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Horizontal Motion of a Body Consisting of Two Symmetric Plates

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Abstract—The problem of motion of a rigid body on a fixed horizontal plane is considered. The body consists of two identical symmetric plates. These plates are connected perpendicularly to one another in such a way that their symmetry axes form a single axis being the symmetry axis of the body. All equilibrium positions of the body on the plane are found and their stability analysis is performed. The particular case of a body consisting of two identical elliptic plates is studied.

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INTRODUCTION

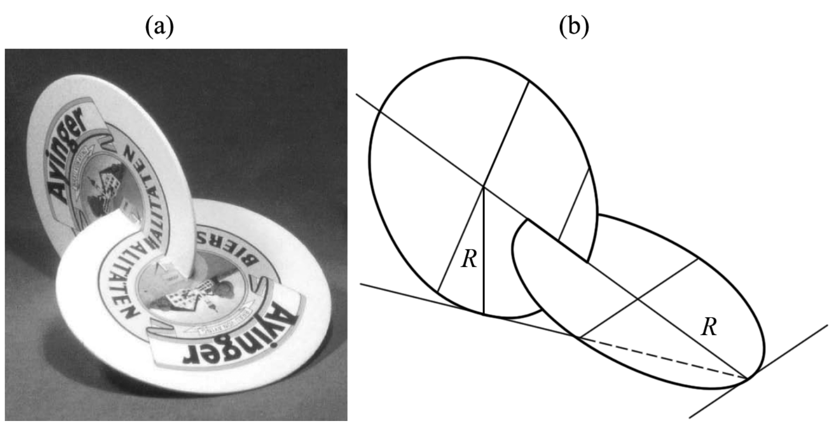

Fig. 1. An oloid and the body described in [4].

Let us consider a rigid body moving without slip on a fixed horizontal plane. The body consists of two identical symmetric plates connected perpendicularly to one another in such a way that their symmetry axes form a single axis being the symmetry axis of the body. During its motion, the body is in contact with the horizontal plane at two points (Fig. 1).

The most known bodies corresponding to the above situation are an oloid [1–3] and the body described in [4]. The oloid illustrated in Fig. 1a consists of two identical disks of radius R connected perpendicularly to one another in such a way that the circumference of one of the disks passes through the center of the other disk and vice versa. The body discussed in [4] is similar to this oloid: it consists of two identical mutually perpendicular disks, but the distance between their centers is equal to $R\sqrt{2}$ (Fig. 1b). The motion of bodies similar to these two ones is also of interest.

The theory proposed in [2, 3] allows one to study the motion of a body on a fixed horizontal plane in the case when this body consists of two identical symmetric plates of arbitrary shape. Based on this theory, in [5] the above problem is studied in the case of two identical elliptic plates connected perpendicularly to one another along the semimajor axis.

In this paper we assume that the shape of symmetric plates is arbitrary. All equilibrium positions of the body under consideration are found. Their stability analysis is performed.

FORMULATION OF THE PROBLEM

Let us consider a rigid body consisting of two identical symmetric plates connected perpendicularly to one another. The body moves on a fixed horizontal plane. Let 2Δ be the distance between the mass centers C_1 and C_2 of the first and second plates. Since the plates are identical, the mass center G of the body is situated at the middle of the segment C_1C_2, i.e., $GC_1 = GC_2 = \Delta$.

In order to describe the motion of the body, as is done in [2, 3, 5], we introduce the following moving coordinate system $Gx_1x_2x_3$: the Gx_2 axis is perpendicular to the plane of the first plate and the Gx_3 axis is perpendicular to the plane of the second plate (Fig. 2). The unit vectors of this coordinate system are denoted by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$.

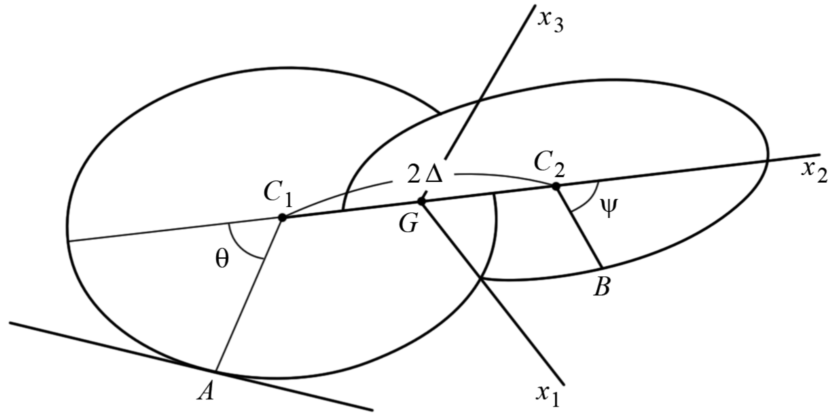

Fig. 2. A body consisting of two symmetric plates.

Let A and B be the points of contact between the body and the horizontal plane. The position of the point A is defined by the angle θ between the line GC_1 and the line C_1A directed from the mass center C_1 to the contact point A (Fig. 2). It is assumed that the shape of the first plate is completely defined if a dependence of the distance C_1A on θ is given: $C_1A = r(\theta)$. Similarly, the position of the point B is defined by the angle ψ between the line GC_2 and the line C_2B directed from the mass center C_2 to the contact point B (Fig. 2). The shape of the second plate is defined by a relation $C_2B = r(\psi)$. Since these two plates are identical, we can assume that their shapes are defined by one and the same function $r(\cdot)$ dependent on the arguments θ and ψ: $C_1A = r(\theta)$ and $C_2B = r(\psi)$. The radius vector of the point A is of the following form with respect to the coordinate system $Gx_1x_2x_3$:

$$\mathbf{GA} = \boldsymbol{r}_1 = r(\theta)\sin\theta\,\boldsymbol{e}_1 - \big(\Delta + r(\theta)\cos\theta\big)\,\boldsymbol{e}_2.$$

For the radius vector of the pont B, we have

$$\mathbf{GB} = \boldsymbol{r}_2 = \big(\Delta + r(\psi)\cos\psi\big)\boldsymbol{e}_2 - r(\psi)\sin\psi\,\boldsymbol{e}_3.$$

Since our mechanical system is a system of one degree of freedom, there exists a relation between θ and ψ. Let us find this relation. During the motion of the body on the plane, the vectors $\boldsymbol{r}_2 - \boldsymbol{r}_1$, ${(\boldsymbol{r}_1)'}_\theta$, and ${(\boldsymbol{r}_2)'}_\psi$ belong to the support plane. The corresponding condition is of the form

$$< \boldsymbol{r}_2 - \boldsymbol{r}_1,\, {(\boldsymbol{r}_1)'}_\theta,\, {(\boldsymbol{r}_2)'}_\psi >= 0,$$

where $< \cdot, \cdot, \cdot >$ indicates the scalar triple product of these vectors. From this condition we can find the desired relation

$$\big(r'_\psi(\psi)\sin\psi + r(\psi)\cos\psi\big)\,\Phi(\theta) + \big(r'_\theta(\theta)\sin\theta + r(\theta)\cos\theta\big)\,\Phi(\psi) = 0, \tag{1}$$

where

$$\Phi(x) = r^2(x) + \Delta r(x)\cos x + \Delta r'_x(x)\sin x.$$

Based on this relation, we can prove the following assertion.

Assertion 1. *The variables θ and ψ cannot be equal to zero simultaneously.*

Proof. Indeed, if we assume that $\psi = \theta = 0$, then relation (1) becomes of the form $r^2(0)\big(\Delta + r(0)\big) = 0$. Since $r(\theta) > 0$ for all the values of θ when the motion on the plane is possible, we conclude that this relation is not valid if $\psi = \theta = 0$. This assertion is proved. □

In our further discussion, we assume that the dependence $\psi = \psi(\theta)$ can be determined from relation (1).

THE POTENTIAL ENERGY OF THE BODY. EQUILIBRIUM OF THE BODY ON THE PLANE

Now we consider the equation of the support plane in the coordinate system $Gx_1x_2x_3$. Let X, Y, and Z be the coordinates of an arbitrary point of this plane with respect to the coordinate system $Gx_1x_2x_3$. The desired equation can be derived from the following condition: during the motion process, the points A and B

and the tangent vector to the boundary of the first plate from the point A belong to the support plane. This condition can be written as

$$\begin{vmatrix} X - r(\theta)\sin\theta & Y + \Delta + r(\theta)\cos\theta & Z \\ -r(\theta)\sin\theta & r(\theta)\cos\theta + r(\psi)\cos\psi + 2\Delta & -r(\psi)\sin\psi \\ r(\theta)\cos\theta + r'_\theta(\theta)\sin\theta & r(\theta)\sin\theta - r'_\theta(\theta)\cos\theta & 0 \end{vmatrix} = 0.$$

Finally, the desired equation takes the following form with respect to the coordinate system $Gx_1x_2x_3$:

$$(r'_\theta(\theta)\cos\theta - r(\theta)\sin\theta)X + (r'_\theta(\theta)\sin\theta + r(\theta)\cos\theta)Y - \frac{\left(r'_\psi(\psi)\cos\psi - r(\psi)\sin\psi\right)\Phi(\theta)}{\Phi(\psi)}Z + \ldots = 0.$$

The absolute term of this equation is not important for our further discussion and is omitted. The unit vector

$$\boldsymbol{n} = \frac{\Phi(\psi)}{\sqrt{\left(r^2(\theta) + (r'_\theta(\theta))^2\right)\Phi^2(\psi) + \left(r'_\psi(\psi)\cos\psi - r(\psi)\sin\psi\right)^2\Phi^2(\theta)}}$$

$$\times\left[(r'_\theta(\theta)\cos\theta - r(\theta)\sin\theta)\boldsymbol{e}_1 + (r'_\theta(\theta)\sin\theta + r(\theta)\cos\theta)\boldsymbol{e}_2 - \frac{\left(r'_\psi(\psi)\cos\psi - r(\psi)\sin\psi\right)\Phi(\theta)}{\Phi(\psi)}\boldsymbol{e}_3\right]$$

is the normal vector to the support plane. Based on the explicit expressions for the vector $\mathbf{GA} = \boldsymbol{r}_1$ and the normal vector $\boldsymbol{n}$, we come to the following formula for the potential energy of the body under consideration:

$$V = Mgz_G = -Mg(\mathbf{GA}\cdot\boldsymbol{n}).$$

Here M is the mass of the body and g is the gravitational acceleration. The explicit form of the potential energy is

$$V = \frac{Mg\,\Phi(\theta)\,\Phi(\psi)}{\sqrt{\left(r^2(\theta) + (r'_\theta(\theta))^2\right)\Phi^2(\psi) + \left(r'_\psi(\psi)\cos\psi - r(\psi)\sin\psi\right)^2\Phi^2(\theta)}}.$$

The critical points of the potential energy correspond to the equilibrium positions on the plane. These critical points can be found from the equation

$$\begin{aligned} &(r(\theta)r'_\theta(\theta) + \Delta r'_\theta(\theta)\cos\theta - \Delta r(\theta)\sin\theta)\Phi(\psi)r(\psi)\sin\psi \\ &+ \frac{\Phi(\theta)r(\theta)r(\psi)\sin\theta\sin\psi}{2}\left(r'_\psi(\psi)\sin\psi + r(\psi)\cos\psi\right) \\ &- \left(r(\psi)r'_\psi(\psi) + \Delta r'_\psi(\psi)\cos\psi - \Delta r(\psi)\sin\psi\right)\Phi(\theta)r(\theta)\sin\theta \\ &- \frac{\Phi(\psi)r(\theta)r(\psi)\sin\theta\sin\psi}{2}(r'_\theta(\theta)\sin\theta + r(\theta)\cos\theta) = 0. \end{aligned} \tag{2}$$

Based on (2), we can formulate the following assertions.

Assertion 2. *If the condition*

$$\Phi(\theta)(r'_\theta(\theta)\sin\theta + r(\theta)\cos\theta) = 0 \tag{3}$$

is valid, then $\psi = \theta$ is an equilibrium position of the body.

Proof. By direct inspection, we see that condition (2) is valid for $\theta = \psi$. Substituting $\psi = \theta$ into (1), we conclude that relation (1) takes the form (3). In other words, the equilibrium position $\psi = \theta$ exists only for those θ that satisfy condition (3). This assertion is proved. □

Assertion 3. *The equilibrium position $\theta = 0$ exists if and only if the point $\theta = 0$ is an extreme point of the function $r(\theta)$.*

Proof. Substituting $\theta = 0$ into (2), we represent it in the form

$$r'_\theta(\theta)\,|_{\theta=0}\,(\Delta + r(0))\,(\Phi(\psi)\, r(\psi)\sin\psi)\,|_{\theta=0} = 0.$$

According to Assertion 1, the function $\Phi(\psi)\, r(\psi)\sin\psi$ cannot be equal to zero for $\theta = 0$. Hence, $\theta = 0$ corresponds to the equilibrium position only if $r'_\theta(\theta)\,|_{\theta=0} = 0$. This assertion is proved. □

In our further discussion, we assume that the function $r(\theta)$ reaches its strict minimum at the point $\theta = 0$. In addition, we assume that this function is even. Based on these assumptions, we can prove the following assertion.

Assertion 4. *If the function $r(\theta)$ is even and the angle θ satisfies* (3), *then $\psi = \theta$ and $\psi = -\theta$ are the equilibrium positions of the body under consideration.*

Proof. If $r(\theta)$ is an even function, then the function $r'_\theta(\theta)$ is odd, whereas the function $r'_\theta(\theta)\sin\theta$ is even. In (1), hence, all the functions are even, whereas relations (1) and (2) remain the same if θ is replaced by $-\theta$. This assertion is proved. □

STABILITY OF EQUILIBRIUM POSITIONS

The stability of the above equilibrium positions depends on the sign of the second derivative of the potential energy V at the corresponding equilibrium position. Based on this second derivative for $\psi = \theta$, we come to the following assertion.

Assertion 5. *The equilibrium position $\psi = \theta$ is stable if*

$$2r(\theta)\sin\theta - \left(r^2(\theta)\sin^2\theta + 2\,(\Delta + r(\theta)\cos\theta)^2\right) K > 0, \tag{4}$$

where K is the curvature of the curve bounding the plate at this equilibrium position.

Based on (4), we can conclude that the equilibrium position $\psi = \theta$ is stable for sufficiently small Δ, i.e., when the mass centers of the plates are close to one another.

Now we find a stability condition for the equilibrium position $\theta = 0$. Note that for $\theta = 0$ we have $\psi = \psi_0$, where ψ_0 can be determined from the equation

$$r^2(\psi_0) + (r(0) + 2\Delta)\left(r'_\psi(\psi_0)\sin\psi_0 + r(\psi_0)\cos\psi_0\right) = 0.$$

Utilizing this equation, we can find $r'_\psi(\psi_0)$ and use it to determine the second derivative of V for $\theta = 0$. As a result, we come to the following assertion.

Assertion 6. *The equilibrium position $\theta = 0$ is stable if*

$$2\Delta^2 + \left(r(0) + 3r(\psi_0)\cos\psi_0 - Kr^2(\psi_0)\sin^2\psi_0\right)\Delta$$
$$+\, r(\psi_0)\left(r(0)\cos\psi_0 + r(\psi_0) - Kr(0)\, r(\psi_0)\sin^2\psi_0\right) > 0, \tag{5}$$

where $K = \dfrac{r(0) - r''(0)}{r^2(0)}$ is the curvature of the curve bounding the plate for $\theta = 0$.

Condition (5) is a second-degree polynomial with respect to Δ. Since Δ^2 increases faster than Δ, we can conclude that the equilibrium position $\theta = 0$ is stable if Δ is large enough. In other words, the equilibrium position $\theta = 0$ is stable if the mass centers of the plates are sufficiently far from one another.

MOTION OF A BODY CONSISTING OF TWO ELLIPTIC PLATES

The above general conclusions on the existence and stability of equilibrium positions of a rigid body consisting of two symmetric plates are illustrated here by the example of two identical elliptic plates whose semiaxes are equal to a and b $(a > b)$. These plates are connected perpendicularly to one another along the larger symmetry axis in such a way that the distance between their mass centers C_1 and C_2 is equal to 2Δ, $\Delta \le a$. Below we use the coordinate system $Gx_1x_2x_3$. The position of the body is described by the angles θ and ψ (Fig. 2). The radius vector of the point A and the radius vector of the point B are of the form

$$\mathbf{GA} = \boldsymbol{r}_1 = \frac{ab\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\,\boldsymbol{e}_1 - \left(\Delta + \frac{ab\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\right)\boldsymbol{e}_2,$$

$$\mathbf{GB} = \boldsymbol{r}_2 = \left(\Delta + \frac{ab\cos\psi}{\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}}\right)\boldsymbol{e}_2 - \frac{ab\sin\psi}{\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}}\boldsymbol{e}_3.$$

In this case, in other words, the function $r(x)$ defining the shape of the plate takes the form

$$r(x) = \frac{ab}{\sqrt{a^2\sin^2 x + b^2\cos^2 x}}.$$

Equation (1) can be rewritten as

$$2\Delta b\cos\theta\cos\psi + a\cos\theta\sqrt{a^2\sin^2\psi + b^2\cos^2\psi} + a\cos\psi\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = 0. \tag{6}$$

The existence condition (2) takes the form

$$\left(2a^2 - b^2 - 2\Delta^2\right)\sin\theta\sin\psi\left(\cos\psi\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} - \cos\theta\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}\right) = 0. \tag{7}$$

From (7) it follows that, if $2a^2 - b^2 - 2\Delta^2 = 0$, the body is in its indifferent equilibrium. The potential energy is constant during the motion of the body. In addition, from (7) we conclude that the body is at equilibrium for $\theta = 0$. Another equilibrium position is defined by the condition

$$\cos\psi\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = \cos\theta\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}.$$

From this condition and condition (6), we see that both of them are valid simultaneously only for $\theta = \pm\psi = \pi/2$.

This equilibrium position is stable if condition (4) is valid. Into this condition we substitute the explicit expression for $r(\theta)$ and the following expression for the curvature:

$$K = \frac{ab\left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{3}{2}}}{\left(a^4\sin^2\theta + b^4\cos^2\theta\right)^{\frac{3}{2}}}.$$

Evaluating the resulting expression for $\theta = \pi/2$, we obtain

$$\frac{b}{a^2}\left(2a^2 - b^2 - 2\Delta^2\right) > 0.$$

Thus, the equilibrium position $\theta = \pm\psi = \pi/2$ is stable if $2a^2 - b^2 - 2\Delta^2 > 0$ and is unstable if $2a^2 - b^2 - 2\Delta^2 < 0$. Hence, the stability of this equilibrium position is possible if the mass centers of the plates are close to one another. A similar conclusion is stated in [5].

Now we study the stability of the equilibrium position $\theta = 0$. According to (6), the angle ψ can be obtained from the following formula for $\theta = 0$:

$$\cos\psi = \cos\psi_0 = -\frac{a^2}{\sqrt{a^4 + 4ab^2\Delta + 4b^2\Delta^2}}.$$

Substituting $\theta = 0$ and $\psi = \psi_0$ into (5), we rewrite this condition as

$$\frac{4\Delta\left(a + \Delta\right)\left(b^2 + 2\Delta^2 - 2a^2\right)}{\left(a + 2\Delta\right)^2} > 0.$$

This means that the equilibrium position $\theta = 0$, $\psi = \psi_0$ is stable if $b^2 + 2\Delta^2 - 2a^2 > 0$ and is unstable if $b^2 + 2\Delta^2 - 2a^2 < 0$. Hence, the stability of this equilibrium position is possible if the mass centers of the plates are far enough from one another.

Thus, we completely studied the stability of equilibrium positions of a body consisting of two elliptic plates.

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