n) = 2^{n+1} - 2^{\left[\frac{n+1}{2}\right]} - 2^{\left[\frac{n-1}{2}\right]} + 1 \)

\[ \beta^{-q,2(q+1)}(St^n) = \binom{n}{\left[\frac{n}{2}\right]} \]

\[ \beta^{-i,2(i+1)}(St^n) = 0 \quad \text{for } i \geq q + 1, \]

where \( q = q(n) = 2^n - 2^{\left[\frac{n}{2}\right]} - 2^{\left[\frac{n-1}{2}\right]} + \left[\frac{n+3}{2}\right] \).

Finally, as graph-associahedra are flag polytopes, we can apply the previous result to studying the loop homology algebra \( H_\ast(\Omega Z P) \) for graph-associahedra \( P \). Namely, due to [8, Theorem 4.3] the number of minimal multiplicative generators of \( H_\ast(\Omega Z P) \) equals to \( \sum_{i=1}^{m-n} \beta^{-i,2(i+1)}(P) \). Then Theorem 2.5 gives us lower bounds for the number of multiplicative generators in the Pontryagin algebra of \( Z P \).

We next consider a particular family of 2-truncated \( n \)-cubes \( P \), one for each dimension \( n \), for which \( Z P \) has a nontrivial Massey product of order \( n \).

**Definition.** Suppose \( I^n \) is an \( n \)-dimensional cube with facets \( F_1, \ldots, F_{2n} \), such that \( F_i \) and \( F_{n+i}, 1 \leq i \leq n \) are parallel (do not intersect). Then \( P \) is a result of a consecutive cut of faces of codimension 2 from \( I^n \), having the following Stanley-Reisner ideal:

\[ I = (v_1v_{n+1}, \ldots, v_nv_{2n}, v_{n+1}v_{n+2}, \ldots, v_{n-1}v_{2n}, \ldots, v_1v_{2n-1}, v_2v_{2n}). \]

This determines uniquely the combinatorial type of \( P \).
Remark. (1) Note that for \( n = 2 \) we have a 2-dimensional cube (the square) and for \( n = 3 \) we have a simple polytope \( P \) for which \( K = K_P \) is a simplicial complex with a nontrivial triple Massey product in \( H^*(Z_K) \) due to the result of Baskakov, see \([2]\).

(2) The 2-truncated cube \( P \) is not a graph-associahedron, due to \([5, \text{Theorem 9.2}], \) as its number of facets \( f_0(P) = \frac{n(n+3)}{2} - 1 < f_0(A^n) = \frac{n(n+3)}{2} \). We can easily construct the building set \( B \) for \( P \) on the vertex set \( S = [n + 1] \) by identifying \( F_i \) with \( \{1, \ldots, i\} \) for \( 1 \leq i \leq n \) and identifying \( F_i \) with \( \{i - n + 1\} \) for \( n + 1 \leq i \leq 2n \). Then we consecutively cut the following faces:

\[
\{1, n\}, \{1, 2, n + 1\} \cdots \{1, 3\}, \{1, 2, 4\}, \ldots, \{1, \ldots, n - 1, n + 1\}.
\]

**Theorem 2.7.** Let \( \alpha_i \in H^3(Z_P) \) be represented by a 3-cocycle \( v_i u_{n+i} \) for \( 1 \leq i \leq n \) and \( n \geq 2 \). Then all Massey products of consecutive elements from \( \alpha_1, \ldots, \alpha_n \) are defined and the whole \( n \)-product is nontrivial.

**Proof.** Let us prove the theorem by induction on \( n \). The base case \( n = 2 \) is trivial: \( \alpha_1 \) and \( \alpha_2 \) are the classes of 3-dimensional spheres in \( Z_P \cong S^3 \times S^3 \) and their cup-product (Massey 2-product) is the fundamental class of \( Z_P \).

We first prove that all Massey products of orders less than \( n \) vanish simultaneously in \( H^{k+1}(Z_P) \cong H[\mathbb{A}[u_1, \ldots, u_n] \otimes k[K], d] \), i.e. contain coboundaries. This can be checked easily using the codimension 2 face cuts we described above (see the Example below).

Now one can see, that the Massey \( n \)-product \( < \alpha_1, \ldots, \alpha_n > \) is defined and any cohomology class, belonging to it, lies in the multigraded component \( H^{(2n-2),(2,\ldots,2)}(Z_P) \) of the moment-angle manifold \( Z_P \) and one of its representatives is the class of the cocycle \( v_1 v_2 u_2 \cdots u_{2n-1} \). Up to sign we have the following equality for a representative \( c \) of an element in \( < \alpha_1, \ldots, \alpha_n > \), see \([12]\):

\[
c = d(c_{1,n+1}) - (-1)^3 v_1 u_{n+1} c_{2,n+1} - v_{1,3} c_{3,n+1} - \cdots - v_{1,n} u_n v_{2n},
\]

where \( (n+1) \times (n+1) \)-matrix \( C \) is upper triangular with zeros on the diagonal and \( c_{i,i+1} = -v_i u_{n+i} \) for \( 1 \leq i \leq n \), such that the following condition holds:

\[
c E_{1,n+1} = d(C) - C
\]

By induction, the indeterminates in the products of orders less than \( n - 1 \) are zero in cohomology of \( Z_P \). By definition of matrix Massey operations, one has: \( d(c_{2,n+1}) \) is a representative in \( < \alpha_2, \ldots, \alpha_n > \) and \( d(c_{1,n}) \) is a representative in \( < \alpha_1, \ldots, \alpha_{n-1} > \). To prove the triviality of the indeterminacies at each step we use induction on \( n \) and the multigrading in \( H^*(Z_P) \). Namely, it is sufficient to show that any multigraded component of the indeterminacy is trivial, see Theorem 1.3. The indeterminacy for the first of the \((n-1)\)-products above lies in the multigraded component of \( v_2 u_3 \cdots u_{2n} \) and the only cocycle there is the coboundary \( d(u_2 \cdots u_{2n}) \). As for the second \((n-1)\)-product above, the indeterminacy lies in the multigraded component of \( v_1 u_2 \cdots u_{2n-1} \) and the only cocycle there is the coboundary \( d(u_1 \cdots u_{2n-1}) \).

Thus, the Massey \( n \)-product \( < \alpha_1, \ldots, \alpha_n > \) is defined and nontrivial, consisting only of the cohomology class of \( v_1 v_2 u_2 \cdots u_{2n-1} \). \( \square \)

**Remark.** Note, that the nontrivial \( n \)-product constructed above is decomposable. Namely, one has: \([v_1 v_2 u_2 \cdots u_{2n-1}] = [v_1 u_{n+1} \cdots u_{2n-1}] [v_2 u_2 \cdots u_n] \).
Example 2.8. Consider the case $n = 4$. Then the Stanley–Reisner ideal of $P$ is
\[ I = (v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_1v_6, v_2v_7, v_3v_8, v_1v_7, v_2v_8) \]
and the cohomology classes $\alpha_i, 1 \leq i \leq 4$ are represented by the cocycles $a_i = v_i u_{4+i}, 1 \leq i \leq 4$. One has:
\[
a_{12} = -d(v_1 u_2 u_5 u_6), \quad a_{23} = -d(v_2 u_3 u_6 u_7), \quad a_{34} = -d(v_2 u_4 u_7 u_8).
\]
Then one has the following cocycle representing a class in $< \alpha_1, \alpha_2, \alpha_3 >$ (here the Massey 2-product of $a$ and $b$ is equal to $\overline{a} \cdot b$, $\overline{a} = (-1)^{|a|}a$):
\[
v_1 u_5 \cdot (-v_2 u_3 u_6 u_7) - v_1 u_2 u_5 u_6 \cdot v_3 u_7 = d(v_1 u_2 u_3 u_5 u_6 u_7)
\]
and the following cocycle representing a class in $< \alpha_2, \alpha_3, \alpha_4 >$:
\[
v_2 u_6 \cdot (-v_3 u_4 u_7 u_8) - v_2 u_3 u_6 u_7 \cdot v_4 u_8 = d(v_2 u_3 u_4 u_6 u_7 u_8).
\]
Thus, the Massey products $< \alpha_1, \alpha_2, \alpha_3 >$ and $< \alpha_2, \alpha_3, \alpha_4 >$ vanish simultaneously and the 4-product $< \alpha_1, \alpha_2, \alpha_3, \alpha_4 >$ is defined. Considering the multigrading in $H^*(\mathbb{Z}_P)$ it is easy to see that the latter 4-product consists of the only class with a representative $v_1 v_8 u_2 \ldots u_7$ in $H^{-6, (2, \ldots, 2)}(\mathbb{Z}_P) \subset H^{-6, 16}(\mathbb{Z}_P) \subset H^{10}(\mathbb{Z}_P)$ and $\mathbb{Z}_P$ is a closed smooth 17-dimensional manifold. More precisely, the representing cocycle $c$ for $< \alpha_1, \alpha_2, \alpha_3, \alpha_4 >$ is equal to:
\[
d(c_{1,5}) - \overline{a}_1 c_{2,5} - \overline{c}_{1,3} c_{3,5} - \overline{c}_{1,4} a_4
\]
which is equal to:
\[
d(c_{1,5}) = d(v_1 u_2 \ldots u_8) + v_1 v_8 u_2 \ldots u_7.
\]
The class $[v_1 v_8 u_2 \ldots u_7] = [v_1 u_5 u_6 u_7] \cdot [v_8 u_2 u_3 u_4]$

Remark. Note, that for $n = 3$ Theorem 2.7 generalizes the construction and the result of the paper [2].

References


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